C. DUMITRESCU V. SELEACU

SMARANDACHE NOTIONS

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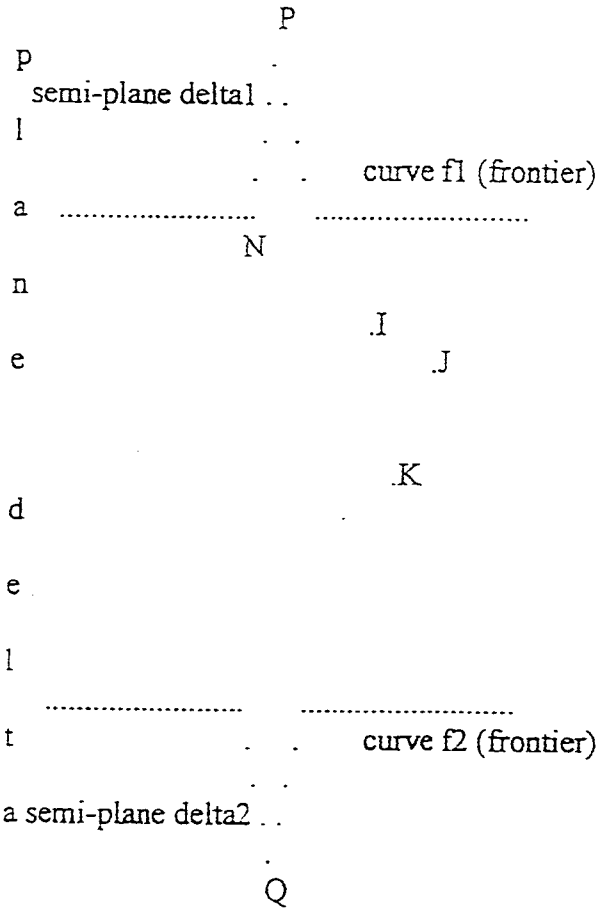

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1998

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Dr. C. Dumitrescu & Dr. V. Seleacu
Department of Mathematics
University of Craiova, Romania;

FOREWARD

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FACTORS AND PRIMES IN TWO SMARANDACHE SEQUENCES

RALF W. STEPHAN

ABSTRACT. Using a personal computer and freely available software, the author factored some members of the Smarandache consecutive sequence and the reverse Smarandache sequence. Nearly complete factorizations are given up to Sm(80) and RSm(80). Both sequences were excessively searched for prime members, with only one prime found up to Sm(840) and RSm(750): RSm(82)= 828180 $\cdots$ 10987654321.

1. INTRODUCTION

Both the Smarandache consecutive sequence, and the reverse Smarandache sequence are described in [S93]. Throughout this article, Sm(n) denotes the nth member of the consecutive sequence, and RSm(n) the nth member of the reverse sequence, e.g. Sm(11)=1234567891011, and RSm(11)=1110987654321.

The Fundamental Theorem of Arithmetic states that every $n \in \mathbb{N}$, $n > 1$ can be written as a product $p_1 p_2 p_3 \ldots p_k$ of a finite number of primes. This "factorization" is unique for n if the p_k are ordered by size. A proof can be found in [R85].

Factorization of large numbers has rapidly advanced in the past decades, both through better algorithms and faster hardware. Although there is still no polynomial-time algorithm known for finding prime factors p_k of composite numbers $n = \prod p_k$, several methods have been developed that allow factoring of numbers with 100 digits or more within reasonable time:

- the elliptic curve method (ECM) by Lenstra [L87], with enhancements by Montgomery [M87][M92] and others, has found factors with up to 49 digits, as of April 1998. Its running time depends on the size of the unknown p, and only slightly on the size of n.
- the quadratic sieve [S87] and the number field sieve [LL93]. The running time of these methods depends on the size of n. Factors with 60-70 digits are frequently found by NFSNet[1].

For $\log p \gg 50$ and $\log n / \log p \approx 2$, sieving methods are faster than ECM. Because ECM time depends on p, which is unknown from the start, it is difficult to predict when a factor will be found. Therefore, when fully factoring a large number, one tries to eliminate small factors first, using conventional sieving and other methods, then one looks for factors with 20, 30, and 40 digits using ECM, and finally, if there is enough computing power, one of the sieving methods is applied.

The primality of the factors and the remaining numbers is usually shown first through a probabilistic test [K81] that has a small enough failure probability like 2^{-50}. Such a prime is called a probable prime. Proving primality can be done using number theory or the ECPP method by Atkin/Morain [AM93].

[1] *URL:* `http://www.dataplex.net/NFSNet/`

In the following, p_n denotes a probable prime of n digits, P_n is a proven prime with n digits, and c_n means a composite number with n digits.

2. Free software

For computations with large numbers, it is not necessary to buy one of the well known Computer Algebra software packages like Maple or Mathematica. There are several multiprecision libraries freely available that can be used with the programming language C. The advantage of using one of these libraries is that they are usually by an order of magnitude faster than interpreted code when compared on the same machine [Z98].

For factoring, we used `science0`[2] and `GMP-ECM`[3]. To write the program for finding prime members of Sm(n) and RSm(n), we used the `GMP`[4] multiprecision library. For proving primality of RSm(82), we used `ECPP`[5].

3. Factorization results

We used `science0` to eliminate small factors of Sm(n) and RSm(n) with $1 < n \leq 80$, and `GMP-ECM` to find factors of up to about 40 digits. The system is a Pentium 200 MHz running Linux[6].

The timings we measured for reducing the probability of a factor with specific size to $1/e$ are given in the following table:

$\log p$	$\log n$	B1	curves	time
20	40	$1.5 \cdot 10^4$	100	7 minutes
30	60	$3 \cdot 10^5$	780	23 hours
40	80	$4 \cdot 10^6$	4800	107 days

TABLE 1. Time to find p with probability $1 - 1/e$ on a Pentium 200 MHz using `GMP-ECM` under Linux

All remaining composites were searched with ECM parameter B1=40000 and 200 curves were computed. Therefore, the probability of a remaining factor with less than 24 digits is less than $1/e$. No primes were proven. The following tables list the results.

[2] *URL:* `http://www.perfsci.com`
[3] *URL:* `http://www.loria.fr/~zimmerma/records/ecmnet.html`
[4] *URL:* `http://www.matematik.su.se/~tege/gmp/`
[5] *URL:* `http://lix.polytechnique.fr/~morain/Prgms/ecpp.english.html`
[6] *URL:* `http://www.linux.org`

n	known factors of Sm(n)
2	$2^2 \cdot 3$
3	$3 \cdot 41$
4	$2 \cdot 617$
5	$3 \cdot 5 \cdot 823$
6	$2^6 \cdot 3 \cdot 643$
7	$127 \cdot 9721$
8	$2 \cdot 3^2 \cdot 47 \cdot 14593$
9	$3^2 \cdot 3607 \cdot 3803$
10	$2 \cdot 5 \cdot 1234567891$
11	$3 \cdot 7 \cdot 13 \cdot 67 \cdot 107 \cdot 630803$
12	$2^3 \cdot 3 \cdot 2437 \cdot p_{10}$
13	$113 \cdot 125693 \cdot 869211457$
14	$2 \cdot 3 \cdot p_{18}$
15	$3 \cdot 5 \cdot p_{19}$
16	$2^2 \cdot 2507191691 \cdot p_{13}$
17	$3^2 \cdot 47 \cdot 4993 \cdot p_{18}$
18	$2 \cdot 3^2 \cdot 97 \cdot 88241 \cdot p_{18}$
19	$13 \cdot 43 \cdot 79 \cdot 281 \cdot 1193 \cdot p_{18}$
20	$2^5 \cdot 3 \cdot 5 \cdot 323339 \cdot 3347983 \cdot p_{16}$
21	$3 \cdot 17 \cdot 37 \cdot 43 \cdot 103 \cdot 131 \cdot 140453 \cdot p_{18}$
22	$2 \cdot 7 \cdot 1427 \cdot 3169 \cdot 85829 \cdot p_{22}$
23	$3 \cdot 41 \cdot 769 \cdot p_{32}$
24	$2^2 \cdot 3 \cdot 7 \cdot 978770977394515241 \cdot p_{19}$
25	$5^2 \cdot 15461 \cdot 31309647077 \cdot p_{25}$
26	$2 \cdot 3^4 \cdot 21347 \cdot 2345807 \cdot 982658598563 \cdot p_{18}$
27	$3^3 \cdot 19^2 \cdot 4547 \cdot 68891 \cdot p_{32}$
28	$2^3 \cdot 47 \cdot 409 \cdot 416603295903037 \cdot p_{27}$
29	$3 \cdot 859 \cdot 24526282862310130729 \cdot p_{26}$
30	$2 \cdot 3 \cdot 5 \cdot 13 \cdot 49269439 \cdot 370677592383442753 \cdot p_{23}$
31	$29 \cdot 2597152967 \cdot p_{42}$
32	$2^2 \cdot 3 \cdot 7 \cdot 45068391478912519182079 \cdot p_{30}$
33	$3 \cdot 23 \cdot 269 \cdot 7547 \cdot 116620853190351161 \cdot p_{31}$
34	$2 \cdot p_{50}$
35	$3^2 \cdot 5 \cdot 139 \cdot 151 \cdot 64279903 \cdot 4462548227 \cdot p_{37}$
36	$2^4 \cdot 3^2 \cdot 103 \cdot 211 \cdot p_{56}$
37	$71 \cdot 12379 \cdot 4616929 \cdot p_{52}$
38	$2 \cdot 3 \cdot 868939563541898787756 43 \cdot p_{43}$
39	$3 \cdot 67 \cdot 311 \cdot 1039 \cdot 62161577813320317996884 69 \cdot p_{36}$
40	$2^2 \cdot 5 \cdot 3169 \cdot 60757 \cdot 579779 \cdot 4362289433 \cdot 79501124416220680469 \cdot p_{26}$
41	$3 \cdot 487 \cdot 493127 \cdot 32002651 \cdot p_{56}$
42	$2 \cdot 3 \cdot 127 \cdot 421 \cdot 22555732187 \cdot 45623714922273271251101 77 \cdot p_{34}$
43	$7 \cdot 17 \cdot 449 \cdot p_{72}$
44	$2^3 \cdot 3^2 \cdot 12797571009458074720816277 \cdot p_{52}$

continued...

n	known factors of $\mathrm{Sm}(n)$
45	$3^2 \cdot 5 \cdot 7 \cdot 41 \cdot 727 \cdot 1291 \cdot 2634831682519 \cdot 379655178169650473 \cdot p_{41}$
46	$2 \cdot 31 \cdot 103 \cdot 270408101 \cdot 374332796208406291 \cdot 3890951821355123413169209 \cdot p_{28}$
47	$3 \cdot 4813 \cdot 679751 \cdot 4626659581180187993501 \cdot p_{53}$
48	$2^2 \cdot 3 \cdot 179 \cdot 1493 \cdot 1894439 \cdot 157719406241884267103235886 57 \cdot p_{46}$
49	$23 \cdot 109 \cdot 3251653 \cdot 2191196713 \cdot 53481597817014258108937 \cdot p_{47}$
50	$2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 211 \cdot 20479 \cdot 160189818494829241 \cdot 46218039785302111919 \cdot p_{44}$
51	$3 \cdot 17708093685609923339 \cdot p_{73}$
52	$2^7 \cdot 43090793230759613 \cdot p_{76}$
53	$3^3 \cdot 7^3 \cdot 127534541853151177 \cdot p_{76}$
54	$2 \cdot 3^6 \cdot 79 \cdot 389 \cdot 3167 \cdot 13309 \cdot 69526661707 \cdot 8786705495566261913717 \cdot p_{51}$
55	$5 \cdot 768643901 \cdot 641559846437453 \cdot 1187847380143694126117 \cdot p_{55}$
56	$2^2 \cdot 3 \cdot c_{102}$
57	$3 \cdot 17 \cdot 36769067 \cdot 2205251248721 \cdot c_{83}$
58	$2 \cdot 13 \cdot c_{105}$
59	$3 \cdot 340038104073949513 \cdot c_{91}$
60	$2^3 \cdot 5 \cdot 97 \cdot 157 \cdot p_{104}$
61	$10386763 \cdot 35280457769357 \cdot p_{92}$
62	$2 \cdot 3^2 \cdot 1709 \cdot 329167 \cdot 1830733 \cdot c_{98}$
63	$3^2 \cdot 17028095263 \cdot c_{105}$
64	$2^2 \cdot 7 \cdot 17 \cdot 19 \cdot 197 \cdot 522673 \cdot 1072389445090071307 \cdot c_{89}$
65	$3 \cdot 5 \cdot 31 \cdot 83719 \cdot c_{113}$
66	$2 \cdot 3 \cdot 7 \cdot 20143 \cdot 971077 \cdot c_{111}$
67	$397 \cdot 183783139772372071 \cdot p_{105}$
68	$2^4 \cdot 3 \cdot 23 \cdot 764558869 \cdot 1811890921 \cdot c_{105}$
69	$3 \cdot 13 \cdot 23 \cdot 8684576204660284317187 \cdot 2812596085975357491750 83 \cdot c_{80}$
70	$2 \cdot 5 \cdot 2411111 \cdot 109315518091391293936799 \cdot c_{100}$
71	$3^2 \cdot 83 \cdot 2281 \cdot c_{126}$
72	$2^2 \cdot 3^2 \cdot 5119 \cdot c_{129}$
73	$37907 \cdot c_{132}$
74	$2 \cdot 3 \cdot 7 \cdot 1788313 \cdot 21565573 \cdot 99014155049267797799 \cdot c_{103}$
75	$3 \cdot 5^2 \cdot 193283 \cdot c_{133}$
76	$2^3 \cdot 828699354354766183 \cdot 213643895352490047310058981 \cdot p_{97}$
77	$3 \cdot 383481022289718079599637 \cdot 8749118329379889989350 21 \cdot c_{97}$
78	$2 \cdot 3 \cdot 31 \cdot 185897 \cdot c_{139}$
79	$73 \cdot 137 \cdot 22683534613064519783 \cdot 1323163358338897421917 73 \cdot c_{102}$
80	$2^2 \cdot 3^3 \cdot 5 \cdot 101 \cdot 10263751 \cdot 12953313401954533664084 89 \cdot p_{115}$

TABLE 2. Factorizations of $\mathrm{Sm}(n)$, $1 < n \leq 80$

n	known factors of RSm(n)
2	$3 \cdot 7$
3	$3 \cdot 107$
4	$29 \cdot 149$
5	$3 \cdot 19 \cdot 953$
6	$3 \cdot 218107$
7	$19 \cdot 402859$
8	$3^2 \cdot 1997 \cdot 4877$
9	$3^2 \cdot 17^2 \cdot 379721$
10	$7 \cdot 28843 \cdot 54421$
11	$3 \cdot p_{12}$
12	$3 \cdot 7 \cdot p_{13}$
13	$17 \cdot 3243967 \cdot 237927839$
14	$3 \cdot 11 \cdot 24769177 \cdot p_{10}$
15	$3 \cdot 13 \cdot 19^2 \cdot 79 \cdot p_{15}$
16	$23 \cdot 233 \cdot 2531 \cdot p_{16}$
17	$3^2 \cdot 13 \cdot 17929 \cdot 25411 \cdot 47543 \cdot 677181889$
18	$3^2 \cdot 11^2 \cdot 19 \cdot 23 \cdot 281 \cdot 397 \cdot 8577529 \cdot 399048049$
19	$17 \cdot 19 \cdot 1462095938449 \cdot p_{14}$
20	$3 \cdot 89 \cdot 317 \cdot 37889 \cdot p_{21}$
21	$3 \cdot 37 \cdot 732962679433 \cdot p_{19}$
22	$13 \cdot 137 \cdot 178489 \cdot 1068857874509 \cdot p_{14}$
23	$3 \cdot 7 \cdot 191 \cdot p_{33}$
24	$3 \cdot 107 \cdot 457 \cdot 57527 \cdot p_{29}$
25	$11 \cdot 31 \cdot 59 \cdot 158820811 \cdot 410201377 \cdot p_{20}$
26	$3^3 \cdot 929 \cdot 1753 \cdot 2503 \cdot 4049 \cdot 11171 \cdot p_{24}$
27	$3^5 \cdot 83 \cdot 3216341629 \cdot 7350476679347 \cdot p_{18}$
28	$23 \cdot 193 \cdot 3061 \cdot 2150553615963932561 \cdot p_{21}$
29	$3 \cdot 11 \cdot 709 \cdot 105971 \cdot 2901761 \cdot 1004030749 \cdot p_{24}$
30	$3 \cdot 73 \cdot 79 \cdot 18041 \cdot 24019 \cdot 32749 \cdot 5882899163 \cdot p_{24}$
31	$7 \cdot 30331061 \cdot p_{45}$
32	$3 \cdot 17 \cdot 1231 \cdot 28409 \cdot 103168496413 \cdot p_{35}$
33	$3 \cdot 7 \cdot 7349 \cdot 9087576403 \cdot p_{42}$
34	$89 \cdot 488401 \cdot 2480227 \cdot 63292783 \cdot 254189857 \cdot 3397595519 \cdot p_{19}$
35	$3^2 \cdot 881 \cdot 1559 \cdot 755173 \cdot 7558043 \cdot 1341824123 \cdot 4898857788363449 \cdot p_{16}$
36	$3^2 \cdot 11^2 \cdot 971 \cdot 1114060688051 \cdot 1110675649582997517457 \cdot p_{24}$
37	$29 \cdot 2549993 \cdot 39692035358805460481 \cdot p_{38}$
38	$3 \cdot 9833 \cdot p_{63}$
39	$3 \cdot 19 \cdot 73 \cdot 709 \cdot 66877 \cdot p_{58}$
40	$11 \cdot 41 \cdot 199 \cdot 537093776870934671843838337 \cdot p_{39}$
41	$3 \cdot 29 \cdot 41 \cdot 89 \cdot 3506939 \cdot 18697991901857 \cdot 59610008384758528597 \cdot p_{28}$
42	$3 \cdot 13249 \cdot 14159 \cdot 25073 \cdot 6372186599 \cdot p_{52}$
43	$52433 \cdot 73638227044684393717 \cdot p_{53}$
44	$3^2 \cdot 7 \cdot 3067 \cdot 114883 \cdot 245653 \cdot 657119070884376607609 39 \cdot p_{41}$

continued...

n	known factors of RSm(n)
45	$3^2 \cdot 23 \cdot 167 \cdot 15859 \cdot 25578743 \cdot p_{65}$
46	$23 \cdot 35801 \cdot 543124946137 \cdot 45223810713458070167393 \cdot p_{43}$
47	$3 \cdot 11 \cdot 31 \cdot 59 \cdot 1102254985918193 \cdot 480842121756396198701982040 1 \cdot p_{38}$
48	$3 \cdot 151 \cdot 457 \cdot 990013 \cdot 246201595862687 \cdot 636339569791857481119613 \cdot p_{39}$
49	$71 \cdot 9777943361 \cdot p_{77}$
50	$3 \cdot 157 \cdot 3307 \cdot 3267926640703 \cdot 771765128032466758284258631297 \cdot p_{43}$
51	$3 \cdot 11 \cdot p_{92}$
52	$7 \cdot 29 \cdot 670001 \cdot 403520574901 \cdot 70216544961751 \cdot 1033003489172581 \cdot p_{47}$
53	$3^4 \cdot 499 \cdot 673 \cdot 6287 \cdot 57653 \cdot 199236731 \cdot 1200017544380023$ $\cdot 1101541941540576883505692003 \cdot p_{31}$
54	$3^3 \cdot 7^4 \cdot 13 \cdot 1427 \cdot 632778317 \cdot 57307460723 \cdot 7103977527461 \cdot 617151073326209 \cdot p_{43}$
55	$357274517 \cdot 460033621 \cdot p_{84}$
56	$3 \cdot 13^2 \cdot 85221254605693 \cdot p_{87}$
57	$3 \cdot 41 \cdot 25251380689 \cdot p_{93}$
58	$11 \cdot 2425477 \cdot 178510299010259 \cdot 377938364291219561$ $\cdot 546572896582343748037 1566249 \cdot p_{40}$
59	$3 \cdot c_{109}$
60	$3 \cdot 8522287597 \cdot p_{101}$
61	$13 \cdot 373 \cdot 63990327212461530651 83 \cdot c_{88}$
62	$3^2 \cdot 11 \cdot 487 \cdot 6870011 \cdot 3921939670009 \cdot 11729917979119$ $\cdot 938364538509696981249417 1823 \cdot p_{50}$
63	$3^2 \cdot 97 \cdot 26347 \cdot 338856918508353449187667 \cdot p_{86}$
64	$397 \cdot 653 \cdot 459162927787 \cdot 27937903937681 \cdot 386877715040952336040363 \cdot p_{65}$
65	$3 \cdot 7 \cdot 23 \cdot 13219 \cdot 24371 \cdot c_{110}$
66	$3 \cdot 53 \cdot 83 \cdot 2857 \cdot 1154129 \cdot 9123787 \cdot p_{103}$
67	$43 \cdot 38505359279 \cdot c_{113}$
68	$3 \cdot 29 \cdot 277213 \cdot 68019179 \cdot 152806439 \cdot 295650514394629363$ $\cdot 14246700953701310411 \cdot p_{67}$
69	$3 \cdot 11 \cdot 71 \cdot 167 \cdot 1481 \cdot 2326583863 \cdot 19962002424322006111361 \cdot p_{89}$
70	$1157237 \cdot 41847137 \cdot 8904924382857569546497 \cdot p_{96}$
71	$3^2 \cdot 17 \cdot 131 \cdot 16871 \cdot 1504047269 \cdot 82122861127 \cdot 1187275015543580261 \cdot p_{87}$
72	$3^2 \cdot 449 \cdot 1279 \cdot p_{129}$
73	$7 \cdot 11 \cdot 21352291 \cdot 1051174717 \cdot 92584510595404843 \cdot 33601392386546341921 \cdot p_{83}$
74	$3 \cdot 177337 \cdot 6647068667 \cdot 31386093419 \cdot 669035576309897 \cdot 4313244765554839 \cdot c_{83}$
75	$3 \cdot 7 \cdot 230849 \cdot 7341571 \cdot 24260351 \cdot 1618133873 \cdot 19753258488427$ $\cdot 46752975870227777 \cdot c_{81}$
76	$53 \cdot c_{142}$
77	$3 \cdot 919 \cdot 571664356244249 \cdot c_{127}$
78	$3 \cdot 17 \cdot 47 \cdot 17795025122047 \cdot c_{131}$
79	$160591 \cdot 274591434968167 \cdot 1050894390053076193 \cdot p_{112}$
80	$3^3 \cdot 11 \cdot 443291 \cdot 1575307 \cdot 19851071220406859 \cdot c_{121}$

TABLE 3. Factorizations of RSm(n), $1 < n \leq 80$

4. SEARCHING FOR PRIMES IN SM AND RSM

Using the GMP library, a fast C program was written to search for primes in Sm(n) and RSm(n). We used the Miller-Rabin [K81] test to check for compositeness.

No primes were found in Sm(n), $1 < n < 840$, and only one probable prime in RSm(n), $1 < n < 750$, namely RSm(82)= 82818079...1110987654321. This number proved prime with ECPP.

5. Acknowledgements and contact information

Thanks go to Paul Zimmermann for discussion and review of the paper. He also contributed one factor to the data.

This work wouldn't have been possible without the open-source software provided by the respective authors: Richard Crandall (`sciences0`), Torbjorn Granlund (`GMP`), Paul Zimmermann (`GMP-ECM`), and François Morain (`ECPP`).

The author can be reached at the E-mail address `stephan@tmt.de` and his homepage is at the URL `http://rws.home.pages.de`.

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Editor's Note:

Professor Eric W. Weisstein, University of Virginia, has extended Staphan's result and proved, with a supercomputer, that

RSm(82) = 8281807978...12111098765432l

is the only Reverse Smarandache Prime in the first 2,739 terms of this sequence;
[can we conjecture this is the only one in the entire sequence?]

and there is no Consecutive Smarandache Prime in the first 3,072 terms of the respective sequence;
[can we conjecture there is no one in the entire sequence?].

He also showed that:
- the Smarandache Concatenation Prime Sequence: 2, 23, 235, 2357, 235711, 23571113, ..., which is prime for terms 1, 2, 4, 128, 174, 342, 435, 1429,... (N.J.A.Sloane, "Encyclopedia of Integers", online: http://www.research.att.com/~njas/sequences), with no others less than 1,634;
[can we conjecture there is a finite number of primes in the entire sequence?].
URL side:
http://www.astro.virginia.edu/~eww6n/math/ConsecutiveNumberSequences.html
- the Smarandache Concatenated Odd Sequence: 1, 13, 135, 1357, 13579, 1357911, 135791113, ..., is prime for terms 2, 10, 16, 34, 49, ..., with no others less than 1,000;
[can we conjecture there is a finite number of primes in the entire sequence?].
- the Smarandache Concatenated Square Sequence: 1, 14, 149, 14916, 1491625, ..., is prime for term 3 only in the first 1,000;
[can we conjecture this is the only one in the entire sequence?]
- the Smarandache Concatenated Cubic Sequence: 1, 18, 1827, 192764, ..., has no prime in the first 1,000 terms;
[can we conjecture there is no one in the entire sequence?].

(Jose Castillo)

To Enjoi is a Permanent Component of Mathematics

†by
C. Dumitrescu and R. Müller

1. The Theorem of Platon

Studying the properties of the proportions the peoples of the antiquity could build using the ruler and the compasses. For example if instead of a square of side a it was required the construction of another square, of side x determined by the condition that the new square has a double area, so

$$x^2 = 2a^2 \tag{1}$$

Pithagora's descendents used to write this relation as

$$\frac{a}{x} = \frac{x}{2a} \tag{2}$$

and used to build an isosceles rectangular triangle having its hypotenuse $2a$.

The celebrated philosopher of the antiquity *Platon* (427 - 347 B.C.) was greatly interested in Mathematics, especialy in connections with the so called "*solid numbers*", that is numbers of the form

$$a \cdot b \cdot c$$

representing a volume.

This sympathy is also due to a famous event even today.

In the Greek city Athens there was an epidemic diseare that killed many peoples. The inhabitants anced the oracle of Delphi (a town in Delos, the smallest of the Ciclade isles) what to do in order to save themselves.

† (1949-1997)

The gods asked the prierts of the temple to replace their cubic altar with a new one having a double volume.

The prierts appealed to the greatest mathematicians of the time to get the solution.

The problem is to calculate the lenght x of the side of a cube such that

$$x^3 = 2a^3$$

That is

$$x = 2^{\frac{1}{3}} a \tag{3}$$

But the peoples of those times didn't know any method to calculate, not even approximatively, the radicals over to two. Only in the fifth century A.D. the Indians used the approximation in order to extract the cubic root:

$$(a^3 + b)^{\frac{1}{3}} \simeq a + \frac{b}{3a^2}$$

where a^3 is the greatest perfect cube not exceding the number $a^3 + b$.

The problem (3) can't be solved using only the rule and the compasses.

Let us observe that this problem is a particular problem on *solid numbers*, and of course it is unsolvable by of *only one* proportion of kind (2).

However Platon observed that this problem could be solved using *two proportions*. Namely, he affirmed that:

Theorem of Platon. *While one simple proportion is enought to connect two plane numbers (numbers of the form $a \cdot b$), three proportions are necessary to connect twoo solide numbers.*

The solution of the problem of Delos is then obtained by Platon approximatively writing

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a} \tag{4}$$

Indeed, from (4) we obtain

$$x^2 ay \quad \text{and} \quad y^2 = 2ax \tag{5}$$

so $x^3 = 2a^3$.

Platon and others [Archytas of Tarent ($\sim$ 380 B.C.), Eudoxus (408 - 355 B.C.), Appollonios of Perga (260 - 170 B.C.)] imaginated approximate solutions of the

equation (4), rather difficult, which, of course, could be simplified in the course of time.

Today, we can easily find an approximate solution to the system (5) through drawning the two parabolas or intersecting one of these parabolas with the circle

$$x^2 + y^2 - 2ax - ay = 0$$

obtained through adding the equations of the two parabolas.

2. A method to construct convergent sequences

The name of *Leonard Euler* (1707 - 1783) is known amoung the young people loving mathematics, especially because of the sequence given by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n \tag{6}$$

It is said that this sequence is monotonous and bounded, converging to a constant $\gamma \in (0,1)$, known as *Euler's constant.*

This constant appears in many occasions in mathematics. For instance if $d(n)$ is the number of (positive) divisors of the positive integer n, then it is proved that

$$\frac{1}{n}\sum_{i=1}^{n} d(i) \simeq \ln n + 2\gamma - 1$$

Considering the sequence (6) and proving his convergence Euler has etablished a connection between the following two sequences

$$b_n = 1 + \frac{1}{2} + ... + \frac{1}{n} \quad \text{and} \quad c_n = \ln n$$

both converging to infinity.

To prove the monotonicity and boundness of the sequence $(a_n)_{n\in\mathbb{N}^*}$ it is used a well known theorem, does to the *count Luis de Lagrange* (1736 - 1813). This method may be generalised in the following way:

Proposition. *Let $f:[1,\infty)\longrightarrow R$ a derivable function with the property that f and f' are monotonous, but of different monotonicity (that is either f increase and f' decrease or f decrease and f' increase).*

Then the sequence

$$x_n = f'(1)+f'(2)+...+f'(n)-f(n) \tag{7}$$

is convergent.

Proof. The proof is analogous with that of Euler's sequence (6).

Indeed, let us suppose that f is increasing and f' is decreasing. For the monotoncity of the sequence $(x_n)_{n\in N^s}$ we obtains:

$$x_{n+1}-x_n = f'(n+1)-(f(n+1)-f(n))$$

and applying the theorem of Lagrange to the function f on the interval $[k,k+1]$ it results:

$$(\exists)\ c_k\in(k,k+1)\quad \text{such that}\quad f(k+1)-f(k)=f'(c_k) \tag{8}$$

and

$$k<c_k<k+1\Longrightarrow f'(k)>f'(c_k)>f'(k+1) \tag{9}$$

so

$$x_{n+1}-x_n = f'(n+1)-f'(c_n)<0$$

because f' is decreasing.

We have now to find a lower bound of the sequence (7). For this we write the implication (9) for every $k=1,2,....$and we get:

$$\begin{array}{l} 1<c_1<2\Longrightarrow f'(1>f'(c_1)>f'(2)\\ 2<c_2<3\Longrightarrow f'(2)>f'(c_2)>f'(3)\\ .....\\ n<c_n<n+1\Longrightarrow f'(n)>f'(c_n)>f'(n+1)\end{array} \tag{10}$$

So,

$$\begin{array}{l} x_n = f'(1)+f'(2)+...+f'(n)-f(n)>\\ >f'(c_1)+f'(c_2)+...+f'(c_n)-f(n)\end{array}$$

Writing now the equalities (8) for $k=1,2,...n$ and adding, it results:

$$f'(c_1) + f'(c_2) + ... + f'(c_n) = f(n+1) - f(1)$$

so $x_n \geq f(n+1) - f(1) - f(n) > -f(1)$ because f is increasing.

Of course, the limit point of this sequence is between $-f(1)$ and $x_1 = f'(1) - f(1)$.

This proposition permet to construct many convergent sequences of the form (7)

Indeed,

1) considering the increasing function $f(x) = 2\sqrt{x}$, whose derivative $f'(x) = 1/\sqrt{x}$ is decreasing, it results that the sequence

$$x_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} - 2\sqrt{n}$$

has a limit point $l \in [-2, -1]$.

2) considering the function $f(x) = \ln(\ln x)$ it results that the sequence

$$x_n = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + ... + \frac{1}{n\ln n} - \ln(\ln n)$$

is convergent to a point $l \in [-\ln(\ln 2), \frac{1}{2\ln 2} - \ln(\ln 2)]$.

3) the sequence

$$x_n = 2(\frac{\ln 2}{2} + \frac{\ln 3}{3} + ... + \frac{\ln n}{n}) - \ln^2 n$$

as well as

$$x_n = \frac{\ln^k 2}{2} + \frac{\ln^k 3}{3} + ... + \frac{\ln^k n}{n} - \frac{\ln^{k+1} n}{k+1}$$

are convergent sequences, and, of course, the reader may construct himself many other convergent sequences, using the same method.

It is interesting to mentione that by means of the same way as in the proof of the above proposition it may be proved the following curious inequalities:

$$1998 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{10^6}} < 1999$$

and, more general,

$$2\cdot 10^k-2<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{10^{2k}}}<2\cdot 10^k-1$$

or these

$$\frac{p}{p-1}(a^{k(p-1)}-1)<1+\frac{1}{2^{1/p}}+\frac{1}{3^{1/p}}+...+\frac{1}{(a^{pk})^{1/p}}<\frac{p}{p-1}(a^{k(p-1)}-\frac{1}{p})$$

3. The Problem of Titeica

The Romanian mathematician *Gh. Titeica* (1873 - 1939) while in a waiting room and because time hardly passed, started drawing circles on a newspaper margin, using a coin.

In this playing with it, he begun to move the coin so that it have a fixed point on the circumference of a cercle. Because he had to wait for a long while, he had the time to find out that drawing three circles in which the coin had a fixed point on the circumference, the circles intersected two by two in three points (called A, B, and C) over which the coin was exactly superposed.

Of course, the three points A, B, and C make a circle. The novelty was that this circle seemed to have the same radius as the circles drawn with the coin.

When he reached home, Titeica proved that indeed:

The Problem of Titeica. *If three circles of the same radius r have a common fixed point M, they still intersect two by two in the points A, B, C which make another circle with the same radius r.*

Proof. Because we have $MC_1 = MC_2 = MC_3$ (see figure below) it results that M is the centre of the circumscribed circle of the triangle determined by the points C_1, C_2, C_3.

Now, it is sufficient to prove the equality (congruence) between this triangle and the triangle determined by the points A, B, C.

We have:

$$AB \equiv C_2C_3 \ (\text{ because } \ \Delta AC_1B \equiv \Delta C_2CC_3)$$
$$AC \equiv C_1C_3 \ (\text{because } \ \Delta AC_2C \equiv \Delta C_1BC_3)$$
$$BC \equiv C_1C_2 \ (\text{because } \ \Delta BC_3C \equiv \Delta C_1AC_2)$$

and the theorem is proved.

4. Hexagons in Pascal's Triangle

The hexagon $AC_2CC_3BC_1$ used in the proof of the problem of *Titeica* is in connection with some cercles. Now we shall make in evidence other hexagons. this time lied with a triangle, the celebrate triangle of *Pascal.*

In 1654 *Blaise Pascal* (1623 - 1662) published the paper *"On an Arithmetical Triangle"* in which studied the properties of the numbers in the triangle

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & & & \\
 & & & & & & & 1 & & 2 & & 1 & & & & & & & \\
 & & & & & & 1 & & 3 & & & & 3 & & 1 & & & & \\
 & & & & & 1 & & 4 & & & & 6 & & & & 4 & & & & 1 & \\
 & & & & 1 & & 5 & & & & 10 & & & & 10 & & & & 5 & & & & 1 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}
$$

constructed such that the $n-row$ contains the elements

$$\left(\begin{array}{c} n \\ 0 \end{array}\right), \left(\begin{array}{c} n \\ 1 \end{array}\right), \ldots, \left(\begin{array}{c} n \\ k-1 \end{array}\right), \left(\begin{array}{c} n \\ k \end{array}\right), \left(\begin{array}{c} n \\ k+1 \end{array}\right), \ldots, \left(\begin{array}{c} n \\ n \end{array}\right)$$

where

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}$$

In the sequel we shall focus the attention on the following elements in this triangle:

$$
\begin{array}{ccccc}
 & \left(\begin{array}{c} n-1 \\ k-1 \end{array}\right) & & \left(\begin{array}{c} n-1 \\ k \end{array}\right) & \\
\left(\begin{array}{c} n \\ k-1 \end{array}\right) & & \left(\begin{array}{c} n \\ k \end{array}\right) & & \left(\begin{array}{c} n \\ k+1 \end{array}\right) \\
 & \left(\begin{array}{c} n+1 \\ k \end{array}\right) & & \left(\begin{array}{c} n+1 \\ k+1 \end{array}\right) &
\end{array}
$$

For simplicity we note

$$A = \left(\begin{array}{c} n-1 \\ k-1 \end{array}\right), B = \left(\begin{array}{c} n \\ k-1 \end{array}\right), C = \left(\begin{array}{c} n+1 \\ k \end{array}\right), D = \left(\begin{array}{c} n+1 \\ k+1 \end{array}\right)$$
$$E = \left(\begin{array}{c} n \\ k+1 \end{array}\right), F = \left(\begin{array}{c} n-1 \\ k \end{array}\right), \quad \text{and} \quad X = \left(\begin{array}{c} n \\ k \end{array}\right)$$

so it results the configuration

$$
\begin{array}{ccc}
 & A \quad F & \\
B & X & E \\
 & C \quad D &
\end{array}
$$

The multiplicative equality

$$A \cdot C \cdot E = B \cdot D \cdot F \tag{11}$$

was found by *V. E. Hoggatt Jr.* and *W. Hansell* [5]. Therefore this configuration is called *"Hoggat-Hansell perfect square hexagon"*.

This hexagon has also the following interesting property, found in [2]:

$$g.c.d.(A, C, E) = g.c.d.(B, D, F) \tag{12}$$

where *g.c.d.* is the abreviation for the greatest common divisor.

The identities (11) and (12) are the first two non-trivial examples of translatable identities of binomial coefficients and are called *"the Star of David theorem"*.

The lower common multiple (*l.c.m.*) counterpart of the identity (12), namely

$$l.c.m.(A, C, E) = l.c.m.(B, D, F) \tag{13}$$

does not hold on Pascal's triangle and it has been a long-standing open question wheter there exists any mathematically non-trivial and/or artistically interesting configurations which give a *translatable l.c.m. identity* of type (12).

S. Ando and D. Sato have proved [2] that the answer to this question is *"yes"*. They have proved that:

Theorem. *(Pisa triple equality theorem) There exists a configuration which gives simultaneously equal product, equal g.c.d. and equal l.c.m. properties on binomial, Fibonacci-binomial and their modified coefficients.*

A *Fibonacci-binomial coefficient* (or *Fibonomial-coefficient*) is the number defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_1 \cdot F_2 \ldots F_n}{F_1 \cdot F_2 \ldots F_k \cdot F_1 \cdot F_2 \ldots F_{n-k}}$$

where F_i is the $i-th$ *Fibonacci number*, i.e.

$$F_1 = F_2 = 1, \quad F_{n+2} = F_n + F_{n+1}, \text{ for } n = 1, 2, \ldots$$

All Fibonomial coefficients are positive integers, and the triangular array of these numbers has a structure similar to Pascal's triangle.

A. P. Hilmann and V. E. Hoggatt Jr. investigated the similarities with Pascal's triangle and showed that the original Star of David theorem also holds on this Fibonacci version of the Pascal-like triangle.

The modified binomial coefficient is defined as

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{(n+1)!}{k!(n-k)!} = (n+1) \left(\begin{array}{c} n \\ k \end{array} \right)$$

It is proved that the translatable product and *l.c.m.* equalities, similar to (11) and (13), but not the *g.c.d.* equality (12), hold for the array of modified binomial coefficients.

The two Pascal like number array can be combined further to define the *modified Fibonacci coefficient*, given by:

$$\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle = \frac{F_1 \cdot F_2 \ldots F_{n+1}}{F_1 \cdot F_2 \ldots F_k \cdot F_1 \cdot F_2 \ldots F_{n-k}} = F_{n+1} \left[\begin{array}{c} n \\ k \end{array} \right]$$

S. Ando and D. Sato announced at the *third International Conference on Fibonacci Numbers and their Applications* (held in Pisa, Italy, July 25 -29, 1988) some interesting results concerning *g.c.d.* and *l.c.m.* properties on configurations like these reproduced below. We mention here only the following:

Theorem (Sakasa - Fuji quadruple equality theorem). *The configuration of Fujiyama* (see below) *has equal g.c.d. and equal l.c.m. properties on Fibonacci - Pascal's triangle, while its upside down configuration* (called *SAKASA - FUJI*, in japonese) *has equal g.c.d. and equal l.c.m. properties on modifies Pascal's and modified Fibonacci - Pascal's triangle.*

Theorem (Universal equality theorem). *The Julia's snowflake and its upside down configuration both give translatable simultaneously equal product* (symbolised below by the Greek letter Π), *equal g.c.d. and equal l.c.m. properties on Pascal's triangle, Fibonacci - Pascal's triangle and modified Fibonacci - Pascal's triangle*

We reproduce here, after [2] these configurations.

S. Ando and D. Sato in their paper explained with amability the terminology used for these configurations.

Thus one of the configurations is named in memoriam of Professor *Julia Robinson* for the friendship and support given to the authors during many years of mathematical associations.

Fujiyama is a highly symetric triangular mountain near Tokio, and *Saskatchewan* is a name of a province in western Canada, where the first non - trivial mutualy exclusive equal *g.c.d.* – *l.c.m.* configuration was constructed.

5. The Smarandache Function

This function is originated from the exiled Romanian Professor *Florentin Smarandache* and it is defined as follows:

For any non − null n, $S(n)$ is the smallest integer such that $S(n)!$ is divisible byn.

To calculate the value of $S(n)$, for a given n, we need to use twoo numerical scale, as we shall see in the following.

A strange addition. A (standard) numerical scale is a sequence

$$(h) \ : \ 1, \ a_1, \ a_2, ..., a_i, ... \tag{14}$$

where $a_i = h^i$, for a fixed $h > 1$.

By means of such a sequence every integer $n \in N$ may be writen as

$$n_{(h)} = \varphi_k a_k + \varphi_{k-1} a_{k-1} + ... + \varphi_0$$

and we can use the notation

$$n_{(h)} = \overline{\varphi_k \varphi_{k-1} ... \varphi_0}$$

The integers φ_i are called *"digits"* and verifie the inequalities

$$0 \leq \varphi_i \leq h - 1$$

For the scale given by the sequence (14) it is trouth the recurence relation

$$a_{i+1} = h \cdot a_i \tag{15}$$

which permet numerical calculus, as additions, substractions, etc.

The standard scale (14) was been generalised, considering *an arbitrary increassing sequence:*

$$(g) \ : \ 1, \ b_1, \ b_2, ..., b_i, ...$$

and knowing a corresponding recurence relation.

For instance the Fibonacci sequence:

$$F_1 = 1, \ F_2 = 2, \ \text{ and } F_{i+1} = F_i + F_{i-1}$$

is such a generalised scale, for which the digits are only the integers 0 and 1.

Another generalised numerical scale is the scale defined by the sequence:

$$[p] \ : \ 1, \ b_1, \ b_2, ..., b_i, ... \tag{16}$$

with

$$b_i = \frac{p^i - 1}{p - 1} \tag{17}$$

and p a prime number.

This scale verifies the recurence

$$b_{i+1} = p \cdot b_i + 1 \tag{18}$$

and is used in the calculus of Smarandache function.

Let us observe that because of the diference between the recurences (15) and (18) we have essentially different ruls for the calculus in the scale $[p]$. To ilustrate these differences let we consider the generalised scale $[5]$:

$$[5] \ : \ 1, \ 6, \ 31, \ 156, \ ...$$

and the integer $m = 150_{(10)}$, which becomes $m_{[5]} = 442$, in the scale $[5]$. Indeed, because

$$a_i(5) \leq 150 \Longleftrightarrow (p^i - 1)/(p - 1) \leq 150 \Longleftrightarrow p^i \leq 150(p - 1) + 1 \Longleftrightarrow$$
$$\Longleftrightarrow i \leq \log_5(150(p - 1) + 1)$$

it results that the greatest $a_i(5)$ for which $a_i(5) \leq 150$ is $a_3(5) = 31$. Then the first digit of the number $m_{[5]}$ is

$$k_3 = [\frac{150}{a_3(5)}] = 4$$

so, $150 = 4a_3(5) + 26$.

For $m_1 = 26$ it results that the greatest $a_i(5)$ for which $a_i(5) \leq 26$ is $a_2(5) = 6$ and the corresponding digit is:

$$k_2 = [\frac{26}{6}] = 4$$

so, $150 = 4a_3(5) + 4a_2(5) + 2 = 442_{[5]}$.

If we consider in addition the numbers:

$$n_{[5]} = 412, \quad r_{[5]} = 44$$

then

$$\begin{aligned} m + n + r = & \ 442 + \\ & 412 \\ & 44 \\ & dcba \end{aligned}$$

From the recurence (18) it results that we need to start the addition from the column corresponding to $a_2(5)$:

$$4a_2(5) + a_2(5) + 4a_2(5) = 5a_2(5) + 4a_2(5)$$

Now, using an unit from the first collumn it results:

$$5a_2(5) + 4a_2(5) = a_3(5) + 4a_2(5), \quad \text{so } b = 4$$

Continuing, $4a_3(5) + 4a_3(5) + a_3(5) = 5a_3(5) + 4a_3(5)$ and using a new unit from the first collumn it results

$$4a_3(5) + 4a_3(5) + a_3(5) = a_4(5) + 4a_3(5), \quad \text{so } c = 4 \text{ and } d = 1$$

Finally, adding the remainder units, $4a_1(5)+2a_1(5) = 5a_1(5)+a_1(5) = 5a_1(5)+1 = a_2(5)$, it results that b must be modified and $a = 0$. So, $m + n + r = 1450_{[5]}$.

An other particularity for the calculus in the scale $[p]$ results from the fact that *in this scale the last non-zero digit may be even p.* This particularity is a consequence of the recurence relation (18).

Which are the numbers with the factorial ending in 1,000 zeros? The answer to this question is in a strong connection with the Smarandache function.

For this reason let us observe first that if

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots . p_t^{\alpha_t} \tag{19}$$

is the decomposition of a given positive integer n into primes, then as an imediate consequence of the definition of S it results

$$S(n) = \max_{i=1,t} (S(p_i^{\alpha_i})) \tag{20}$$

Now, for $n = 10^{1,000}$ it results that $S(n)!$ is a multiple of $10^{1,000}$ and it is the smallest positive integer with this property.

We have

$$S(10^{1,000}) = S(2^{1,000} \cdot 5^{1,000}) = \max(S(2^{1,000}), S(5^{1,000})) = S(5^{1,000})$$

Indeed, for the calculus of $S(p^{\alpha})$ we can use the formula:

$$S(p^{\alpha}) = p(\alpha_{[p]})_{(p)}$$

which signify that the value of the function S for p^{α} is obtained multiplying by p the number obtained *writting* the exponent α in the generalised scale $[p]$ and *reading* it in the scale (p).

So, we have:

$$S(2^{1,000}) = 2((1,000)_{[2]})_{(2)} = 2((111111100)_{[2]})_{(2)} = 508$$
$$S(5^{1,000}) = 5(11201_{[5]})_{(5)} = 4005$$

and it results that $n = 4005$ is the smallest positive integer who's factorial end in $1,000$ zeros.

The next integers with this property are 4006, 4007, 4008, and 4009, because the factorial of 4010 has $1,001$ zeros.

Smarandache magic square. For $n \geq 2$ let A be a set of n^2 elements and l a $n-$arry law defined on A. The *Smarandache magic square of order n* is a 2 square array of rows of elements of A arranged so that the law l applied to each horizontal and vertical row and diagonal give the same result.

Mike R. Mudge, considering such squares, poses the following questions (see Smarandache Function Journal, Vol. 7, No. 1, 1996):

1) Can you find such magic square of order at least 3 or 4, when A is a set of prime numbers and l the addition?

2) Same question when A is a set of square numbers, or cube numbers, or special numbers. For example Fibonacci or Lucas numbers, triangular numbers, Smarandache quotients (i.e. $q(m)$ is the smallest k such that mk is a factorial).

An interesting law may be

$$l(a_1, a_2, ..., a_n) = a_1 + a_2 + -a_3 + a_4 - a_5 + ...$$

Now some examples of *Smarandache Magic Square:*

If A is a set of *prime numbers* and l is the operation of addition such magic squares, with the constant in brackets, are:

$$\left|\begin{array}{ccc} 83 & 89 & 41 \\ 29 & 71 & 113 \\ 101 & 53 & 59 \\ & (213) & \end{array}\right|, \quad \left|\begin{array}{ccc} 101 & 491 & 251 \\ 431 & 281 & 131 \\ 311 & 71 & 461 \\ & (843) & \end{array}\right|, \quad \left|\begin{array}{ccc} 71 & 461 & 311 \\ 521 & 281 & 41 \\ 251 & 101 & 491 \\ & (843) & \end{array}\right|$$

$$\left|\begin{array}{ccc} 113 & 149 & 257 \\ 317 & 173 & 29 \\ 89 & 197 & 233 \\ & (519) & \end{array}\right|, \quad \left|\begin{array}{ccccc} 97 & 907 & 557 & 397 & 197 \\ 367 & 167 & 67 & 877 & 677 \\ 997 & 647 & 337 & 137 & 37 \\ 107 & 157 & 967 & 617 & 307 \\ 587 & 277 & 227 & 127 & 937 \\ & & (2155) & & \end{array}\right|$$

The *multiplication* magic square

$$\left|\begin{array}{ccc} 18 & 1 & 12 \\ 4 & 6 & 9 \\ 3 & 36 & 2 \\ & (216) & \end{array}\right|$$

is such that the constant 216 may be obtained by multiplication of the elements in any *row/column/principal diagonal.*

A geometric magic square is obtained using elements which are a given base raised to the powers of the corresponding elements of a magic square it is clearly a multiplication magic square.

For instance, considering

$$\left|\begin{array}{ccc} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \\ & (15) & \end{array}\right| \text{ and base 2 it results } \left|\begin{array}{ccc} 256 & 2 & 64 \\ 8 & 32 & 128 \\ 16 & 512 & 4 \\ & (2^{15}) & \end{array}\right|$$

Talisman Magic Squares are a relatively new concept, contain the integers from 1 to n^2 in such a way that the difference between any integer and its neighbours (either *row-, column-*, or *diagonal-wise*) is greather than some given constant:

5	15	9	12
10	1	6	3
13	16	11	14
2	8	4	7

(2)

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The Convergence of Smarandache Harmonic Series

*Sabin Tabirca** *Tatiana Tabirca***

* Buckinghamshire University College, Computer Science Division

** "Transilvania" University of Brasov, Computer Science Department

The aim of this article is to study the series $\sum_{n\geq 2}\frac{1}{S^m(n)}$ called Smarandache harmonic series. The article shows that the series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ is divergent and studies from the numerical point of view the sequence $a_n=\sum_{i=2}^{n}\frac{1}{S^2(i)}-\ln(n)$.

1. Introduction

The studies concerning the series with Smarandache numbers have been done recently and represents an important research direction on Smarandache's notions. The question of convergence or divergence were resolved for several series and the sums of some series were proved to be irrational.

The most important study in this area has been done by Cojocaru [1997]. He proved the following results:

1. If $(x_n)_{n\geq 0}$ is an increasing sequence then the series $\sum_{n\geq 0}\frac{x_{n+1}-x_n}{S(x_n)}$ is divergent.

   As a direct consequence, the following series $\sum_{n\geq 2}\frac{1}{S(n)}$, $\sum_{n\geq 1}\frac{1}{S(2\cdot n+1)}$ and $\sum_{n\geq 1}\frac{1}{S(4\cdot n+1)}$ are divergent.

2. The series $\sum_{n\geq 2}\frac{n^{\alpha}}{S(2)\cdot S(3)\cdot\ldots\cdot S(n)}$ is convergent and the sum $\sum_{n\geq 2}\frac{1}{S(2)\cdot S(3)\cdot\ldots\cdot S(n)}$ is in then interval $\left(\frac{71}{100},\frac{101}{100}\right)$.

3. The series $\sum_{n\geq 0}\frac{1}{S(n)!}$ is converges to a number in the interval $\left(\frac{717}{1000},\frac{1253}{1000}\right)$.

4. The series $\sum_{n\geq 0}\frac{S(n)}{n!}$ converges to an irrational number.

Jozsef [1997] extended Cojocaru's result and proved that the series $\sum_{n\geq 0}(-1)^n\cdot\frac{S(n)}{n!}$ also converges to an irrational number.

2. Divergence of the Series $\sum_{n\geq 2}\frac{1}{S^2(n)}$

In this section, the divergence of the series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ will be proved based on an inequality which we shall establish in Lemma 1.

Lemma 1.

$$\lim_{n\to\infty} n\cdot\left(\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}\right)=\frac{1}{4} \tag{1}$$

Proof

The proof is based on the well-known formula

$$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{(2i+1)^2}=\frac{\pi^2}{8} \tag{2}$$

and on a double inequality for the quantity $n\cdot\left(\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}\right)$.

Let m be a natural number such that $m>n$. We then have

$$\sum_{i=n}^{m}\frac{1}{(2i+1)\cdot(2i-1)}=\frac{1}{2}\cdot\sum_{i=n}^{m}\left(\frac{1}{2i-1}-\frac{1}{2i+1}\right)=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2m+1}\right) \tag{3}$$

$$\sum_{i=n}^{m}\frac{1}{(2i+1)\cdot(2i+3)}=\frac{1}{2}\cdot\sum_{i=n}^{m}\left(\frac{1}{2i+1}-\frac{1}{2i+3}\right)=\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2m+3}\right) \tag{4}$$

The difference $\sum_{i=0}^{m}\frac{1}{(2i+1)^2}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}=\sum_{i=n+1}^{m}\frac{1}{(2i+1)^2}$ is studied using (3-4) to obtain the inequalities (5-6).

$$\sum_{i=n+1}^{m}\frac{1}{(2i+1)^2}<\sum_{i=n+1}^{m}\frac{1}{(2i+1)(2i-1)}=\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2m+1}\right) \tag{5}$$

$$\sum_{i=n+1}^{m}\frac{1}{(2i+1)^2}>\sum_{i=n+1}^{m}\frac{1}{(2i+1)(2i+3)}=\frac{1}{2}\left(\frac{1}{2n+3}-\frac{1}{2m+3}\right) \tag{6}$$

Therefore, the inequality

$$\frac{1}{2}\left(\frac{1}{2n+3}-\frac{1}{2m+3}\right)<\sum_{i=0}^{m}\frac{1}{(2i+1)^2}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}<\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2m+1}\right) \tag{7}$$

holds for all $m>n$. If $m\to\infty$ then the inequality (7) becomes

$$\frac{1}{2(2n+3)}<\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}<\frac{1}{2(2n+1)} \quad \text{and} \tag{8}$$

$$\frac{n}{2(2n+3)}<n\left(\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}\right)<\frac{n}{2(2n+1)}. \tag{9}$$

The inequality (9) gives the limit $\lim_{n\to\infty} n\cdot\left(\frac{\pi^2}{8}-\sum_{i=0}^{n}\frac{1}{(2i+1)^2}\right)=\frac{1}{4}$.

♣

In Lemma 2 we will prove an inequality for Smarandache's function.

Lemma 2.

$$S(2^k\cdot n)\le n\ (\forall n>2k>1). \tag{10}$$

Proof

Because $n>2k$ the product $n!=1\cdot 2\cdot\ldots\cdot n$ contains the factors 2, 4, ..., $2k$. Therefore, the divisibility $n!=1\cdot 2\cdot\ldots\cdot n=2^k\cdot n\cdot m\,\mathfrak{M}\,2^k\cdot n$ holds resulting in the inequality $S(2^k\cdot n)\le n$.

♣

In the following, we analyse the summation $a_n=\sum_{i=1}^{2^n}\frac{1}{S^2(2^n+i)}$, where $n>0$.

Let us define the sets

$$A_n=\left\{i=\overline{1,2^n}\,|\,S(2^n+i)\le\frac{2^n+i}{(2^n,i)}\right\} \text{ and } B_n=\left\{i=\overline{1,2^n}\,|\,S(2^n+i)>\frac{2^n+i}{(2^n,i)}\right\} \tag{11}$$

which is a partition of the set $\left\{i=\overline{1,2^n}\right\}$.

Lemma 3.

If $i=2^k\cdot j$ satisfies the following conditions:

- $k \le n - \log_2(n) - 1$ (12)
- j is a odd number so that $j < 2^{n-k}$ (13)

then $i = 2^k \cdot j \in A_n$.

Proof

If k satisfies $k \le n - \log_2(n) - 1$ then $n - k \ge \log_2(n) + 1$ and the inequality

$$2^{n-k} + 1 \ge 2^{\log_2(n)+1} + 1 = 2n + 1 > 2k \tag{14}$$

holds.

Applying Lemma 2 and (14), the following inequality

$$S(2^n + i) = S(2^k(2^{n-k} + j)) \le 2^{n-k} + j = \frac{2^n + 2^k \cdot j}{(2^n, 2^k \cdot j)}$$

is found to be true. Therefore, the relationship $i = 2^k \cdot j \in A_n$ holds.

♣

Let $C_n = \left\{ 2^k \cdot j = \overline{1,2^n} \middle| k \le n - \log_2(n) - 1,\ j\ odd,\ j < 2^{n-k} \right\}$ be the set of numbers which satisfies the conditions of Lemma 3. Thus, the inclusion $C_n \subseteq A_n$ holds.

Theorem 1 shows an inequality satisfied by the sequence a_n.

Theorem 1.

$$(\forall \varepsilon > 0)(\exists N_\varepsilon > 0)(\forall n > N_\varepsilon) a_n = \sum_{i=1}^{2^n} \frac{1}{S^2(2^n + i)} > (\frac{1}{4} - \varepsilon)\frac{1}{n-1}. \tag{15}$$

Proof

Let $\varepsilon > 0$ be a positive number.

The summation $a_n = \sum_{i=1}^{2^n} \frac{1}{S^2(2^n + i)}$ is split into two parts as follows

$$a_n = \sum_{i=1}^{2^n} \frac{1}{S^2(2^n + i)} = \sum_{i \in A_n} \frac{1}{S^2(2^n + i)} + \sum_{i \in B_n} \frac{1}{S^2(2^n + i)} > \sum_{i \in A_n} \frac{1}{S^2(2^n + i)}. \tag{16}$$

Because $C_n \subseteq A_n$, the inequality $\sum_{i \in A_n} \frac{1}{S^2(2^n + i)} > \sum_{i \in C_n} \frac{1}{S^2(2^n + i)}$ holds.

Consequently ,

$$a_n > \sum_{i \in C_n} \frac{1}{S^2(2^n + i)} \tag{17}$$

is true.

If $i = 2^h \cdot j \in C_n \subseteq A_n$ then $S(2^n + i) \le \frac{2^n + i}{(2^n, i)} = 2^{n-k} + j$ holds. This inequality is applied in (17) resulting in

$$a_n > \sum_{2^k \cdot j \in C_n} \frac{1}{\left(2^{n-k} + j\right)^2} = \sum_{k \le n - \log_2(n) - 1} \sum_{j\, odd, j < 2^{n-k}} \frac{1}{\left(2^{n-k} + j\right)^2} \tag{18}$$

The right side of (18) is equivalent to the following summations

$$k = 0 \Rightarrow \frac{1}{(2^n + 1)^2} + \frac{1}{(2^n + 3)^2} + \ldots + \frac{1}{(2^{n+1} - 1)^2} +$$

$$k = 1 \Rightarrow + \frac{1}{(2^{n-1} + 1)^2} + \frac{1}{(2^{n-1} + 3)^2} + \ldots + \frac{1}{(2^n - 1)^2} +$$

...

$$k = n - (\lceil \log_2(n) \rceil + 1) \Rightarrow \frac{1}{(2^{\lceil \log_2(n) \rceil + 1} + 1)^2} + \frac{1}{(2^{\lceil \log_2(n) \rceil + 1} + 3)^2} + \ldots + \frac{1}{(2^{\lceil \log_2(n) \rceil + 2} - 1)^2} +$$

therefore, the sum is equal to $\sum_{i = 2^{\lceil \log_2(n) \rceil}}^{2^n - 1} \frac{1}{(2i+1)^2}$.

The inequality (13) becomes

$$a_n > \sum_{i = 2^{\lceil \log_2(n) \rceil}}^{2^n - 1} \frac{1}{(2i+1)^2} = \sum_{i=1}^{2^n - 1} \frac{1}{(2i+1)^2} - \sum_{i=1}^{2^{\lceil \log_2(n) \rceil} - 1} \frac{1}{(2i+1)^2}. \tag{19}$$

Based on Lemma1, a natural number N_ε can be found so that the inequalities (20-21) hold simultaneous true for all $n > N_\varepsilon$.

$$\frac{\pi^2}{8} - \left(\frac{1}{4} - \varepsilon\right) \cdot \frac{1}{2^n - 1} > \sum_{i=1}^{2^n - 1} \frac{1}{(2i+1)^2} > \frac{\pi^2}{8} - \left(\frac{1}{4} + \varepsilon\right) \cdot \frac{1}{2^n - 1} \tag{20}$$

$$\frac{\pi^2}{8} - \left(\frac{1}{4} - \varepsilon\right) \cdot \frac{1}{2^{\lceil \log_2(n) \rceil} - 1} > \sum_{i=1}^{2^{\lceil \log_2(n) \rceil} - 1} \frac{1}{(2i+1)^2} > \frac{\pi^2}{8} - \left(\frac{1}{4} + \varepsilon\right) \cdot \frac{1}{2^{\lceil \log_2(n) \rceil} - 1} \tag{21}$$

Using (20-21), the inequality (19) is transformed as follows

$$a_n > \sum_{i=1}^{2^n - 1} \frac{1}{(2i+1)^2} - \sum_{i=1}^{2^{\lceil \log_2(n) \rceil} - 1} \frac{1}{(2i+1)^2} > \frac{\pi^2}{8} - \left(\frac{1}{4} + \varepsilon\right) \frac{1}{2^n - 1} - \frac{\pi^2}{8} + \left(\frac{1}{4} - \varepsilon\right) \frac{1}{2^{\lceil \log_2(n) \rceil} - 1} \Rightarrow$$

$$a_n > \left(\frac{1}{4}-\varepsilon\right)\frac{1}{2^{[\log_2(n)]}-1}-\left(\frac{1}{4}+\varepsilon\right)\frac{1}{2^n-1} > \left(\frac{1}{4}-\varepsilon\right)\frac{1}{2^{[\log_2(n)]}-1} \Rightarrow$$

$$a_n > \left(\frac{1}{4}-\varepsilon\right)\frac{1}{2^{\log_2(n)}-1} = \left(\frac{1}{4}-\varepsilon\right)\frac{1}{n-1} \qquad (22).$$

The inequality (22) is true for all $n > N_\varepsilon$.

♣

The divergence of the series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ is proved based on the inequality (22).

Theorem 2.

The series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ is divergent.

Proof

Theorem 1 is applied starting from the obvious equation $\sum_{n\geq 2}\frac{1}{S^2(n)} = \sum_{n\geq 1} a_n$.

Let $\varepsilon>0$ be a positive number. There exists a number $N_\varepsilon>0$ so that the inequality $a_n > \left(\frac{1}{4}-\varepsilon\right)\frac{1}{n-1}$ holds for all $n > N_\varepsilon$. The divergence of the series is given by $\sum_{n\geq 2}\frac{1}{S^2(n)} = \sum_{n\geq 1} a_n \geq \sum_{n\geq N_\varepsilon} a_n \geq \left(\frac{1}{4}-\varepsilon\right)\cdot\sum_{n\geq N_\varepsilon}\frac{1}{n-1} = \infty$.

♣

Consequence 1.

If $m\leq 2$ then the series $\sum_{n\geq 2}\frac{1}{S^m(n)}$ is divergent.

Proof

The statement follows directly from divergence of the series $\sum_{n\geq 2}\frac{1}{S^2(n)}$ and the inequality $\sum_{n\geq 2}\frac{1}{S^2(n)} \leq \sum_{n\geq 2}\frac{1}{S^m(n)}$.

♣

3. About the Sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$

In this section the sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$ is evaluated and some remarks concerning the sequence values are made[1].

n	a_n	n	a_n	n	a_n
500	-3.14	100000	11.19	500000	31.15
1000	-2.97	200000	17.95	1000000	47.74
1500	-2.75	300000	23.09	1500000	56.80
2000	-2.55	400000	27.38	2000000	66.05
2500	-2.35	500000	31.15	2500000	74.14
3000	-2.14	600000	34.53	3000000	81.45
3500	-1.95	700000	37.63	3500000	88.13
4000	-1.79	800000	40.51	4000000	94.34
4500	-1.60	900000	43.20	4500000	100.15
5000	-1.44	1000000	45.74	5000000	105.63

Table 1. The values for the sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$

Because $\sum_{n\geq 2} \frac{1}{S^2(n)}$ is divergent, it is natural to find the convergence order for the series.

Firstly, we evaluate the sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$ and its values are presented in Table 1. Analysing the results from Table 1, the following remarks are obvious:

1. The sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$ can be considered *pseudo*-monotone.

[1] Numerical results presented in the tables have been calculated by Henry Ibstedt. The algorithm and its implementation will be included in *Computer Analysis of Number Sequences, H.Ibstedt, American Research Press (to appear summer 1998).*

2. The sequence $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)$ satisfies the inequality $a_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n) > 0 \ \forall n: 50000 < n < 5000000$. If the inequality holds for all n>50000 then it is evident that $\sum_{n\geq 2} \frac{1}{S^2(n)}$ diverges.

3. Because (the values of) the sequence a_n is *pseudo*-increasing we conjecture that $\lim_{n\to\infty}\left(\sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n)\right) = \infty$.

Secondly, the sequence $b_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n) - \ln(\ln(n))$ is evaluated in Table 2.

n	b_n	n	b_n	n	b_n
500	-3.14	100000	4.83	500000	1.83
1000	0.17	200000	3.08	1000000	1.26
1500	0.21	300000	2.43	1500000	1.02
2000	0.2	400000	2.07	2000000	0.87
2500	0.21	500000	1.83	2500000	0.77
3000	0.2	600000	1.65	3000000	0.7
3500	0.18	700000	1.52	3500000	0.65
4000	0.17	800000	1.4	4000000	0.61
4500	0.18	900000	1.33	4500000	0.57
5000	0.16	1000000	1.26	5000000	0.53

Table 2. The values for the sequence $b_n = \sum_{i=2}^{n} \frac{1}{S^2(i)} - \ln(n) - \ln(\ln(n))$.

This sequence is more unpredictable than the sequence a_n. The only thing, which can be remarked is the decreasing behaviour. We have not been able to predict if this sequence is convergent yet.

4. Conclusions

A proof more simple than the proof presented in this article can be obtained using a convergence test similar to the *condensation* test [Nicolescu *et.al.* 1974]. According to this test, if $(a_n)_{n>0}$ is a decreasing sequence of positive numbers then the series $\sum_{n>0} a_n$ is convergent if and only if the series $\sum_{n>0} 2^n \cdot a_{2^n}$ is convergent. The sequence $\left(\frac{1}{S^m(n)}\right)_{n>1}$ satisfies that $\sum_{n>0} 2^n \cdot \frac{1}{S^m(2^n)}$ is divergent. In spite of that, we cannot conclude that the series $\sum_{n>1} \frac{1}{S^m(n)}$ is divergent because the sequence $\left(\frac{1}{S^m(n)}\right)_{n>1}$ is not decreasing.

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SMARANDACHE ALGEBRAIC STRUCTURES

by Raul Padilla
Dept. of Physical Sciences
University of Tarapaca
Arica, Chile

A few notions are introduced in algebra in order to better study the congruences. Especially the Smarandache semigroups are very important for the study of congruences.

1) The SMARANDACHE SEMIGROUP is defined to be a semigroup A such that a proper subset of A is a group (with respect with the same induced operation).
By proper subset we understand a set included in A, different from the empty set, from the unit element -- if any, and from A.

For example, if we consider the commutative multiplicative group
SG = {18^2, 18^3, 18^4, 18^5} (mod 60)
we get the table:

x	24 12 36 48
24	36 48 24 12
12	48 24 12 36
36	24 12 36 48
48	12 36 48 24

Unitary element is 36.

Using the Smarandache's algorithm [see 2] we get that
18^2 is congruent to 18^6 (mod 60).

Now we consider the commutative multiplicative semigroup
SS = {18^1, 18^2, 18^3, 18^4, 18^5} (mod 60)
we get the table:

x	18	24 12 36 48
18	24	12 36 48 24

24	12	36 48 24 12
12	36	48 24 12 36
36	48	24 12 36 48
48	24	12 36 48 24

Because SS contains a proper subset SG, which is a group, then SS is a Smarandache Semigroup. This is generated by the element 18. The powers
of 18 form a cyclic sequence: 18, 24,12,36,48, 24,12,36,48,... .

Similarly are defined:

2) The SMARANDACHE MONOID is defined to be a monoid A such that a proper subset of A is a group (with respect with the same induced operation).
By proper subset we understand a set included in A, different from the empty set, from the unit element -- if any, and from A.

3) The SMARANDACHE RING is defined to be a ring A such that a proper subset of A is a field (with respect with the same induced operation).
By proper subset we understand a set included in A, different from the empty set, from the unit element -- if any, and from A.

We consider the commutative additive group M={0,18^2,18^3,18^4,18^5} (mod 60) [using the module 60 residuals of the previous powers of 18],
M={0,12,24,36,48}, unitary additive unit is 0.
(M,+,x) is a field.
While (SR,+,x)={0,6,12,18,24,30,36,42,48,54} (mod 60) is a ring whose proper subset {0,12,24,36,48} (mod 60) is a field.
Therefore (SR,+,x) (mod 60) is a Smarandache Ring.
This feels very nice.

4) The SMARANDACHE SUBRING is defined to be a Smarandache Ring B which is a proper subset of s Smarandache Ring A (with respect with the same induced operation).

5) The SMARANDACHE IDEAL is defined to be an ideal A such that a proper subset of A is a field (with respect with the same induced operation).
By proper subset we understand a set included in A, different from the

empty set, from the unit element -- if any, and from A.

6) The SMARANDACHE SEMILATTICE is defined to be a lattice A such that a proper subset of A is a lattice (with respect with the same induced operation).
By proper subset we understand a set included in A, different from the empty set, from the unit element -- if any, and from A.

7) The SMARANDACHE FIELD is defined to be a field (A,+,x) such that a proper subset of A is a K-algebra (with respect with the same induced operations, and an external operation).
By proper subset we understand a set included in A, different from the empty set, from the unit element -- if any, and from A.

8) The SMARANDACHE R-MODULE is defined to be an R-MODULE (A,+,x) such that a proper subset of A is a S-algebra (with respect with the same induced operations, and another "x" operation internal on A), where R is a commutative unitary Smarandache ring and S its proper subset field.
By proper subset we understand a set included in A, different from the empty set, from the unit element -- if any, and from A.

9) The SMARANDACHE K-VECTORIAL SPACE is defined to be a K-vectorial space (A,+,.) such that a proper subset of A is a K-algebra (with respect with the same induced operations, and another "x" operation internal on A), where K is a commutative field.
By proper subset we understand a set included in A, different from the empty set, from the unit element -- if any, and from A.

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SMARANDACHE CONTINUED FRACTIONS

by Jose Castillo, Navajo Community College,
Tsaile, Arizona, USA

Abstract:
Open problems are studied using Smarandache type sequences in the composition of simple and general continued fractions.

Key Words:
Simple and General Continued Fractions, Smarandache Simple and Continued Fractions

1) A Smarandache Simple Continued Fraction is
a fractionn of the form:

$$a(1) + \cfrac{1}{a(2) + \cfrac{1}{a(3) + \cfrac{1}{a(4) + \cfrac{1}{a(5) + \ldots}}}}$$

where a(n), for n >= 1, is a Smarandache type Sequence or Sub-Sequence.

2) And a Smarandache General Continued Fraction is
a fraction of the form:

$$a(1) + \cfrac{b(1)}{a(2) + \cfrac{b(2)}{a(3) + \cfrac{b(3)}{a(4) + \cfrac{b(4)}{a(5) + \ldots}}}}$$

where a(n) and b(n), for n >= 1, are both Smarandache type Sequences or Sub-Sequences.

(Over 200 such sequences are listed in Sloane's database of Encyclopedia of Integer sequences -- online).

For example:
a) if we consider the smarandache consecutive sequence:
1, 12, 123, 1234, 12345, ..., 123456789101112, ...
we form a smarandache simple continued fraction:

$$1 + \cfrac{1}{12 + \cfrac{1}{123 + \cfrac{1}{1234 + \cfrac{1}{12345 + \ldots}}}}$$

b) if we include the smarandache reverse sequence:
1, 21, 321, 4321, 54321, ..., 121110987654321, ...
to the previous one we get a smarandache general continued fraction:

$$1 + \cfrac{1}{12 + \cfrac{21}{123 + \cfrac{321}{1234 + \cfrac{4321}{12345 + \ldots}}}}$$

With a mathematics software it is possible to calculate such continued fractions to see which ones of them converge, and eventually to make conjectures, or to algebraically prove those converging towards certain constants.

Open Problem: Are the previous two examples of continued fractions convergent?

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SMARANDACHE PARADOXIST GEOMETRY

by

Sandy P. Chimienti
Mathematics and Science Department
University of New Mexico
Gallup, NM 87301, USA

Mihaly Bencze
6, Hatmanului Street
2212 Sacele 3
Jud. Brasov, Romania

Abstract:
This new geometry is important because it generalizes and unites in the same time all together: Euclid, Lobachevsky/Bolyai/Gauss, and Riemann geometries. And separates them as well!
It is based on the first four Euclid's postulates, but the fifth one is replaced so that there exist various straight lines and points exterior to them in such a way that none, one, more, and infinitely many parallels can be drawn through the points in this mixted smarandacheian space.

Key Words: Non-Euclidean Geometry, Euclidean Geometry, Lobacevskyian Geometry, Riemannian Geometry, Smarandache Geometries, Geometrical Model

Introduction:
A new type of geometry has been constructed by F.Smarandache[5] in 1969 simultaneously in a partial euclidean and partial non-euclidean space by a replacement of the Euclid's fifth postulate (axiom of parallels) with the following five-statement proposition:

a) there are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line;
[1 parallel]

b) there are at least a straight line and a point exterior to it in this space for which only a finite number of lines $l_1, ..., l_k$ ($k >= 2$) pass through the point and do not intersect the initial line;
[2 or more (in a finite number) parallels]

c) there are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersects the initial line;
[0 parallels]

d) there are at least a straight line and a point exterior to it in this space for which an infinite number of lines that pass through the point (but not all of them) do not intersect the initial line;
[an infinite number of parallels, but not all lines passing throught]

e) there are at least a straight line and a point exterior to it in this space for which any line that passes through the point does not intersect the initial line;
[an infinite number of parallels, all lines passing throught the point]

I have found a partial geometrical model, different from Popescu's [1], by putting together the Riemann sphere (Ellyptic geometry), tangent to the Beltrami disk (Hyperbolic geometry), which is tangent to a plane (Euclidean geometry). But is it any better one?
(because this doesn't satisfy all the above required axioms).

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SMARANDACHE NON-GEOMETRY

by

Sandy P. Chimienti
Mathematics and Science Department
University of New Mexico
Gallup, NM 87301, USA

Mihaly Bencze
6, Hatmanului Street
2212 Sacele 3
Jud. Brasov, Romania

Abstract:
All Euclid's five postulates are denied in this new geometry.

Key Words: Euclidean Geometry, Non-Euclidean Geometry, Smarandache Geometries, Geometrical Model

Introduction:
We introduce this curious geometry, created in 1969 by F.Smarandache[4], and ask for the readers' feedback in finding a model to satisfy the below "axioms".

1. It is not always possible to draw a line from an arbitrary point to another arbitrary point.

2. It is not always possible to extend by continuity a finite line to an infinite line.

3. It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.

4. Not all the right angles are congruent.

5. If a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angles, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angles.

Conclusion:
We thought at a discontinous space to satisfy the first three axioms, but didn't find yet a corresponding definition for the "right angle".

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SMARANDACHE COUNTER-PROJECTIVE GEOMETRY

by

Sandy P. Chimienti
Mathematics and Science Department
University of New Mexico
Gallup, NM 87301, USA

Mihaly Bencze
6, Hatmanului Street
2212 Sacele 3
Jud. Brasov, Romania

Abstract:
All three axioms of the projective geometry are denied in this new geometry.

Key Words: Projective Geometry, Smarandache Geometries, Geometrical Model

Introduction:
This type of geometry has been constructed by F.Smarandache[4] in 1969.

Let P, L be two sets, and r a relation included in PxL. The elements of P are called points, and those of L lines. When (p, l) belongs to r, we say that the line l contains the point p.
For these, one imposes the following COUNTER-AXIOMS:

(I) There exist: either at least two lines, or no line, that contains two given distinct points.

(II) Let p1, p2, p3 be three non-collinear points, and q1, q2 two distinct points. Suppose that {p1, q1, p3} and {p2, q2, p3} are collinear triples. Then the line containing p1, p2, and the line containing q1, q2 do not intersect.

(III) Every line contains at most two distinct points.

We consider that in a discontinuous space one can construct a model to this geometry.

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SMARANDACHE ANTI-GEOMETRY

by

Sandy P. Chimienti
Mathematics and Science Department
University of New Mexico
Gallup, NM 87301, USA

Mihaly Bencze
6, Hatmanului Street
2212 Sacele 3
Jud. Brasov, Romania

Abstract: This is an experimental geometry. All Hilbert's 20 axioms of the Euclidean GGeometry are denied in this vanguardist geometry of the real chaos! What is even more intriguing? F.Smarandache[5] has even found in 1969 a model of it!

Key Words: Hilbert's Axioms, Euclidean Geometry, Non-Euclidean Geometry, Smarandache Geometries, Geometrical Model

Introduction:

Here it is exposed the Smarandache Anti-Geometry:

It is possible to entirely de-formalize Hilbert's groups of axioms of the Euclidean Geometry, and to construct a model such that none of his fixed axioms holds.

Let's consider the following things:

- a set of <points>: A, B, C, ...
- a set of <lines>: h, k, l, ...
- a set of <planes>: alpha, beta, gamma, ...

and

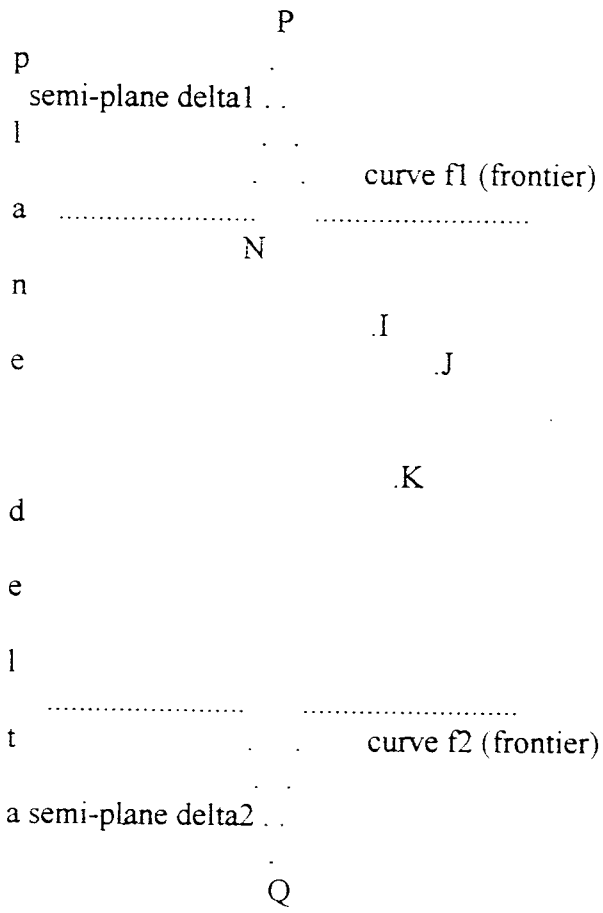
- a set of relationships among these elements: "are situated", "between", "parallel", "congruent", "continuous", etc.

Then, we can deny all Hilbert's twenty axioms [see David Hilbert, "Foundations of Geometry", translated by E. J. Townsend, 1950; and Roberto Bonola, "Non-Euclidean Geometry", 1938].
There exist cases, within a geometric model, when the same axiom is verified by certain points/lines/planes and denied by others.

GROUP I. ANTI-AXIOMS OF CONNECTION:

I.1. Two distinct points A and B do not always completely determine a line.

Let's consider the following model MD:
get an ordinary plane delta, but with an infinite hole in of the following shape:



Plane delta is a reunion of two disjoint planar semi-planes;
f1 lies in MD, but f2 does not;
P, Q are two extreme points on f that belong to MD.

One defines a LINE l as a geodesic curve: if two points A, B that belong to MD lie in l, then the shortest curve lied in MD between A and B lies in l also.
If a line passes two times through the same point, then it is called double point (KNOT).

One defines a PLANE alpha as a surface such that for any two points A, B that lie in alpha and belong to MD there is a geodesic which passes through A, B and lies in alpha also.

Now, let's have two strings of the same length:
one ties P and Q with the first string s1 such that the curve s1 is folded in two or more different planes and s1 is under the plane delta;
next, do the same with string s2, tie Q with P, but over the plane delta and such that s2 has a different form from s1;
and a third string s3, from P to Q, much longer than s1.
s1, s2, s3 belong to MD.

Let I, J, K be three isolated points -- as some islands, i.e. not joined with any other point of MD, exterior to the plane delta.

This model has a measure, because the (pseudo-)line is the shortest way (lenth) to go from a point to another (when possible).

Question 37:

Of course, this model is not perfect, and is far from the best. Readers are asked to improve it, or to make up a new one that is better.

(Let A, B be two distinct points in delta1-f1. P and Q are two points on s1, but they do not completely determine a line, referring to the first axiom of Hilbert, because A-P-s1-Q are different from B-P-s1-Q.)

I.2. There is at least a line l and at least two distinct points A and B of l, such that A and B do not completely determine the line l.

(Line A-P-s1-Q are not completely determined by P and Q in the previous construction, because B-P-s1-Q is another line passing through P and Q too.)

I.3. Three points A, B, C not situated in the same line do not always completely determine a plane alpha

(Let A, B be two distinct points in delta1-f1, such that A, B, P are not co-linear. There are many planes containing these three points: delta1 extended with any surface s containing s1, but not cutting s2 in between P and Q, for example.)

I.4. There is at least a plane, alpha, and at least three points A, B, C in it not lying in the same line, such that A, B, C do not completely determine the plane alpha.

(See the previous example.)

I.5. If two points A, B of a line l lie in a plane alpha, doesn't mean that every point of l lies in alpha.

(Let A be a point in delta1-f1, and B another point on s1 in between P and Q. Let alpha be the following plane: delta1 extended with a surface s containing s1, but not cutting s2 in between P and Q, and tangent to delta2 on a line QC, where C is a point in delta2-f2.
Let D be point in delta2-f2, not lying on the line QC.
Now, A, B, D are lying on the same line A-P-s1-Q-D, A, B are in the plane alpha, but D do not.)

I.6. If two planes alpha, beta have a point A in common, doesn't mean they have at least a second point in common.

(Construct the following plane alpha: a closed surface containing s1 and s2, and intersecting delta1 in one point only, P. Then alpha and delta1 have a single point in common.)

I.7. There exist lines where lies only one point, or planes where lie only two points, or space where lie only three points.

(Hilbert's I.7 axiom may be contradicted if the model has discontinuities.
Let's consider the isolated points area.
The point I may be regarded as a line, because it's not possible to add any new point to I to form a line.
One constructs a surface that intersects the model only in the points I and J.)

GROUP II. ANTI-AXIOMS OF ORDER:

II.1. If A, B, C are points of a line and B lies between A and C, doesn't mean that always B lies also between C and A.

[Let T lie in s1, and V lie in s2, both of them closer to Q, but different from it. Then:
P, T, V are points on the line P-s1-Q-s2-P
(i.e. the closed curve that starts from the point P and lies in s1 and passes through the point Q and lies back to s2 and ends in P),
and T lies between P and V
-- because PT and TV are both geodesics --,
but T doesn't lie between V and P
-- because from V the line goes to P and then to T, therefore P lies between V and T.]

[By definition: a segment AB is a system of points lying upon a line between A and B (the extremes are included).

Warning: AB may be different from BA;
for example:
the segment PQ formed by the system of points starting with P, ending with Q, and lying in s1, is different from the segment QP formed by the system of points starting with Q, ending with P, but belonging to s2.
Worse, AB may be sometimes different from AB;
for example:
the segment PQ formed by the system of points starting with P, ending with Q, and lying in s1, is different from the segment PQ formed by the system of points starting with P, ending with Q,

but belonging to s2.]

II.2. If A and C are two points of a line, then:
there does not always exist a point B lying between A and C,
or there does not always exist a point D such that C lies between A and D.

[For example:
let F be a point on f1, F different from P,
and G a point in delta1, G doesn't belong to f1;
draw the line l which passes through G and F;
then:
there exists a point B lying between G and F
-- because GF is an obvious segment --,
but there is no point D such that F lies between G and D -- because GF is right bounded in F
(GF may not be extended to the other side of F, because otherwise the line will not remain a geodesic anymore).]

II.3. There exist at least three points situated on a line such that:
one point lies between the other two,
and another point lies also between the other two.

[For example:
let R, T be two distinct points, different from P and Q, situated on the line P-s1-Q-s2-P, such that the lenghts PR, RT, TP are all equal;
then:
R lies between P and T,
and T lies between R and P;
also P lies between T and R.]

II.4. Four points A, B, C, D of a line can not always be arranged:
such that B lies between A and C and also between A and D,
and such that C lies between A and D and also between B and D.

[For examples:
- let R, T be two distinct points, different
from P and Q, situated on the line P-s1-Q-s2-P such
that the lenghts PR, RQ, QT, TP are all equal,
therefore R belongs to s1, and T belongs to s2;
then P, R, Q, T are situated on the same line:
such that R lies between P and Q, but not between
P and T
-- because the geodesic PT does not pass through
R --,
and such that Q does not lie between P and T
-- because the geodesic PT does not pass through
Q --,
but lies between R and T;
- let A, B be two points in delta2-f2 such that A, Q, B
are colinear, and C, D two points on s1, s2 respectively,
all of the four points being different from P and Q;
then A, B, C, D are points situated on the same line
A-Q-s1-P-s2-Q-B, which is the same with line
A-Q-s2-P-s1-Q-B, therefore we may have two different
orders of these four points in the same time:
A, C, D, B and A, D, C, B.]

II.5. Let A, B, C be three points not lying in the same
line, and l a line lying in the same plane ABC and
not passing through any of the points A, B, C.
Then, if the line l passes through a point of the
segment AB, it doesn't mean that always the line l
will pass through either a point of the segment BC
or a point of the segment AC.

[For example:
let AB be a segment passing through P in the
semi-plane delta1, and C a point lying in delta1
too on the left side of the line AB;
thus A, B, C do not lie on the same line;
now, consider the line Q-s2-P-s1-Q-D, where D is
a point lying in the semi-plane delta2 not on f2:
therefore this line passes through the point P of
the segment AB, but do not pass through any point
of the segment BC, nor through any point of the
segment AC.]

GROUP III. ANTI-AXIOM OF PARALLELS.

In a plane alpha there can be drawn through a point
A, lying outside of a line l, either no line,
or only one line, or a finite number of lines,
or an infinite number of lines which do not intersect
the line l. (At least two of these situations should occur.)
The line(s) is (are) called the parallel(s) to l
through the given point A.

[For examples:
- let l0 be the line N-P-s1-Q-R, where N is a point
  lying in delta1 not on f1, and R is a similar
  point lying in delta2 not on f2,
  and let A be a point lying on s2,
  then: no parallel to l0 can be drawn through A
  (because any line passing through A, hence through
  s2, will intersect s1, hence l0, in P and Q);
- if the line l1 lies in delta1 such that l1 does
  not intersect the frontier f1, then:
  through any point lying on the left side of l1
  one and only one parallel will pass;
- let B be a point lying in f1, different from P,
  and another point C lying in delta1, not on f1;
  let A be a point lying in delta1 outside of BC;
  then: an infinite number of parallels to the
  line BC can be drawn through the point A.

Theorem. There are at least two lines l1, l2 of a
plane, which do not meet a third line l3 of the
same plane, but they meet each other,
(i.e. if l1 is parallel to l3, and l2 is parallel
to l3, and all of them are in the same plane,
it's not necessary that l1 is parallel to l2).
[For example:
consider three points A, B, C lying in f1, and
different from P, and D a point in delta1 not on
f1; draw the lines AD, BE and CE such that
E is a point in delta1 not on f1 and both BE
and CE do not intersect AD;
then: BE is parallel to AD, CE is also parallel
to AD, but BE is not parallel to CE because the
point E belong to both of them.]

GROUP IV. ANTI-AXIOMS OF CONGRUENCE

IV.1. If A, B are two points on a line l, and A' is a point upon the same or another line l', then: upon a given side of A' on the line l', we can not always find only one point B' so that the segment AB is congruent to the segment A'B'.

[For examples:
- let AB be segment lying in delta1 and having no point in common with f1, and construct the line C-P-s1-Q-s2-P (noted by l') which is the same with C-P-s2-Q-s1-P, where C is a point lying in delta1 not on f1 nor on AB;
take a point A' on l', in between C and P, such that A'P is smaller than AB;
now, there exist two distinct points B1' on s1 and B2'on s2, such that A'B1' is congruent to AB and A'B2' is congruent to AB,
with A'B1' different from A'B2';
- but if we consider a line l' lying in delta1 and limited by the frontier f1 on the right side (the limit point being noted by M),
and take a point A' on l', close to M, such that A'M is less than A'B', then: there is no point B' on the right side of l' so that A'B' is congruent to AB.]

A segment may not be congruent to itself!

[For example:
- let A be a point on s1, closer to P,
and B a point on s2, closer to P also;
A and B are lying on the same line A-Q-B-P-A which is the same with line A-P-B-Q-A,
but AB meseared on the first representation of the line is strictly greater than AB meseared on the second representation of their line.]

IV.2. If a segment AB is congruent to the segment A'B' and also to the segment A"B", then not always the segment A'B' is congruent to the segment A"B".

[For example:
- let AB be a segment lying in delta1-f1, and consider the line C-P-s1-Q-s2-P-D, where C, D are two distinct points in delta1-f1 such that C, P, D are colinear. Suppose that the segment AB is congruent to the segment CD (i.e. C-P-s1-Q-s2-P-D). Get also an obvious segment A'B' in delta1-f1, different from the preceding ones, but congruent to AB.
Then the segment A'B' is not congruent to the segment CD (considered as C-P-D, i.e. not passing through Q.)

IV.3. If AB, BC are two segments of the same line l which have no points in common aside from the point B,
and A'B', B'C' are two segments of the same line or of another line l' having no point other than B' in common, such that AB is congruent to A'B' and BC is congruent to B'C',
then not always the segment AC is congruent to A'C'.

[For example:
let l be a line lying in delta1, not on f1,
and A, B, C three distinct points on l, such that AC is greater than s1;
let l' be the following line: A'-P-s1-Q-s2-P
where A' lies in delta1, not on f1,
and get B' on s1 such that A'B' is congruent to AB, get C' on s2 such that BC is congruent to B'C' (the points A, B, C are thus chosen);
then: the segment A'C' which is first seen as A'-P-B'-Q-C' is not congruent to AC,
because A'C' is the geodesic A'-P-C' (the shortest way from A' to C' does not pass through B') which is strictly less than AC.]

Definitions. Let h, k be two lines having a point O in common. Then the system (h, O, k) is called the angle of the lines h and k in the point O.
(Because some of our lines are curves, we take the angle of the tangents to

the curves in their common point.)

The angle formed by the lines h and k situated in the same plane, noted by <(h, k), is equal to the arithmetic mean of the angles formed by h and k in all their common points.

IV.4. Let an angle (h, k) be given in the plane alpha, and let a line h' be given in the plane beta. Suppose that in the plane beta a definite side of the line h' be assigned, and a point O'. Then in the plane beta there are one, or more, or even no half-line(s) k' emanating from the point O' such that the angle (h, k) is congruent to the angle (h', k'), and at the same time the interior points of the angle (h', k') lie upon one or both sides of h'.

[Examples:

- Let A be a point in delta1-f1, and B, C two distinct points in delta2-f2; let h be the line A-P-s1-Q-B, and k be the line A-P-s2-Q-C; because h and k intersect in an infinite number of points (the segment AP), where they normally coincide -- i.e. in each such point their angle is congruent to zero, the angle (h, k) is congruent to zero. Now, let A' be a point in delta1-f1, different from A, and B' a point in delta2-f2, different from B, and draw the line h' as A'-P-s1-Q-B'; there exist an infinite number of lines k', of the form A'-P-s2-Q-C' (where C' is any point in delta2-f2, not on the line QB'), such that the angle (h, k) is congruent to (h', K'), because (h', k') is also congruent to zero, and the line A'-P-s2-Q-C' is different from the line A'-P-s2-Q-D' if D' is not on the line QC'.
- If h, k, and h' are three lines in delta1-P, which intersect the frontier f1 in at most one point, then there exist only one line k' on a given part of h' such that the angle (h, k) is congruent to the angle (h', k').

- *Is there any case when, with these hypotheses, no k' exists ?
- Not every angle is congruent to itself; for example:
  <(s1, s2) is not congruent to <(s1, s2)
  [because one can construct two distinct lines:
  P-s1-Q-A and P-s2-Q-A, where A is a point in delta2-f2, for the first angle, which becomes equal to zero;
  and P-s1-Q-A and P-s2-Q-B, where B is another point in delta2-f2, B different from A, for the second angle, which becomes strictly greater than zero!].

IV. 5. If the angle (h, k) is congruent to the angle (h', k',) and the angle (h", k"), then the angle (h', k') is not always congruent to the angle (h", k").

(A similar construction to the previous one.)

IV. 6. Let ABC and A'B'C' be two triangles such that
AB is congruent to A'B',
AC is congruent to A'C',
<BAC is congruent to <B'A'C'.
Then not always
<ABC is congruent to <A'B'C'
and <ACB is congruent to <A'C'B'.

[For example:
Let M, N be two distinct points in delta2-f2, thus obtaining the triangle PMN;
Now take three points R, M', N' in delta1-f1, such that RM' is congruent to PM, RN' is congruent to RN, and the angle (RM', RN') is congruent to the angle (PM, PN). RM'N' is an obvious triangle.
Of course, the two triangle are not congruent, because for example PM and PN cut each other twice -- in P and Q -- while RM' and RN' only once -- in R.
(These are geodesical triangles.)]

Definitions:

Two angles are called supplementary if they have the same vertex, one side in common, and the other sides not common form a line.

A right angle is an angle congruent to its supplementary angle.

Two triangles are congruent if its angles are congruent two by two, and its sides are congruent two by two.

Propositions:

A right angle is not always congruent to another right angle.

For example:
Let A-P-s1-Q be a line, with A lying in delta1-f1, and B-P-s1-Q another line, with B lying in delta1-f1 and B not lying in the line AP;
we consider the tangent t at s1 in P, and B chosen in a way that <(AP, t) is not congruent to <(BP, t);
let A', B' be other points lying in delta1-f1 such that <APA' is congruent to <A'P-s1-Q, and <BPB' is congruent to <B'P-s1-Q.
Then:
- the angle APA' is right, because it is congruent to its supplementary (by construction);
- the angle BPB' is also right, because it is congruent to its supplementary (by construction);
- but <APA' is not congruent to <BPB', because the first one is half of the angle A-P-s1-Q, i.e. half of <(AP, t), while the second one is half of the B-P-s1-Q, i.e. half of <(BP, t).

The theorems of congruence for triangles [side, side, and angle in between; angle, angle, and common side; side, side, side] may not hold either in the Critical Zone (s1, s2, f1, f2) of the Model.

Property:
The sum of the angles of a triangle can be:
- 180 degrees, if all its vertexes A, B, C are lying, for example, in delta1-f1;

- strictly less than 180 degrees [any value in the interval (0, 180)],
for example:
let R, T be two points in delta2-f2 such that Q does not lie in RT, and S another point on s2;
then the triangle SRT has <(SR, ST) congruent to 0 because SR and ST have an infinite number of common points (the segment SQ), and <QTR + <TRQ congruent to 180-<TQR [by construction we may vary <TQR in the interval (0, 180)];
- even 0 degree!
let A be a point in delta1-f1, B a point in delta2-f2, and C a point on s3, very close to P;
then ABC is a non-degenerate triangle (because its vertexes are non-colinear), but <(A-P-s1-Q-B, A-P-s3-C) = <(B-Q-s1-P-A, B-Q-s1-P-s3-C) = <(C-s3-P-A, C-s3-P-s1-Q-B) = 0
(one considers the lenth C-s3-P-s1-Q-B strictly less than C-s3-B);
the area of this triangle is also 0 !
- more than 180 degrees,
for example:
let A, B be two points in delta1-f1, such that <PAB + <PBA + <(s1, s2; in Q) is strictly greater than 180 degrees;
then the triangle ABQ, formed by the intersection of the lines A-P-s2-Q, Q-s1-P-B, AB will have the sum of its angles strictly greater than 180 degrees.

Definition:
A circle of center M is a totality of all points A for which the segments MA are congruent to one another.

For example, if the center is Q, and the lenth of the segments MA is chosen greater than the lenth of s1, then the circle is formed by the arc of circle centered in Q, of radius MA, and lying in delta2, plus another arc of circle centered in P, of radius MA-lenth of s1, lying in delta1.

GROUP V. ANTI-AXIOM OF CONTINUITY (ANTI-ARCHIMEDEAN AXIOM)

Let A, B be two points. Take the points A1, A2, A3, A4, ... so that A1 lies between A and A2, A2 lies between

A1 and A3, A3 lies between A2 and A4, etc. and the segments AA1, A1A2, A2A3, A3A4, ... are congruent to one another.
Then, among this series of points, not always there exists a certain point An such that B lies between A and An.

For example:
let A be a point in delta1-f1, and B a point on f1, B different from P;
on the line AB consider the points A1, A2, A3, A4, ... in between A and B, such that AA1, A1A2, A2A3, A3A4, etc. are congruent to one another;
then we find that there is no point behind B (considering the direction from A to B), because B is a limit point (the line AB ends in B).

The Bolzano's (intermediate value) theorem may not hold in the Critical Zone of the Model.

Can you readers find a better model for this anti-geometry?

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On certain new inequalities and limits for the Smarandache function

József Sándor

Department of Mathematics, Babes-Bolyai University,
3400 Cluj - Napoca, Romania

I. Inequalities

1) If $n \geq 4$ is an even number, then $S(n) \leq \frac{n}{2}$.

—Indeed, $\frac{n}{2}$ is integer, $\frac{n}{2} > 2$, so in $(\frac{n}{2})! = 1 \cdot 2 \cdot 3 \cdots \frac{n}{2}$ we can simplify with 2, so $n \mid (\frac{n}{2})!$.

This simplies clearly that $S(n) \leq \frac{n}{2}$.

2) If $n > 4$ is an even number, then $S(n^2) \leq n$

—By $n! = 1 \cdot 2 \cdot 3 \cdots \frac{n}{2} \cdots n$, since we can simplify with 2, for $n > 4$ we get that $n^2 | n!$. This clearly implies the above stated inequality. For factonials, the above inequality can be much improved, namely one has:

3) $S\left((m!)^2\right) \leq 2m$ and more generally, $S\left((m!)^n\right) \leq n \cdot m$ for all positive integers m and n.

—First remark that $\frac{(mn)!}{(m!)^n} = \frac{(mn)!}{m!(mn-m)!} \cdot \frac{(mn-m)!}{m!(mn-2m)!} \cdots \frac{(2m)!}{m! \cdot m!} =$

$= C_{2m}^m \cdot C_{3m}^m \cdots C_{nm}^m$, where $C_n^k = \binom{n}{k}$ denotes a binomial coefficient. Thus $(m!)^n$ divides $(mn)!$, implying the stated inequality. For $n = 2$ one obtains the first part.

4) Let $n > 1$. Then $S\left((n!)^{(n-1)!}\right) \leq n!$

—We will use the well-known result that the product of n consecutive integers is divisible by $n!$. By $(n!)! = 1 \cdot 2 \cdot 3 \cdots n \cdot \left((n+1)(n+2) \cdots 2n\right) \cdots \left((n-1)!\text{-}1\right) \cdots (n-1)!\, n$

each group is divisible by $n!$, and there are $(n-1)!$ groups, so $(n!)^{(n-1)!}$ divides $(n!)!$. This gives the stated inequality.

5) For all m and n one has $[S(m), S(n)] \leq S(m \cdot S(n)) \leq [m, n]$. where $[a, b]$ denotes the

$\ell \cdot c \cdot m$ of a and b.

—If $m = \prod P_i^{ai}$, $n = \prod q_j^{bj}$ are the canonical representations of m, resp. n, then it is well-known that $S(m) = S\left(P_i^{ai}\right)$ and $S(n) = S(q_j^{bj})$, where $S\left(P_i^{ai}\right) = \max\left\{S\left(P_i^{ai}\right) : i = 1, \cdots, r\right\}$; $S(q_j^{bj}) = \max\left\{S(q_j^{bj}) : j = 1, \cdots, h\right\}$, with r and h the number of prime divisors of m, resp. n. Then clearly $[S(m), S(n)] \leq S(m) \cdot S(n) \leq {P_i}^{ai} \cdot q_j^{bj} \leq [m, n]$

6) $\left(S(m), S(n)\right) \geq \frac{S(m) \cdot S(n)}{mn} \cdot (m, n)$ for all m and n

—Since $\left(S(m), S(m)\right) = \frac{S(m) \cdot S(n)}{[S(m), S(n)]} \geq \frac{S(m) \cdot S(n)}{[m, n]} = \frac{S(m) \cdot S(n)}{nm} \cdot (m, n)$

by 5) and the known formula $[m,n] = \frac{mn}{(m,n)}$.

7) $\frac{\left(S(m), S(n)\right)}{(m,n)} \geq \left(\frac{S(mn)}{mn}\right)^2$ for all m and n

—Since $S(mn) \leq m\, S(n)$ and $S(mn) \leq n\, S(m)$ (See [1]), we have $\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)\, S(n)}{mn}$, and the result follows by 6).

8) We have $\left(\frac{S(mn)}{mn}\right)^2 \leq \frac{S(m)\, S(n)}{mn} \leq \frac{1}{(mn)}$

—This follows by 7) and the stronger inequality from 6), namely $S(m)\, S(n) \leq [m\, n] = \frac{mn}{(m,n)}$

Corollary $S(mn) \leq \frac{mn}{\sqrt{mn}}$

9) Max $\left\{S(m), S(n)\right\} \geq \frac{S(mn)}{(mn)}$ for all m, n; where (m,n) denotes the $g \cdot c \cdot d$ of m and n.

—We apply the known result: max $\left\{S(m), S(n)\right\} = S([m, n])$ On the other hand, since [m, n] | m · n, by Corollary 1 from our paper [1] we get $\frac{S(mn)}{mn} \leq \frac{S([m,n])}{[m,n]}$.

Since $[m, n] = \frac{mn}{(m,n)}$,

The result follows:

Remark. Inequality g) compliments Theorem 3 from [1],

namely that max $\left\{S(m), S(n)\right\} \leq S(mn)$.

10) Let $d(n)$ be the number of divisors of n. Then $\frac{S(n!)}{n!} \leq \frac{S\left(n^{d(n)/2}\right)}{n^{d(n)/2}}$

—We will use the known relation $\prod_{k|n} k = n^{d(n)/2}$, where the product is extended over all divisors k of n. Since this product divides $\prod_{k \leq n} k = n!$, by Corollary 1 from [1] we can write $\frac{S(n!)}{n!} \leq \frac{S(\prod_{k/n} k)}{\prod_{k/n} k}$, which gives the desired result.

Remark If n is of the form m^2, then $d(n)$ is odd, but otherwise $d(n)$ is even. So, in each case $n^{d(n)/2}$ is a positive integer.

11) For infinitely many n we have $S(n+1) < S(n)$, but for infinitely many m one has $S(m+1) > S(m)$.

—This is a simple application of 1). Indeed, let $n = p - 1$, where $p \geq 5$ is a prime. Then, by 1) we have $S(n) = S(p-1) \leq \frac{P-1}{2} < p$. Since $p = S(p)$, we have $S(p-1) < S(p)$. Let in the same manner $n = p + 1$. Then, as above, $S(p+1) \leq \frac{P+1}{2} < p = S(p)$.

12) Let p be a prime. Then $S(p!+1) > S(p!)$ and $S(p!-1) > S(p!)$

—Clearly, $S(p!) = p$. Let $p! + 1 = \prod q_j^{\partial j}$ be the prime factorization of $p! + 1$. Here each $q_j > p$, thus $S(p!+1) = S(q_j^{\partial j})$ (for certain j) $\geq S(p^{\partial j}) \geq S(p) = p$. The same proof applies to the case $p! - 1$.

Remark: This offers a new proof for M).

13) Let P_k be the kth prime number. Then $S(p_1 p_2 \ldots P_k + 1) > S(p_1 p_2 \cdots P_k)$ and $S(p_1 p_2 \ldots P_k - 1) > S(p_1 p_2 \cdots P_k)$

—Almost the same proof as in 12) is valid, by remarking that $S(p_1 p_2 \cdots P_k) = P_k$ (since $p_1 < p_2 < \cdots < p_k$).

14) For infinitely many n one has $\left(S(n)^2\right) < S(n-1) \cdot S(n+1)$ and for infinitely many m, $\left(S(m)\right)^2 > S(m-1) \cdot S(m+1)$.

—By $S(p+1) < p$ and $S(p-1) < p$ (See the proof in 11) we have

$\frac{S(p+1)}{S(p)} < \frac{S(p)}{S(p)} < \frac{S(p)}{S(p-1)}$. Thus $\Big(S(p)\Big)^2 > S(p-1) \cdot S(p+1)$.

On the other hand, by putting $x_n = \frac{S(n+1)}{S(n)}$, we shall see in part II,

that $\limsup_{n\to\infty} x_n = +\infty$. Thus $x_{n-1} < x_n$ for infinitely many n, giving

$\Big(S(n)\Big)^2 < S(n-1) \cdot S(n+1)$.

II. Limits:

1) $\liminf_{n\to\infty} \frac{S(n)}{n} = 0$ and $\limsup_{n\to\infty} \frac{S(n)}{n} = 1$

—Clearly, $\frac{S(n)}{n} > 0$. Let $n = 2^m$. Then, since $S(2^m) \leq 2m$, and $\lim_{m\to\infty} \frac{2m}{2m} = 0$, we have

$\lim_{m\to\infty} \frac{S(2^m)}{2^m} = 0$, proving the first part. On the other hand, it is well known that $\frac{S(n)}{n} \leq 1$.

For $n = p_k$ (the kth prime), one has $\frac{S(p_k)}{p_k} = 1 \to 1$ as $k \to \infty$, proving the second part.

Remark: With the same proof, we can derive that $\liminf_{n\to\infty} \frac{S(n^r)}{n} = 0$ for all integers r.

—As above $S(2^{kr}) \leq 2kr$, and $\frac{2kr}{2^k} \to 0$ as $k \to \infty$ (r fixed), which gives the result.

2) $\liminf_{n\to\infty} \frac{S(n+1)}{S(n)} = 0$ and $\limsup_{n\to\infty} \frac{S(n+1)}{S(n)} = +\infty$

—Let p_r denote the rth prime. Since $(p_1 \cdots p_r, 1) = 1$, Dirichlet's theorem on arithmetical

progressions assures the existence of a prime p of the form $p = a \cdot p_1 \cdots p_r - 1$.

Then $S(p+1) = S(ap_1 \cdots p_r) \leq a \cdot S(p_1 \cdots p_r)$ by $S(mn) \leq mS(n)$ (see [1])

But $S(p_1 \cdots p_r) = max\{p_1, \cdots, p_r\} = p_r$. Thus $\frac{S(p+1)}{S(p)} \leq \frac{ap_r}{ap_1 \cdots p_r - 1} \leq$

$\frac{p_r}{p_1 \cdots p_r - 1} \to 0$ as $r \to \infty$. This gives the first part.

Let now p be a prime of the form $p = bp_1 \cdots p_r + 1$.

Then $S(p-1) = S(bp_1 \cdots p_r) \leq b\, S(p_1 \cdots p_r) = b \cdot p_r$,

and $\frac{S(p-1)}{S(p)} \leq \frac{bp_r}{bp_1 \cdots p_r + 1} \leq \frac{p_r}{p_1 \cdots p_r} \to 0$ as $r \to \infty$.

3) $\liminf_{n\to\infty} \Big[S(n+1) - S(n)\Big] = -\infty$ and $\limsup_{m\to\infty} \Big[S(n+1) - S(n)\Big] = +\infty$

—We have $S(p+1) - S/p) \leq \frac{p+1}{2} - p = \frac{-p+1}{2} \to -\infty$ for an odd prime

p (see 1) and 11)$\Big)$. On the other hand, $S(p) - S(p-1) \geq p - \frac{p-1}{2} = \frac{p+1}{2} \to \infty$

(Here $S(p) = p$), where $p-1$ is odd for $p \geq 5$. This finishes the proof.

4) Let $\sigma(n)$ denotes the sum of divisors of n. Then $\liminf_{n\to\infty} \frac{S\left(\sigma(n)\right)}{n} = 0$

—This follows by the argument of 2) for $n = p$. Then $\sigma(\varphi) = p+1$ and $\frac{S(p+1)}{p} \to 0$, where

$\{p\}$ is the sequence constructed there.

5) Let $\varphi(n)$ be the Enter totient function. Then $\liminf_{n\to\infty} \frac{S\left(\varphi(n)\right)}{n} = 0$

—Let the set of primes $\{p\}$ be defined as in 2). Since $\varphi(n) = p-1$ and $\frac{S(p-1)}{p} = \frac{S(p-1)}{S(p)} \to 0$,

the assertion is proved. The same result could be obtained by taking $n = z^k$. Then, since

$\varphi(2^k) = 2^{k-1}$, and $\frac{S(2^{k-1})}{2^k} \leq \frac{2\cdot(k-1)}{2^k} \to o$ as $k \to \infty$, the assertion follows:

6) $\liminf_{n\to\infty} \frac{S\left(S(n)\right)}{n} = 0$ and $\max_{n\in\mathbb{N}} \frac{S\left(S(n)\right)}{n} = 1$.

—Let $n = p!$ (p prime). Then, since $S(p!) = p$ and $S(p) = p$, from $\frac{p}{p!} \to 0$ $(p \to \infty)$

we get the first result. Now, clearly $\frac{S\left(S(n)\right)}{n} \leq \frac{S(n)}{n} \leq 1$. By letting $n = p$ (prime), clearly

one has $\frac{S\left(S(p)\right)}{p} = 1$, which shows the second relation.

7) $\liminf_{n\to\infty} \frac{\sigma\left(S(n)\right)}{S(n)} = 1$.

—Clearly, $\frac{\sigma(k)}{k} > 1$. On the other hand, for $n = p$ (prime), $\frac{\sigma\left(S(p)\right)}{S(p)} = \frac{p+1}{p} \to 1$ as $p \to \infty$.

8) Let Q(n) denote the greatest prime power divisor of n. Then $\liminf_{n\to\infty} \frac{\varphi\left(S(n)\right)}{\partial(n)} = 0$.

—Let $n = p_1^k \cdots p_r^k$ $(k > 1, \text{fixed})$. Then, clearly $\partial(n) = p_r^k$.

By $S(n) = S(p_r^k)$ $\left(\text{since } S(p_r^k) > S(p_i^k) \text{ for } i < k\right)$ and $S(p_r^k) = j \cdot p_r$, with $j \leq k$ (which is known) and by $\varphi(j\, p_k) \leq j \cdot \varphi(p_r) \leq k(p_r - 1)$, we get $\frac{\varphi\left(S(n)\right)}{\partial(n)} \leq \frac{k\cdot(p_r-1)}{p_r^k} \to 0$ as $r \to \infty$ (k fixed).

9) $\lim\limits_{\substack{m\to\infty \\ m \text{ even}}} \frac{S(m^2)}{m^2} = 0$

—By 2) we have $\frac{S(m^2)}{m^2} \leq \frac{1}{m}$ for $m > 4$, even. This clearly inplies the above remark.

Remark. It is known that $\frac{S(m)}{m} \leq \frac{2}{3}$ if $m \neq 4$ is composite. From $\frac{S(m^2)}{m^2} \leq \frac{1}{m} < \frac{2}{3}$ for $m > 4$, for the composite numbers of the perfect squares we have a very strong improvement.

10) $\liminf\limits_{n\to\infty} \frac{\sigma\left(S(n)\right)}{n} = 0$

—By $\sigma(n) = \sum\limits_{d/n} d = n\sum\limits_{d/n} \frac{1}{d} \leq n\sum\limits_{d=1}^{n} \frac{1}{d} < n \cdot (2\ log\, n)$, we get $\sigma(n) < 2n \log n$ for $n > 1$. Thus $\frac{\sigma\left(S(n)\right)}{n} < \frac{2\, S(n)\, log\, S(n)}{n}$. For $n = 2^k$ we have $S(2^k) \leq 2k$, and since $\frac{4k\ log\, 2k}{2^k} \to 0$ $(k \to \infty)$, the result follows.

11) $\lim\limits_{n\to\infty} \sqrt[n]{S(n)} = 1$

—This simple relation follows by $1 \leq S(n) \leq n$, so $1 \leq \sqrt[n]{S(n)} \leq \sqrt[n]{n}$; and by $\sqrt[n]{n} \to 1$ as $n \to \infty$. However, 11) is one of a (few) limits, which exists for the Smarandache function.

Finally, we shall prove that:

12) $\limsup\limits_{n\to\infty} \frac{\sigma\left(n\, S(n)\right)}{n\, S(n)} = +\infty$.

—We will use the facts that $S(p!) = p$, $\frac{\sigma(p!)}{p!} = \sum_{d|p!} \frac{1}{d} \geq 1 + \frac{1}{2} + \cdots + \frac{1}{p} \to \infty$ as $p \to \infty$, and the inequality $\sigma(ab) \geq a\,\sigma(b)$ (see [2]).

Thus $\frac{\sigma(S(p!)p!}{p! \cdot S(p!)} \geq \frac{S(p!) \cdot \sigma(p!)}{p! \cdot p} = \frac{\sigma(p!)}{p!} \to \infty$. Thus, for the sequence $\{n\} = \{p!\}$, the results follows.

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THE FACTORIAL SIGNATURE OF NATURAL NUMBERS

by

Ion Bălăcenoiu

In this paper we define the factorial signature for natural numbers and with this we obtain several results.

1. DEFINITION The system $(j_1, j_2, \dots, j_r) \in N^{*r}$ is a system of factorial exponents if $\exists\, s \in N^*$ so that $s! = p_1^{j_1} \cdot p_2^{j_2} \cdot \dots \cdot p_{\pi(s)}^{j_{\pi(s)}}$, where $2 = p_1 < 3 = p_2 < \dots < p_{\pi(s)} \le s$, $\pi(s) = r$.

Obviously, for every natural number $s > 1$ there exists a system of factorial exponents $(j_1, j_2, \dots, j_{\pi(s)})$.

Because $s! = \prod_{i=1}^{\pi(s)} p_i^{e_{p_i}(s)}$, where $e_{p_i}(s)$ are Legendre's exponents, it is true that: $e_{p_1}(s) \ge e_{p_2}(s) \ge \dots e_{p_{\pi(s)}}(s) = 1$.

Therefore for every system of factorial exponents $(j_1, j_2, \dots, j_r)$ it results that $j_1 \ge j_2 \ge \dots \ge j_r = 1$.

It exists a finite number of system of factorial exponents with r components. Indeed, they correspond those natural numbers with the property: $p_r ! \le s! < p_{r-1} !$

If $(j'_1, j'_2, \dots, j'_r)$ and $(j''_1, j''_2, \dots, j''_r)$ are systems of factorial exponent corresponding as n respectively m, then

$n < m \Rightarrow j'_1 \le j''_1,\ j'_2 \le j''_2, \dots, j'_{r-1} \le j''_{r-1}, j'_r = j''_r = 1.$

If $\pi(n) = \pi(n+1)$, then $n + 1$ is a composite number and their systems of factorial exponents have the same number of components.

If $n+1$ is a prime number, then $\pi(n+1) = \pi(n) + 1$ and if $(j_1, j_2, \ldots, j_{\pi(n)} = 1)$ is the system of exponents of adequate factorial for n, then the system of exponents of adequate factorial for $n+1$ is:

$$(j_1, j_2, \ldots, j_{\pi(n)} = 1,\ j_{\pi(n+1)} = 1)$$

Two systems of factorial exponents with r components, adequate as two different natural numbers, have different components and equal components, too.

2. DEFINITION Let $n \in N$, $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots p_{i_t}^{\alpha_{i_t}}$, and let s be the smallest positive integer such that $s!$ is divisible by n. Then the factorial signature for n (denoted by $s.f.(n!)$) is: $s.f.(n) = \left\{ p_{i_{k_1}}^{\alpha_{i_{k_1}}}, p_{i_{k_2}}^{\alpha_{i_{k_2}}}, \ldots, p_{i_{k_r}}^{\alpha_{i_{k_r}}} \right\}$ where $\left\{ p_{i_{k_1}}^{\alpha_{i_{k_1}}}, p_{i_{k_2}}^{\alpha_{i_{k_2}}}, \ldots, p_{i_{k_r}}^{\alpha_{i_{k_r}}} \right\}$ is the largest subset for $\left\{ p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \ldots, p_{i_t}^{\alpha_{i_t}} \right\}$ so that there are $\beta_{i_{k_j}} \geq \alpha_{i_{k_j}} \geq 1$, $j \in \overline{1, r}$ with $p_{i_{k_j}}^{\beta_{i_{k_j}}} \nmid (s-1)!$ and $p_{i_{k_j}}^{\beta_{i_{k_j}}} \mid s!$.

It is considered $s.f.(0) = \varnothing$, $s.f.(1) = \{1\}$.

Obviously: $e_{p_{i_{k_j}}}(s-1) < \beta_{i_{k_j}} \leq e_{p_{i_{k_j}}}(s)$, $j = \overline{1, r}$.

3. DEFINITION The type of the factorial signature for n is noted $T[s.f.(n)]$ and $T[s.f.(0)] = 0$, $T[s.f.(n)] = s$, for $n > 0$, where s is the smallest positive integer such that $n \mid s!$.

4. EXAMPLE

a) Let $n = 120 = 2^3 \times 3 \times 5$, therefore $p_1 = 2$, $p_2 = 3$, $p_3 = 5$; $\alpha_1 = 3$, $\alpha_2 = 1$, $\alpha_3 = 1$. Obviously the smallest positive integer s thus so that $n \mid s!$ is $s = 5$. Indeed, $s.f.(120) = \{5\}$ because $\{5\}$ is the largest subset of $\{2^3, 3, 5\}$ in the sense that (see definition 2) it exist $\beta_3 = \alpha_3 = 1$ so that $5^{\beta_3} \nmid 4!$ and $5^{\beta_3} \mid 5!$.

b) Let $n = p^{\alpha}$, then $s.f.(p^{\alpha}) = \{p^{\alpha}\}$ and $T[s.f.(p^{\alpha})] = s$ iff $e_p(s-1) < \alpha \leq e_p(s)$.

5. PROPOSITION Let $n = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_t}^{\alpha_{i_t}}$, $p_{i_1} < p_{i_2} < \ldots < p_{i_t}$, with $s.f.(n) = \left\{ p_{i_{k_1}}^{\alpha_{i_{k_1}}}, p_{i_{k_2}}^{\alpha_{i_{k_2}}}, \ldots, p_{i_{k_r}}^{\alpha_{i_{k_r}}} \right\}$, and $T[s.f.(n)] = s > 1$ then it exists at least an element $p_{i_{k_j}}^{\alpha_{i_{k_j}}}$, $j = \overline{1, r}$ so that

$e_{p_{i_{k_j}}}(s-1) < \alpha_{i_{k_j}} \le e_{p_{i_{k_j}}}(s)$ and $T\left[s.f.\left(p_{i_{k_j}}^{\alpha_{i_{k_j}}}\right)\right] = s.$

Proof. Let $p_{i_{k_1}} < p_{i_{k_2}} < \ldots < p_{i_{k_r}}$.

Because $T[s.f.(n)] = s > 1$ it results that $n \mid s!$ and it exists $\beta_{i_{k_j}} \ge \alpha_{i_{k_j}} \ge 1$ so that $e_{p_{i_{k_j}}}(s-1) < \beta_{i_{k_j}} \le e_{p_{i_{k_j}}}(s)$.

If does not exist $j \in \overline{1, r}$ so that $e_{p_{i_{k_j}}}(s-1) < \alpha_{i_{k_j}} \le e_{p_{i_{k_j}}}(s)$, then $p_{i_{k_r}} < p_{\pi(s)}$ because $p_{i_{k_r}} = p_{\pi(s)}$ it implies that $\alpha_{i_{k_r}} = e_{p_{i_{k_r}}}(s) = e_{p_{\pi(s)}}(s) = 1 = \beta_{i_{k_r}}$ and $e_{p_{\pi(s)}}(s-1) = 0$.

Using $\alpha_{i_{k_j}} \le \beta_{i_{k_j}} \le e_{p_{i_{k_j}}}(s)$ it results that $\alpha_{i_{k_j}} \le e_{p_{i_{k_j}}}(s-1)$, $j = \overline{1, r}$. Thus we have $T[s.f.(n)] \le s - 1 < s$, which is not possible.

Therefore it exists $j \in \overline{1, r}$ so that $e_{p_{i_{k_j}}}(s-1) < \alpha_{i_{k_j}} \le e_{p_{i_{k_j}}}(s)$ and in consequence $T\left[s.f.\left(p_{i_{k_j}}^{\alpha_{i_{k_j}}}\right)\right]_{\alpha_{i_{k_j}}} = s.$

We can observe that $p_{i_{k_j}}$ indicates the type $T[s.f.(n)]$.

6. DEFINITION The complement until a factorial (see [2]), is $b : N^* \to N^*$, $b(n) = k$, where k is the smallest positive integer so that $n\,b(n)$ is a factorial. Thus $n\,b(n) = m!$.

Obviously, if $n\,b(n) = m!$, then $m!$ is the smallest factorial divisible by n, therefore $n\,b(n) = [\eta(n)]!$ where η is Smarandache function see [1].

It is easy to see that $b(n!) = 1$ and $b(p) = (p-1)!$ p is a prime number.

Because $\eta(n!) = n$ it results $b(n) = \dfrac{[\eta(n)]!}{\eta(n!)}$.

7. PROPOSITION Let p be a prime number and $p > m$, then $b(m! \cdot p) = \dfrac{(p-1)!}{m!}$

Proof. Obviously, $p!$ is the smallest factorial divisible by $m!\,p$.

Therefore $b(m! \cdot p) = \dfrac{p!}{m!\,p} = \dfrac{(p-1)!}{m!}$

8. PROPOSITION

$T[s.f.(n)] = s$ iff $nb(n) = s!$

Proof. Obviously, $T[s.f.(n)] = s \Leftrightarrow s!$ is the smallest factorial divisible by $n \Leftrightarrow n\,b(n) = s!$.

9. DEFINITION We define the echivalent relation: $s.f.(n) \approx s.f.(m) \Leftrightarrow$ $\Leftrightarrow$ $nbn = mb(m)$.

We note $\hat{s!} = \{n \in N^* / nb(n) = s!\}$.

10. REMARK Obviously, if $s.f.(n) = s.f.(m)$ then $s.f.(n) \approx s.f.(m)$. If $s.f.(n) \approx s.f.(m)$ it does not result that $s.f.(n) = s.f.(m)$. If $s.f.(n) = s.f.(m)$ it does not result that $n = m$. If $n\, b(n) = s!$ it results that $s.f.(n) \approx s.f.(s!)$ because $s!\, b(s!) = s!$.

We also observe that $T[s.f.(n)] = s \Leftrightarrow n \in \hat{s!}$.

If p is a prime number, then $p \in \hat{p!}$ because $p\, b(p) = p!$. It is easy to see that $s.f.(p) = s.f.(p!) = \{p\}$.

Because $p! = p_1^{e_{p_1}(p)} \cdot p_2^{e_{p_2}(p)} \cdot \ldots p$, where $2 = p_1 < 3 = p_2 < \ldots < p$, it results $\hat{p!} = \{p, p_1p, p_2p, p_1^2p, p_1p_2p, \ldots, p_1^{e_{p_1}(p)} \cdot p_2^{e_{p_2}(p)} \cdot \ldots \cdot p\}$.

If $s = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_k}^{\alpha_{i_k}}$, then $s.f.(s!) = \{p_{i_1}^{e_{p_{i_1}}(s)}, p_{i_2}^{e_{p_{i_2}}(s)}, \ldots, p_{i_k}^{e_{p_{i_k}}(s)}\}$

11. PROPOSITION If $(n, m) = 1$ and $n, m \in \hat{s!}$ then $n \cdot m \in \hat{s!}$ and $s.f.(n \cdot m) = [s.f.(n)] \cup [s.f.(m)]$.

Proof. Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_t^{\alpha_t}$ and $m = q_1^{\gamma_1} \cdot q_2^{\gamma_2} \cdot \ldots \cdot q_h^{\gamma_h}$ be the cannonical decomposition of n and m. Obviously, because $(n, m) = 1$ it results $p_i \neq q_j$ for $i \in \overline{1, t}$, $j \in \overline{1, h}$.

Let $s.f.(n) = \{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \ldots, p_{i_r}^{\alpha_{i_r}}\}$ and $s.f.(m) = \{q_{j_1}^{\gamma_{j_1}}, \ldots, q_{j_k}^{\gamma_{j_k}}\}$.

Because $n, m \in \hat{s!}$ it results that s is the smallest positive integer so that $n \mid s!$, $m \mid s!$ and it exists $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_r}$ and $\delta_{j_1}, \delta_{j_2}, \ldots, \delta_{j_k}$ respectively so that $\beta_{i_u} \geq \alpha_{i_u} \geq 1$, $u \in \overline{1, r}$ and $\delta_{j_v} \geq \gamma_{j_v} \geq 1$, $v \in \overline{1, k}$ and

$$p_{i_u}^{\beta_{i_u}} \nmid (s-1)!, \quad p_{i_u}^{\beta_{i_u}} \mid s!$$

$$q_{j_v}^{\delta_{j_v}} \nmid (s-1)!, \quad q_{j_v}^{\delta_{j_v}} \mid s!$$

In $(n, m) = 1$ and $n \mid s!$, $m \mid s!$ it results that $n\, m \mid s!$. Because s is the smallest natural number such as $n \mid s!$ and $n\, m \mid s!$ it results that s is the smallest natural number such that $s!$ is divisible by $n \cdot m$, therefore $T[s.f.(nm)] = s$, so that $nm \in \hat{s!}$.

Obviously $nm = p_1^{\alpha_1} \cdot p_2^{\alpha 2} \ldots p_t^{\alpha_t} \cdot q_1^{\gamma_1} \cdot q_2^{\gamma_2} \ldots q_h^{\gamma_h}$ and

$$s.f.(n \cdot m) = \{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \ldots, p_{i_r}^{\alpha_{i_r}}, q_{j_1}^{\gamma_{j_1}}, \ldots, q_{j_k}^{\gamma_{j_k}}\} = [s.f.(n)] \cup [s.f.(m)]$$

12. REMARK The proposition 11 can be also formulated in this way: if $n\,b(n) = s!$, $m\,b(m) = s!$ and $(n, m) = 1$, then $nmb(nm) = s!$.

It results $b(n \cdot m) = \dfrac{b(n)b(m)}{s!}$ if $(n, m) = 1$ and $n, m \in \hat{s!}$.

13. PROPOSITION

If $(n, m) = 1$ and $s.f.(n) \approx s.f.(m)$, then $b(n \cdot m) = (b(n), b(m))$.

Proof. Let $T[s.f.(m)] = s$, because $(n, m) = 1$ and $s.f.(n) \approx s.f.(m)$ then it results:

$$nb(n) = mb(m) = nmb(nm) = s!,$$

therefore $b(n \cdot m) = \dfrac{s!}{nm}$. Let us consider $d = (b(n), b(m))$, $b(n) = d \cdot a$ and $b(m) = d \cdot b$, where $(a, b) = 1$. Then $nb(m) = mb(m)$ implies that $na = mb$.

Because $(a, b) = 1$ it results $a \mid m$ and $b \mid n$, then we can write $n = hb$, then $hba = mb$, so that $m = ah$. Since $1 = (n, m) = (hb, ha) = h(a, b) = h$ it results $n = b$, $m = a$.

Then $(b(n), b(m)) = d = \dfrac{s!}{na} = \dfrac{s!}{nm} = b(n \cdot m)$.

14. PROPOSITION

Let $n = q_1^{\gamma_1} \cdot q_2^{\gamma_2} \cdot \ldots \cdot q_t^{\gamma_t}$ and $s.f.(n) = \{q_{j_1}^{\gamma_{j_1}}, \ldots, q_{j_r}^{\gamma_{j_r}}\}$. If $n \in \hat{s!}$ and $s.f.(s!) = \left\{p_{i_1}^{e_{p_{i_1}}(s)}, p_{i_2}^{e_{p_{i_2}}(s)}, \ldots, p_{i_k}^{e_{p_{i_k}}(s)}\right\}$, then $\{q_{j_1}, q_{j_2}, \ldots, q_{j_r}\} \subset \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$.

Proof. Because $s.f.(s!) = \left\{p_{i_1}^{e_{p_{i_1}}(s)}, p_{i_2}^{e_{p_{i_2}}(s)}, \ldots, p_{i_k}^{e_{p_{i_k}}(s)}\right\}$ it results $p_{i_h}^{e_{p_{i_h}}(s)} \nmid (s-1)!$ and $p_{i_h}^{e_{p_{i_h}}(s)} \mid s!$ for $h = \overline{1, k}$, therefore $p_{i_h} \mid s$, thus we have $s = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \ldots \cdot p_{i_k}^{\alpha_{i_k}}$, where $1 \le \alpha_{i_h} \le e_{p_{i_h}}(s)$ for $h = \overline{1, k}$.

Because $n \in \hat{s!}$ it exists $\beta_{i_m} \ge \gamma_{i_m} \ge 1$, for $m = \overline{1, r}$ so that $q_{j_m}^{\beta_{j_m}} \mid (s-1)!$ and $q_{j_m}^{\beta_{j_m}} \mid s!$, thus $q_{j_m} \mid s$. Therefore $\{q_{j_1}, q_{j_2}, \ldots, q_{j_r}\} \subset \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$.

15. DEFINITION Let $n, m \in \hat{s!}$ and $s.f.(n) = \{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \ldots, p_{i_r}^{\alpha_{i_r}}\}$, $s.f.(m) = \{q_{j_1}^{\gamma_{j_1}}, q_{j_2}^{\gamma_{j_2}}, \ldots, q_{j_k}^{\gamma_{j_k}}\}$ then $s.f.(n) \subseteq s.f.(m)$ iff

$\{p_{i_1}, p_{i_2}, \dots, p_{i_r}\} \subset \{q_{j_1}, q_{j_2}, \dots, q_{j_k}\}$ and for every $p_{i_s} = q_{j_t}$ it implies $\alpha_{i_s} \le \gamma_{j_t}$.

16. REMARK Obviously "$\tilde{\subseteq}$" is a partial order relation in the set of factorial signatures of numbers which belongs to $\hat{s}!$. For any $n \in \hat{s}!$ it results $s.f.(n) \tilde{\subseteq} s.f.(s!)$, so that $s\cdot f(s!)$ is the maximal element. If $s = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \dots \cdot p_{i_u}^{\alpha_{i_u}}$ then the minimal elements in the factorial signatures of numbers which belongs to $\hat{s}!$ are:

$$s.f.\left\{p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}\right\} = \left\{p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}\right\}, \quad h \in \overline{1, u}$$

because $p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1} \in \hat{s}!$ and for any $x \in \hat{s}!$ so that $s.f.(x) \tilde{\subseteq} s.f.\left(p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}\right)$ it results $s.f.(x) = \left\{p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}\right\}$

17. PROPOSITION For any $m \in N$, $\eta^{-1}(m) = m!$, where η is Smarandache function.

Proof. Let $n \in \hat{m}!$, then $nb(n) = [\eta(n)]! = m!$, therefore $\eta(n) = m$, or $n \in \eta^{-1}(m)$. Conversely, if $n \in \eta^{-1}(m)$ it results $\eta(n) = m$, that $nb(n) = [\eta(n)]!$ and therefore $n \in \hat{m}!$.

18. DEFINITIONS In $\hat{s}!$ it is considered the equivalent relation: $n \approx m \Leftrightarrow s.f.(n) = s.f.(m)$. The echivalent class for n is $\tilde{n} = \{m \in \hat{s}! \mid s.f.(n) = s.f.(m)\}$. The set of equivalent classes in $\hat{s}!$ it noted with $\hat{\hat{s}}!$. In $\hat{\hat{s}}!$ it is considered partial order relation $\tilde{n} \tilde{\le} \tilde{m} \Leftrightarrow s.f.(n) \tilde{\subseteq} s.f.(m)$.

19. REMARK Each class $\tilde{n} \in \hat{\hat{s}}!$ is a set of elements which belongs to $\hat{s}!$, and it is total ordered in the sense of the relation $\le$ It is also finite, therefore it has a minimum and a maximum. If $s.f.(n) = \{p_{i_1}^{\alpha_{i_1}}, p_{i_2}^{\alpha_{i_2}}, \dots, p_{i_r}^{\alpha_{i_r}}\}$, then in the class $\tilde{n}$ the smallest number is $\bar{n} = p_{i_1}^{\alpha_{i_1}} \cdot p_{i_2}^{\alpha_{i_2}} \cdot \dots \cdot p_{i_r}^{\alpha_{i_r}}$ and the other numbers of $\tilde{n}$ are $A \cdot \bar{n}$, with $A = p_{h_1}^{\varepsilon_1} \cdot p_{h_2}^{\varepsilon_2} \cdot \dots \cdot p_{h_k}^{\varepsilon_k}$, with p_{h_j}'s and $p_{h_j} \mid s!$ and $0 \le \varepsilon_j \le e_{p_{h_j}}(s)$, $j = \overline{1, k}$, where $\left\{p_{h_1}^{e_{p_{h_1}}(s)}, \dots, p_{h_k}^{e_{p_{h_k}}(s)}\right\} = \left\{p_1^{e_{p_1}(s)}, \dots, p_{\pi(s)}^{1}\right\} - s.f.(s!)$.

The largest number of $\tilde{\tilde{n}}$ is:

$$\bar{n}\, p_{h_1}^{e_{p_{h_1}}(s)} \cdot p_{h_2}^{e_{p_{h_2}}(s)} \cdots \cdot p_{h_k}^{e_{p_{h_k}}(s)} .$$

Minimal elements of $\hat{\hat{s!}}$, in the sense of the partial order relation $\tilde{\leq}$, are the classes which respectively have the elements: $p_{i_h}^{e_{p_{i_h}}(s)-\alpha_{i_h}+1}$, $h = \overline{1, r}$. The maximal class of $\hat{\hat{s!}}$ has $s!$ as element.

If $\tilde{\tilde{n}} \tilde{\leq} \tilde{\tilde{m}}$ and $\tilde{\tilde{n}} \neq \tilde{\tilde{m}}$, the absolute value of the difference between two different numbers in the class $\tilde{\tilde{m}}$ is larger than the smallest between absolute values of differences between two different numbers of the class $\tilde{\tilde{n}}$.

If $\tilde{\tilde{n}} \tilde{\leq} \tilde{\tilde{m}}$ and $\tilde{\tilde{n}} \neq \tilde{\tilde{m}}$, the absolute value of the difference between a number of $\tilde{\tilde{n}}$ and a number of $\tilde{\tilde{m}}$ is larger or equal than the smallest number of $\tilde{\tilde{n}}$ and therefore it is larger or equal than the smallest number of the minimal class comparable (in the sense of the partial order relation $\tilde{\leq}$) with $\tilde{\tilde{n}}$.

20. EXAMPLE Let $s = 12 = 2^2 \cdot 3$, then $s! = 12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, $s.f.(12!) = \{2^{10}, 3^5\}$.

Let us consider the set of natural numbers with the factorial signature of type 12, so that $\hat{\hat{12!}} = \{n \in N / nb(n) = 12!\} = \{n \in N / s.f.(n) = s.f.(12!)\}$.

Obviously $\eta^{-1}(12) = \hat{\hat{12!}}$.

The minimal elements of $\hat{\hat{12!}}$, in the sense of the partial order relation $\tilde{\leq}$, are: $\tilde{\tilde{2}}^{10-2-1} = \tilde{\tilde{2}}^{9}$ and $\tilde{\tilde{3}}^{5-1-1} = \tilde{\tilde{3}}^{5}$.

Factorial signatures of numbers of $\hat{\hat{12!}}$ are ordered in the following way:

$$\{2^9\} \tilde{\subseteq} \left\{ \begin{array}{l} \{2^{10}\} \tilde{\subseteq} \{2^{10},3\} \tilde{\subseteq} \{2^{10},3^2\} \tilde{\subseteq} \{2^{10},3^3\} \tilde{\subseteq} \{2^{10},3^4\} \tilde{\subseteq} \\ \{2^9,3\} \tilde{\subseteq} \{2^9,3^2\} \tilde{\subseteq} \{2^9,3^3\} \tilde{\subseteq} \{2^9,3^4\} \tilde{\subseteq} \end{array} \right\} \{2^9,3^5\} \tilde{\subseteq}$$

$$\{3^5\} \tilde{\subseteq} \{3^5,2\} \tilde{\subseteq} \{3^5,2^2\} \tilde{\subseteq} \ldots \tilde{\subseteq} \{3^5,2^8\} \tilde{\subseteq}$$

Classes of numbers of $\hat{\hat{12!}}$ are presented in next table:

$n \rightarrow$	s.f.(n)	s.f.(n)	$\leftarrow n$	s.f.(n)	$\leftarrow n$
		$\{2^9\}$	2^9	$\{3^5\}$	3^5
			$2^9 \cdot 5$		$3^5 \cdot 5$
			$2^9 \cdot 7$		$3^5 \cdot 7$
			$2^9 \cdot 11$		$3^5 \cdot 11$
			$2^9 \cdot 5^2$		$3^5 \cdot 5^2$
			$2^9 \cdot 5 \cdot 7$		$3^5 \cdot 5 \cdot 7$
			$2^9 \cdot 5 \cdot 11$		$3^5 \cdot 5 \cdot 11$
			$2^9 \cdot 7 \cdot 11$		$3^5 \cdot 7 \cdot 11$
			$2^9 \cdot 5 \cdot 7 \cdot 11$		$3^5 \cdot 5 \cdot 7 \cdot 11$
			$2^9 \cdot 5^2 \cdot 7 \cdot 11$		$3^5 \cdot 5^2 \cdot 7 \cdot 11$
2^{10}	$\{2^{10}\}$	$\{2^9, 3\}$	$2^9 \cdot 3$	$\{3^5, 2\}$	$3^5 \cdot 2$
$2^{10} \cdot 5$			$2^9 \cdot 3 \cdot 5$		$3^5 \cdot 2 \cdot 5$
$2^{10} \cdot 7$			$2^9 \cdot 3 \cdot 7$		$3^5 \cdot 2 \cdot 7$
$2^{10} \cdot 11$			$2^9 \cdot 3 \cdot 11$		$3^5 \cdot 2 \cdot 11$
$2^{10} \cdot 5^2$			$2^9 \cdot 3 \cdot 5^2$		$3^5 \cdot 2 \cdot 5^2$
$2^{10} \cdot 5 \cdot 7$			$2^9 \cdot 3 \cdot 5 \cdot 7$		$3^5 \cdot 2 \cdot 5 \cdot 7$
$2^{10} \cdot 5 \cdot 11$			$2^9 \cdot 3 \cdot 5 \cdot 11$		$3^5 \cdot 2 \cdot 5 \cdot 11$
$2^{10} \cdot 7 \cdot 11$			$2^9 \cdot 3 \cdot 7 \cdot 11$		$3^5 \cdot 2 \cdot 7 \cdot 11$
$2^{10} \cdot 5 \cdot 7 \cdot 11$			$2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 11$		$3^5 \cdot 2 \cdot 5 \cdot 7 \cdot 11$
$2^{10} \cdot 5^2 \cdot 7 \cdot 11$			$2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$		$3^5 \cdot 2 \cdot 5^2 \cdot 7 \cdot 11$
$2^{10} \cdot 3$	$\{2^{10}, 3\}$	$\{2^9, 3^2\}$	$2^9 \cdot 3^2$	$\{3^5, 2^2\}$	$3^5 \cdot 2^2$
- - - -			- - - - -		- - - - -
$2^{10} \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$			$2^9 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$		$3^5 \cdot 2^2 \cdot 5^2 \cdot 7 \cdot 11$
- - - - - - - - - - -	- - - - - -	- - - - - - -	- - - - - - - - - - - -	- - - - - - -	- - - - - - - - - - -
$2^{10} \cdot 3^3$	$\{2^{10}, 3^3\}$	$\{2^9, 3^4\}$	$2^9 \cdot 3^4$	$\{3^5, 2^8\}$	$3^5 \cdot 2^8$
- - - -			- - - - -		- - - - -
$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$			$2^9 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$		$3^5 \cdot 2^8 \cdot 5^2 \cdot 7 \cdot 11$
$2^{10} \cdot 3^4$	$\{2^{10}, 3^4\}$	$\{2^9, 3^5\}$	$2^9 \cdot 3^5$		
- - - -			- - - - -		
$2^{10} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$			$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$		
$2^{10} \cdot 3^5$	$\{2^{10}, 3^5\}$				
- - - -					
$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$					

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CURRENT ADDRESS

Brazda lui Novac, Bloc G_5, Ap. 161
1100, Craiova
Romania

The Pseudo-Smarandache Function and the Classical Functions of Number Theory

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233
e-mail 71603.522@compuserve.com

Abstract: The Pseudo-Smarandache function has a simple definition: Given any integer $n > 0$, the value of the Pseudo-Smarandache function is the smallest integer m such that n evenly divides the sum $1 + 2 + 3 + \ldots + m$. In this paper, several problems concerning this function will be presented and solved. Most will involve the standard number theory functions such as Euler's phi function and the sum of divisors function.[1]

The Pseudo-Smarandache function has the definition

Given any integer $n \geq 1$, the value of the Pseudo-Smarandache function Z(n) is the smallest integer m such that n evenly divides

$$\sum_{k=1}^{m} k$$

Note that this summation is equivalent to the expression

$$\frac{m(m+1)}{2}$$

The purpose of this paper will be to present some theorems concerning the interactions of this function with the classical theorems of number theory.

Basic Theorems

Theorem 1: If p is an odd prime, then $Z(p) = p - 1$.

Proof: Clearly, p divides

$$\frac{(p-1)p}{2}$$

and there is no smaller number that satisfies the definition.

Theorem 2: $Z(2^k) = 2^{k+1} - 1$.

Proof: Since only one of m and m+1 is even, it follows that $Z(2^k)$ is the smallest ratio

$$\frac{m(m+1)}{2}$$

where the even number contains k + 1 instances of 2. This number is clearly 2^{k+1} and the value of m is smallest when $m + 1 = 2^{k+1}$.

Definition: Given any integer $n \geq 2$, the Euler phi function $\phi(n)$ is the number of integers k, $1 \leq k < n$, such that k and n are relatively prime.

Our first theorem concerning the combination of ϕ and Z is trivial.

Theorem : There are an infinite number of integers n such that $\phi(n) = Z(n)$.

Proof: It is well-known that if p is an odd prime $\phi(p) = p - 1$.

So we modify the statement to make it harder.

Modified theorem: There are an infinite number of composite integers n such that $\phi(n) = Z(n)$.

Proof: Let n = 2p, where p is an odd prime of the form p = 4k + 1. It is well-known that this is an infinite set.

Consider the fraction

$$\frac{(p-1)p}{2}.$$

Replacing p by the chosen form

$$\frac{(4k+1-1)(4k+1)}{2} = \frac{4k(4k+1)}{2} = 2k(4k+1)$$

Clearly,

$$2(4k+1) \mid 2k(4k+1)$$

and p = 4k + 1 is the smallest such number. Therefore,
$Z(2p) = p - 1$. It is well-known that $\phi(2p) = p - 1$ for p an odd prime.

Unsolved Question: Is there another infinite set of composite numbers such that $Z(n) = \phi(n)$?

Another equation involving these two functions has an infinite family of solutions.

Theorem: There are an infinite number of solutions to the expression

$$Z(n) + \phi(n) = n.$$

Proof: Let $n = 2^{2j} + 2^{2j-1}$, where $j \geq 1$. Factoring it, we have

$n = 2^{2j} * 3$. Using the well-known formula for the computation of the phi function

$$\phi(n) = (2 - 1)2^{2j-1}(3 - 1)3^0 = 2^{2j}$$

It is easy to verify that if k is odd,

$$3 \mid 2^k + 1.$$

From this, it follows that

$$2^{2j} * 3 \mid \frac{2^{2j-1}(2^{2j-1} + 1)}{2}$$

and it is easy to see that 2^{2j-1} is the smallest such m. Therefore,

$$Z(2^{2j} * 3) = 2^{2j+1}$$

and

$$Z(n) + \phi(n) = n.$$

Unsolved Question: Is there another infinite family of solutions to the equation

$$Z(n) + \phi(n) = n?$$

Another classic number theory function is the sigma or sum of divisors function.

Given any integer $n \geq 1$, $\sigma(n)$ is the sum of all the divisors of n.

Theorem: There are an infinite number of solutions to the equation

$$\sigma(n) = Z(n).$$

Proof: It has already been proven that $Z(2^k) = 2^{k+1} - 1$. It is well-known that $\sigma(p^k) = p^{k-1} - 1$.

A computer search up through n = 10,000 yielded no solutions not of this type.

Unsolved Question: Is there another infinite family of solutions to the equation

$$\sigma(n) = Z(n)?$$

The final classic function of number theory is the number of integral divisors function.

Definition: For $n \geq 1$, the divisors function d(n) is the number if integers m, where $1 \leq m \leq n$, such that m evenly divides n.

Question: How many solutions are there to the equation

$$Z(n) = d(n)?$$

A computer search up through n = 10,000 yielded only the solutions n = 1, 3 and 10.

Question: How many solutions are there to the equation

$$Z(n) + d(n) = n?$$

A computer search up through n = 10,000 yielded only the solution n = 56, as d(56) = 8 and Z(56) = 48.

It is unknown if there are any additional solutions to this problem.

There are many other problems involving the classic functions that can be defined. One such example is

Question: How many solutions are there to the equation

$$Z(n) + \phi(n) = d(n)?$$

A computer search up through 10,000 failed to find a single solution.

The author continues to work on this set of problems and hopes to present additional solutions in the future.

1. This paper was presented at the Spring, 1998 meeting of the Iowa section of The Mathematical Association of America.

TWO FUNCTIONS IN NUMBER THEORY AND SOME UPPER BOUNDS FOR THE SMARANDACHE'S FUNCTION

Sabin Tabirca Tatiana Tabirca

The aim of this article is to introduce two functions and to give some simple properties for one of them. The function's properties are studied in connection with the prime numbers. Finally, these functions are applied to obtain some inequalities concerning the Smarandache's function.

1. Introduction

In this section, the main results concerning the Smarandache and Euler's functions are review. Smarandache proposed [1980] a function $S: N^* \to N$ defined by $S(n) = \min\{k \mid k! \mid n\}$. This function satisfies the following main equations:

1. $(n,m) = 1 \Rightarrow S(n \cdot m) = \min\{S(n), S(m)\}$

   (1)

2. $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots p_s^{k_s} \Rightarrow S(n) = \min\left\{S(p_1^{k_1}), S(p_2^{k_2}), \ldots, S(p_s^{k_s})\right\}$

   (2)

3. $(\forall\, n > 1)\ S(n) \le n$

   (3)

and the equality in the inequality (3) is obtained if and only if n is a prime number. The research on the Smarandache's function has been carried out in several directions. One of these direction studies the average function $\overline{S}: N^* \to N$ defined by $\overline{S}(n) = \dfrac{\sum_{i=1}^{n} S(i)}{n}$. Tabirca [1997] gave the following two upper bounds for this function $(\forall\, n > 5)\ \overline{S}(n) \le \dfrac{3}{8} \cdot n + \dfrac{1}{4} - \dfrac{2}{n}$ and $(\forall\, n > 23)\ \overline{S}(n) \le \dfrac{21}{72} \cdot n + \dfrac{1}{12} - \dfrac{2}{n}$ and conjectured that $(\forall\, n > 1)\ \overline{S}(n) \le \dfrac{2 \cdot n}{\ln}$.

Let $\varphi: N^* \to N$ be the Euler function defined by $\varphi(n) = card\{k = 1, n \mid (k,n) = 1\}$. The main properties of this function are review below:

1. $(n,m) = 1 \Rightarrow \varphi(n \cdot m) = \varphi(n) \cdot \varphi(m)$

(4)

2. $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots p_s^{k_s} \cdot \Rightarrow \varphi(n) = n \cdot \prod_{i=1}^{s} (1 - \frac{1}{p_i})$

(5)

3. $\varphi(\frac{n}{m}) = card\{k = 1,n | (k,n) = m\}$.

(6)

It is known that if $f : N^* \to N$ is a multiplicative function then the function $g : N^* \to N$ defined by $g(n) = \sum_{d|n} f(d)$ is multiplicative as well.

2. The functions ψ_1, ψ_2

In this section two functions are introduced and some properties concerning them are presented.

Definition 1.

Let ψ_1, ψ_2 be the functions defined by the formulas

1. $\psi_1 : N^* \to N$, $\psi_1(n) = \sum_{i=1}^{n} \frac{n}{(i,n)}$

(7)

2. $\psi_2 : N^* \to N$, $\psi_2(n) = \sum_{i=1}^{n} \frac{i}{(i,n)}$.

(8)

i	$\psi_1(i)$	$\psi_2(i)$	I	$\psi_1(i)$	$\psi_2(i)$	i	$\psi_1(i)$	$\psi_2(i)$
1	1	1	11	111	56	21	301	151
2	3	2	12	77	39	22	333	167
3	7	4	13	157	79	23	507	254
4	11	6	14	129	65	24	301	151
5	21	11	15	147	74	25	521	261
6	21	11	16	171	86	26	471	236

3								
7	43	22	17	273	137	27	547	274
8	43	22	18	183	92	28	473	237
9	61	31	19	343	172	29	813	407
10	63	32	20	231	116	30	441	221

Table 1. Table of the functions ψ_1, ψ_2

Remarks 1.

1. These function are correctly defined based on the implication $\frac{n}{(i,n)}, \frac{i}{(i,n)} \in N \Rightarrow \sum_{i=1}^{n} \frac{n}{(i,n)}, \sum_{i=1}^{n} \frac{i}{(i,n)} \in N$.
2. If p is prime number, then the equations $\psi_1(p) = p^2 - p + 1$ and $\psi_2(p) = \frac{p(p-1)}{2} + 1$ can be easy verified.
3. The values of these functions for the first 30 natural numbers are shown in Table 1. From this table, it is observed that the values of ψ_1 are always odd and moreover the equation $\psi_2(n) = \left\lceil \frac{\psi_1(n)}{2} \right\rceil$ seems to be true.

Proposition 1 establishes a connection between ψ_1 and φ.

Proposition 1

If $n>0$ is an integer number, then the equation

$$\psi_1(n) = \sum_{d|n} d \cdot \varphi(d) \tag{9}$$

holds.

Proof

Let $A_d = \{i = \overline{1,n} \mid (i,n) = d\}$ be the set of the elements which satisfy $(i,n)=d$.

The following transformations of the function ψ_1 holds.

$$\psi_1(n)=\sum_{i=1}^{n}\frac{n}{(i,n)}=\sum_{d\mid n}\sum_{i\in A_d}\frac{n}{(i,n)}=\sum_{d\mid n}\sum_{i\in A_d}\frac{n}{d}=\sum_{d\mid n}\frac{n}{d}\sum_{i\in A_d}1=\sum_{d\mid n}\frac{n}{d}\cdot|A_d|. \tag{10}$$

Using (6) the equation (10) gives $\psi_1(n)=\sum_{d\mid n}\frac{n}{d}\cdot\varphi\left(\frac{n}{d}\right)$.

Changing the index of the last sum, the equation (9) is found true.♣

The function $g(n)=n\varphi(n)$ is multiplicative resulting in that the function $\psi_1(n)=\sum_{d\mid n}d\cdot\varphi(d)$ is multiplicative. Therefore, it is sufficiently to find a formula for $\psi_1(p^k)$, where p is a prime number.

Proposition 2.

If p is a prime number and $k\geq 1$ then the equation

$$\psi_1(p^k)=\frac{p^{2k+1}+1}{p+1} \tag{11}$$

holds.

Proof

The equation (11) is proved based on a direct computation, which is described below.

$$\psi_1(p^k)=\sum_{d\mid p^k}d\cdot\varphi(d)=1+\sum_{i=1}^{k}p^i\cdot\varphi(p^i)=1+\left(1-\frac{1}{p}\right)\cdot\sum_{i=1}^{k}p^{2i}=$$

$$=1+\left(1-\frac{1}{p}\right)\cdot p^2\cdot\frac{p^{2k}-1}{p^2-1}=1+p\cdot\frac{p^{2k}-1}{p-1}=\frac{p^{2k+1}+1}{p-1}$$

Therefore, the equation (11) is true.♣

Theorem 1.

If $n=p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots p_s^{k_s}$ is the prime numbers decomposition of n, then the formula

$$\psi_1\left(\prod_{i=1}^{s}p_i^{k_i}\right)=\prod_{i=1}^{s}\frac{p_i^{2k_i+1}+1}{p_i+1} \tag{12}$$

holds.

Proof

The proof is directly found based on Proposition1 1 and on the multiplicative property of ψ_1.

Obviously, if p is a prim number then $\psi_1(p) = \frac{p^3+1}{p+1} = p^2 - p + 1$ holds finding again the equation from Remark 1.2. If $n = p_1 \cdot p_2 \cdot \ldots \cdot p_s$ is a product of prime numbers then the following equation is true.

$$\psi_1(n) = \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_s) = \prod_{i=1}^{s}\left(p_i^2 - p_i + 1\right) \tag{13}$$

Proposition3.

$$\left(\forall\, n > 1\right) \sum_{i=1,\,(i,n)=1}^{n} i = \frac{n \cdot \varphi(n)}{2} \tag{14}$$

Proof

This proof is made based on the *Inclusion & Exclusion* principle.

Let $D_p = \{i = 1,2,\ldots,n \mid p \mid n\}$ be the set which contains the multiples of p.

This set satisfies

$$D_p = p \cdot \left\{1,2,\ldots,\frac{n}{p}\right\} \text{ and } \sum_{i \in D_p} i = p \cdot \sum_{i=1}^{\frac{n}{p}} i = p \cdot \frac{\frac{n}{p} \cdot \left(\frac{n}{p}+1\right)}{2} = \frac{n}{2} \cdot \left(\frac{n}{p}+1\right).$$

Let $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots p_s^{k_s}$ be the prime number decomposition of n.

The following intersection of sets

$$D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}} = \{i = 1, 2, \ldots, n \mid p_{j_1} \mid n \wedge p_{j_2} \mid n \wedge \ldots \wedge p_{j_m} \mid n\}$$

is evaluated as follows

$$D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}} = \{i = 1, 2, \ldots, n \mid p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m} \mid n\} = D_{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}}$$

Therefore, the equation

$$\sum_{i \in D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}}} i = \sum_{i \in D_{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}}} i = \frac{n}{2} \cdot \left(\frac{n}{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}} + 1\right) \tag{14}$$

holds.

The *Inclusion & Exclusion* principle is applied based on

$$D = \{i = 1, 2, \ldots, n \mid (i,n) = 1\} = \{1, 2, \ldots, n\} - \sum_{j=1}^{s} D_{p_k}$$

and it gives

$$\sum_{i\le n,\,(i,n)=1} i = \sum_{i=1}^{n} i - \sum_{m=1}^{n} (-1)^{m-1} \cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \; \sum_{i\in D_{p_{j_1}}\cap D_{p_{j_2}}\cap\ldots\cap D_{p_{j_m}}} i \tag{15}$$

Applying (14), the equation (15) becomes

$$\sum_{i\le n,\,(i,n)=1} i = \sum_{i=1}^{n} i + \sum_{m=1}^{n} (-1)^{m} \cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \; \frac{n}{2}\cdot\left(\frac{n}{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}+1\right). \tag{16}$$

The right side of the equation (16) is simplified by reordering the terms as follows

$$\sum_{i\le n,\,(i,n)=1} i = \frac{n^2}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^{m}\cdot\sum_{1\le j_1<j_2<\ldots<j_m\le n}\frac{1}{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}\right)+\frac{n}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^{m}\cdot\sum_{1\le j_1<j_2<\ldots<j_m\le n}1\right)$$

$$\sum_{i\le n,\,(i,n)=1} i = \frac{n^2}{2}\cdot\prod_{m=1}^{s}\left(1-\frac{1}{p_{j_m}}\right)+\frac{n}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^{m}\cdot\binom{n}{m}\right)=\frac{n^2}{2}\cdot\prod_{m=1}^{s}\left(1-\frac{1}{p_{j_m}}\right)=\frac{n}{2}\cdot\varphi(n).$$

Therefore, the equation (14) holds.♣

Obviously, the equation (14) does not hold for n=1 because $\sum_{i=1,\,(i,1)=1}^{1} i = 1$ and $\frac{n\cdot\varphi(n)}{2}=\frac{1}{2}$.

Based on Proposition 3, the formula of the second function is found.

Proposition 4.

The following equation

$$(\forall\, n>1)\ \psi_2(n)=\frac{\psi_1(n)+1}{2} \tag{17}$$

holds.

Proof

Let $I_{n,d}=\{i=1,2,\ldots,n \mid (i,n)=d\}$ be the set of indices which satisfy $(i,n)=d$. Obviously, the following equation

$$(\forall\, d\mid n)\ I_{n,d}=d\cdot I_{\frac{n}{d},1} \tag{18}$$

holds. Based on (18), the sum $\sum_{i=1}^{n}\frac{i}{(i,n)}$ is transformed as follows

$$\psi_2(n)=\sum_{i=1}^{n} i\cdot(i,n)^{-1}=\sum_{d\mid n} d^{-1}\cdot\sum_{i\in I_{n,d}} i=\sum_{d\mid n} d^{-1}\cdot d\cdot\sum_{i_1\in I_{\frac{n}{d},1}} i_1=\sum_{d\mid n}\sum_{i_1\in I_{\frac{n}{d},1}} i_1 . \tag{19}$$

Proposition 3 is applied for any divisor $d\neq n$ and the equation (19) becomes

$$\psi_2(n)=\sum_{d\mid n}\sum_{i_1\in I_{\frac{n}{d},1}} i_1 = 1+\sum_{n\neq d\mid n}\frac{\frac{n}{d}\varphi\left(\frac{n}{d}\right)}{2}=1+\frac{1}{2}\sum_{n\neq d\mid n}\left(\frac{n}{d}\right)\cdot\varphi\left(\frac{n}{d}\right). \qquad (20)$$

Completing the last sum and changing the index, the equation (20) is transformed as follows

$$\psi_2(n)=1+\frac{1}{2}\sum_{1\neq d\mid n} d\cdot\varphi(d)=1-\frac{1}{2}+\frac{1}{2}\sum_{d\mid n} d\cdot\varphi(d)=\frac{1}{2}+\frac{1}{2}\cdot\psi_1(n)$$

resulting in that (17) is true. ♣

Remarks 2

1. Based on the equation $\psi_2(n)=\frac{\psi_1(n)+1}{2}$, it is found that $\psi_1(n)=2\cdot\psi_2(n)-1$ is always an odd number and that the equation $\psi_2(n)=\left\lceil\frac{\psi_1(n)}{2}\right\rceil$ holds.

2. If $n=p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots p_s^{k_s}>1$ is the prime numbers decomposition of n, then the formula $\psi_2(\prod_{i=1}^{s}p_i^{k_i})=\frac{1}{2}+\frac{1}{2}\cdot\prod_{i=1}^{s}\frac{p_i^{2\cdot k_i+1}+1}{p_i+1}$ holds.

3. Upper bounds for the Smarandache's function

In this section, an application of the functions ψ_1,ψ_2 is presented. Based on these function an inequality concerning the Smarandache's function is proposed and some upper bounds for $\overline{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)$ are deduced.

Let p_1 =2, p_2=3, ..., p_m ,... be the set of the prime numbers.

Proposition 5.

$$\left(\forall\, i\geq p_m\right)\left(\forall\, j=\overline{1,p_1\cdot p_2\cdot\ldots\cdot p_m}\right) S(p_1\cdot p_2\cdot\ldots\cdot p_m\cdot i+j)\leq\frac{p_1\cdot p_2\cdot\ldots\cdot p_m\cdot i+j}{(p_1\cdot p_2\cdot\ldots\cdot p_m, j)} \qquad (21)$$

Proof

Let i, j be two numbers such that $i\geq p_m$ and $j=\overline{1,p_1\cdot p_2\cdot\ldots\cdot p_m}$.

Let us suppose that $(p_1\cdot p_2\cdot\ldots\cdot p_m, j)=p_{i_1}\cdot p_{i_2}\cdot\ldots\cdot p_{i_s}$ and $j=p_{i_1}\cdot p_{i_2}\cdot\ldots\cdot p_{i_s}\cdot j_1$.

Based on the inequality $\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1 \geq i + 1 \geq p_m + 1$, we find that the product $\left(\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1\right)! = 1 \cdot 2 \cdot \ldots \cdot \left(\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1\right)$ contains the factors p_{i_1}, p_{i_2}, ..., p_{i_s} and $\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1$.

The following divisibility holds

$$\left(\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1\right)! \mid p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s} \cdot \left(\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1\right) = \frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j}{(p_1 \cdot p_2 \cdot \ldots \cdot p_m, j)}$$

therefore, the inequality (21) is found true.♣

Proposition 6.

$$(\forall\, i \geq p_m) \quad \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} S(p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j) \leq i \cdot \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) + \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \qquad (22)$$

Proof

The equation (21) is applied for this proof as follows:

$$\sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} S(p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j) \leq \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j}{(p_1 \cdot p_2 \cdot \ldots \cdot p_m, j)} =$$

$$= i \cdot \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{(p_1 \cdot p_2 \cdot \ldots \cdot p_m, j)} + \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \frac{j}{(p_1 \cdot p_2 \cdot \ldots \cdot p_m, j)}$$

Applying the definitions of the functions ψ_1, ψ_2 , the inequality (22) is found true.♣

Theorem 2.

The following inequality

$$\overline{S}(n) = \frac{1}{n} \cdot \sum_{j=1}^{n} S(i) \leq \frac{\psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m)}{2 \cdot (p_1 \cdot p_2 \cdot \ldots \cdot p_m)^2} \cdot n + \frac{2 \cdot \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) - \frac{1}{2}}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} + \frac{1}{n} \cdot C_m \qquad (23)$$

is true for all $n > p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2$, where

$$C_m = \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) - \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot \frac{(p_m - 1) \cdot p_m}{2} - \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot (p_m - 1) \qquad (24)$$

is a constant which does not depend on n.

Proof

Proposition 6 is used for this proof.

Let n be a number such that $n > p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2$. The sum $\sum_{i=1}^{n} S(i)$ is split into two sums as follows

$$\sum_{i=1}^{n} S(i) = \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \sum_{i=p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2 + 1}^{n} S(i) \le \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \sum_{i=p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2 + 1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m \left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil} S(i) =$$

$$= \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \sum_{i=p_m}^{\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil - 1} \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} S(p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j).$$

For the second sum the inequality (22) is applies resulting in the following inequality

$$\sum_{i=1}^{n} S(i) \le \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \sum_{i=p_m}^{\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil - 1} \left[i \cdot \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) + \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \right]. \qquad (25)$$

Calculating the last sum, the inequality (25) becomes

$$\sum_{i=1}^{n} S(i) \le \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot \frac{\left(\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil - 1 \right) \cdot \left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil}{2} +$$

$$+ \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot \left(\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil - 1 \right) - \sum_{i=1}^{p_m - 1} \left[i \cdot \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) + \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \right].$$

Based on the double inequality $\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil - 1 < \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \le \left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil$, we find

$$\sum_{i=1}^{n} S(i) \le \frac{\psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m)}{2 \cdot (p_1 \cdot p_2 \cdot \ldots \cdot p_m)^2} \cdot n^2 + \frac{\frac{1}{2} \cdot \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) + \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m)}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \cdot n + C_m.$$

Dividing by n and using Proposition 4, the equation (22) is found true. ♣

4. Conclusions

The inequality (22) extends the results presented by Tabirca [1997] and generates several inequalities concerning the function S, which are presented in the following:

- m=1 $\Rightarrow (n>4)\ \bar{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le 0.375\cdot n+0.75+\frac{5}{n}$

- m=2 $\Rightarrow (n>18)\ \bar{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le 0.29167\cdot n+1.76+\frac{24}{n}$

- m=3 $\Rightarrow (n>150)\ \bar{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le 0.245\cdot n+7.35-\frac{1052}{n}$

- m=4 $\Rightarrow (n>1470)\ \bar{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le 0.215\cdot n+45.15-\frac{176859}{n}$

The coefficients of n from the above inequalities are decreasing and the inequalities are stronger and stronger. Therefore, it is natural to investigate other upper bounds such us the bound proposed by Tabirca [1997] $\bar{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le\frac{2\cdot n}{\ln n}$. Ibstedt based on an UBASIC program [Ibstedt 1997] proved that the inequality $\bar{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le\frac{n}{\ln n}$ holds for natural numbers less than 5000000. A proof for this results has not been found yet.

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THE SMARANDACHE PERIODICAL SEQUENCES

by

M.R. Popov, student Chandler College

1. Let N be a positive integer with not all digits the same, and N' its digital reverse.

Then, let N1 = abs (N - N'), and N1' its digital reverse. Again, let N2 = abs (N1 - N1'), N2' its digital reverse, and so on, where abs x is the absoluth value of x.

After a finite number of steps one finds an Nj which is equal to a previous Ni, therefore the sequence is periodical (because if N has, say, n digits, all other integers following it will have n digits or less, hence their number is limited, and one applies the Dirichlet's box principal).

For examples:

a. If one starts with N = 27, then N' = 72;
abs (27 - 72) = 45; its reverse is 54;
abs (45 - 54) = 09, ...
thus one gets: 27, 45, 09, 81, 63, 27, 45, ... ;
the Lenth of the Period LP = 5 numbers (27, 45, 09, 81, 63),
and the Lenth of the Sequence 'till the first repetition
occurs LS = 5 numbers either.

b. If one starts with 52, then one gets:
52, 27, 45, 09, 81, 63, 27, 45, ... ;
then LP = 5 numbers, while LS = 6.

c. If one starts with 42, then one gets:
42, 18, 63, 27, 45, 09, 81, 63, 27, ... ;
then LP = 5 numbers, while LS = 7.

For the sequences of integers of two digits, it seems like:
LP = 5 numbers (27, 45, 09, 81, 63; or a circular permutation of them), and $5 <= LS <= 7$.

Question 1:

Find the Lenth of the Period (with its corresponding numbers), and the Lenth of the Sequence 'till the first repetition occurs for:
the integers of three digits, and the integers of four digits.
(It's easier to write a computer program in these cases to check the LP and LS.)

An example for three digits:
321, 198, 693, 297, 495, 099, 891, 693, ... ;
(similar to the previous period, just inserting 9 in the middle of each number).
Generalization for sequences of numbers of n digits.

2. Let N be a positive integer, and N' its digital reverse.

For a given positive integer c, let N1 = abs (N' - c), and N1' its digital reverse. Again, let N2 = abs (N1' - c), N2' its digital reverse, and so on.

After a finite number of steps one finds an Nj which is equal to a previous Ni, therefore the sequence is periodical (same proof).

For example:
If N = 52, and c = 1, then one gets:
52, 24, 41, 13, 30, 02, 19, 90, 08, 79, 68, 85, 57, 74, 46, 63, 35, 52, ... ;
thus LP = 18, LS = 18.

Question 2:

Find the Lenth of the Period (with its corresponding numbers), and the Lenth of the Sequence 'till the first repetition occurs (with a given non-null c) for:
the integers of two digits,
and the integers of three digits.
(It's easier to write a computer program in these cases to check the LP and LS.)

Generalization for sequences of numbers of a n digits.

3. Let N be a positive integer with n digits a1a2 ... an, and c a given integer > 1.

Multiply each digit ai of N by c, and replace ai with the last digit of the product ai x c, say it is bi. Note N1 = b1b2 ... bn, do the same procedure for N1, and so on.

After a finite number of steps one finds an Nj which is equal to a previous Ni, therefore the sequence is a periodical (same proof).

For example:
If N = 68 and c = 7:
68, 26, 42, 84, 68, ... ;
thus LP = 4, LS = 4.

Question 3:

Find the Lenth of the Period (with its corresponding numbers), and the Lenth of the Sequence 'till the first repetition occurs (with a given c) for:
the integers of two digits,
and the integers of three digits.
(It's easier to write a computer program in these cases to check the LP and LS.

Generalization for sequence of numbers of n digits.

4.1. Smarandache generalized periodical sequence:

Let N be a positive integer with n digits a1a2 ...an. If f is a function defined on the set of integers with n digits or less, and the values of f are also in the same set, then:

there exist two natural numbers $i < j$ such that

$$f(f(\ldots f(s) \ldots)) = f(f(f(\ldots f(s) \ldots))),$$

where f occurs i times in the left side, and j times in the right side of the previous equality.

Particularizing f, one obtains many periodical sequences.

Say:

If N has two digits a1a2, then: add 'em (if the sum is greater than 10, add the resulted digits again), and substract 'em (take the absolute value) -- they will be the first, and second digit respectively of N1. And same procedure for N1.

Example:

75, 32, 51, 64, 12, 31, 42, 62, 84, 34, 71, 86, 52, 73, 14, 53, 82, 16, 75, ..

4.2. More General:

Let S be a finite set, and f: S ----> S a function. Then:

for any element s belonging to S, there exist two natural numbers $i < j$ such that

$$f(f(\ldots f(s) \ldots)) = f(f(f(\ldots f(s) \ldots))),$$

where f occurs i times in the left side, and j times in the right side of the previous equality.

Reference:

F. Smarandache, "Sequences of Numbers", University of Craiova Symposium of Students, December 1975.

THE PRIMES IN SMARANDACHE POWER PRODUCT SEQUENCES

Maohua Le and Kejian Wu
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract

For any positive integer k, let A_k be the Smarandache k-power product sequence. In this paper we prove that if k is an odd integer, with k>1, then A_k contains only one prime 2.

In [1], Iacobescu defined the sequence $\{1+c_1c_2...c_n\}_{n=1}^{\infty}$, is the Smarandache cubic product sequence, where c_n is the n-th cubic number. Simultaneous, he posed the following question:

Question: Hou many primes are in the sequence $\{1+c_1c_2...c_n\}_{n=1}^{\infty}$?

We nou give a general definition as follows:

For any positive integers k, n let

$$(1) \qquad a_k(n) = 1+1^k2^k...n^k,$$

and let $A_k = \{a_k(n)\}_{n=1}^{\infty}$. Then A_k is called the Smarandache k-power product sequence. In this paper we prove the following result:

Theorem. If k is an odd integer, with k>1, then A_k contains only one prime 2.

Clearly, the above result completely solves Iacobescu's question.

Proof of Theorem. We see from (1) that

$$(2) \qquad a_k(n) = 1 + (n!)^k.$$

If k is an odd integer, with k>1, then from (2) we get

$$(3) \quad a_k(n) = 1^k + (n!)^k$$
$$= (1+n!)(1 - n! + (n!)^2 - ... - (n!)^{k-2} + (n!)^{k-1}).$$

When n = 1, $a_k(1) = 2$ is a prime.

When n > 1, since

$1 - n! > 1$ and $1 - n! + (n!)^2 - \ldots - (n!)^{k-2} - (n!)^{k-1} =$

$$((n!)^{k-1} - (n!)^{k-2}) + \ldots + ((n!)^2 - n!) + 1 > 1,$$

we find from (3) that $a_k(n)$ is not a prime. Thus, the sequence A_k contains only one prime 2. The theorem is proved.

Reference:

1. F. Iacobescu, "Smarandache partition type and other sequences", Bulletin of Pure and applied Sciences, 16E(1997), No.2, 237-240.

A NOTE ON THE PRIMES IN SMARANDACHE UNARY SEQUENCE

Maohua Le and Kejian Wu
Zhanjiang Normal college, Zhanjiang, Guangdong, P.R.China

Abstract

For any positive integer n, let p_n be the n-th prime, and let $u(n) = (10^{p_n} - 1)/9$. In this note we prove that if $p_n \equiv 1, 13$, or $19 \pmod{20}$, and $2p_n+1$ is also a prime, then $u(n)$ is not a prime.

For any positive integer n, let p_n be n-th prime, and let $u(n) = (10^{p_n} - 1)/9$. Then the sequence $U = \{u(n)\}_{n=1}^{\infty}$ is called the Smarandache unary sequence (see [2]).
It is an odd question that if U contain infinit many primes?
In this note we prove the following result:

Theorem. If $p_n \equiv 1, 13$, or $19 \pmod{20}$, and $2p_n+1$ is also a prime, then $u(n)$ is not a prime.

By using the above result, we see that both $u(12)$ and $u(15)$ are not primes.

Proof of Theorem. Let $q = 2p_n+1$. By Fermat's theorem (see[1], Theorem 71]), if q is a prime, then we have

$$10^{q-1} \equiv 1 \pmod{q}. \tag{1}$$

From (1), we get

$$(10^{p_n} + 1)(10^{p_n} - 1) \equiv 0 \pmod{q}. \tag{2}$$

Since q is a prime, we have either

$$q \mid 10^{p_n} + 1 \tag{3}$$

or

$$q \mid 10^{p_n} - 1, \tag{4}$$

by (2).

We nou assume that p_n satisfies $p_n \equiv 1, 13$, or $19 \pmod{20}$.

Then p_n is an odd prime. Hence, if (3) holds, then we have

(5)
$$\left(\frac{-10}{q}\right) = 1,$$

where (-10/9) is the Legendre symbol. Since $q = 2p_n+1$,
we hawe $q \equiv 3 \pmod 4$ and $(-1/q) = -1$. Therefore, we obtain from (5) that

(6) $\quad (10/q) = (2/q)(5/q) = -1.$

Hoewer, since $q \equiv 3, 27$, or $39 \pmod{40}$ if $p_n \equiv 1, 13$, or $19 \pmod{20}$ respectively, we have

(7) $$(2/q) = \begin{cases} -1, \\ 1, \end{cases} \qquad (5/q) = \begin{cases} -1, \text{ if } q \equiv 3 \text{ or } 27 \pmod{40}; \\ 1, \text{ if } q \equiv 39 \pmod{40}. \end{cases}$$

We find for (7) that (10/q) = 1, which contradicts (6). It implies that (3) does not hold. Thus, by (4), we get

(8) $\quad q \mid 9u(n)$.

Notice that $q \neq 9$ and $1<q<u(n)$. We see from (8) that $q \mid u(n)$ and u(n) is not a prime. The theorem is proved.

References:

1. G.H.Hardy and E.M.Wright, "An Introduction to the Theory of Numbers", Oxford University Press, 1937.
2. F.Iacobescu, "Smarandache partition type and other sequences", Bulletin of Pure and Applied Sciences, 16E (1997), No. 2, 237-240.

SMARANDACHE CONCATENATED POWER DECIMALS AND THEIR IRRATIONALITY

Yongdong Guo and Maohua Le
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract

In this paper we prove that all Smarandache concatenated k-power decimals are irrational numbers.

For any positive integer k, we define the Smarandache concatenated k-power decimal α_k as follows:

(1)
$$\alpha_1 = 0.1234567891011..., \quad \alpha_2 = 0.149162536496481100121...$$
$$\alpha_3 = 0.18276412521634351272910001331..., \ ..., \text{ etc.}$$

In this peper we discuss the irrationally of α_k. We prove the following result:

Theorem. For any positive integer k, α_k is an irrational number.

Proof. We nou suppose that α_k is a rational number. Then, by [1, Theorem 135], α_k is an infinite periodical decimal such that

(2) $$\alpha_k = 0.a_1...a_r\,\overline{a_{r+1}...a_{r+t}},$$

were r, t are fixed integers, with $r \geq 0$ and $t > 0$, $a_1, ..., a_r, a_{r+1}, ..., a_{r+t}$ are integers satisfying $0 \leq a_i \leq 9$ $(i = 1, 2, ..., r+t)$.
However, we see from (1) that there exist arbitrary many continuous zeros in the expansion of α_k. Therefore, we find from (2) that $a_{r+1} = ... = a_{r+t} = 0$. It implies that α_k is a finite decimal; a contradiction. Thus, α_k must be an irrational number. The theorem is proved.

Finally, we pose a further question as follows:

Question. Is α_k a transcedental number for any positive integer k?

By an old result of Mahler [2], the answer of our question is positive for k=1.

References:

1. G.H.Hardy and E.M.Wright, "An Introduction to the Theory of Numbers", Oxford University Press, Oxford, 1938.
2. K.Mahler, "Aritmetische Eigenschaften einer Klasse von Dezimalbruchen", Nederl. Akad. Wetesch. Proc., Ser.A, 40 (1937), 421-428.

ON THE PERFECT SQUARES IN SMARANDACHE CONCATENATED SQUARE SEQUENCE

Kejian Wu and Maohua Le
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract

Let n be positive integer, and let s(n) denote the n-th Smarandache concatenated squre number. In this paper we prove that if $n \equiv 2, 3, 4, 7, 8, 9, 11, 12, 14, 16, 17, 18, 20, 21, 22,$ or $25 \pmod{27}$, then s(n) is not a square.

In [1], Marimutha defined the Smarandache concatenated square sequence $\{s(n)\}_{n=1}^{\infty}$ as follows:

(1) $s(1) = 1,\ s(2) = 14,\ s(3) = 149,\ s(4) = 14916,$
$s(5) = 1491625, \ldots .$

Then we called s(n) the n-th Smarandache concatenated square number. Marimutha [1] conjectured that s(n) is never a perfect square. In this paper we prove the following result:

Theorem.

If $n \equiv 2, 3, 4, 7, 8, 9, 11, 12, 14, 16, 17, 18, 20, 21, 22,$ or $25 \pmod{27}$, then s(n) is not a perfect square.

The above result implies that the density of perfect squares in Smarandache concatenated square sequence is at most 11/27.

Prof of Theorem. We now assume that s(n) is a perfect square. Then we have

$$s(n) = x^2, \tag{2}$$

were x is a positive integer. Notice that $10^k \equiv 1 \pmod 9$ for any positive integer k. We get from (1) and (2) that

$$s(n) \equiv 1^2 + 2^2 + \ldots + n^2 \equiv 1/6\ n(n+1)(2n+1) \equiv x^2 \pmod 9. \tag{3}$$

It implies that

(4) $n(n+1)(2n+1) \equiv 6x^2 \pmod{27}$.

If $n \equiv 2 \pmod{27}$, then from (4) we get $2*3*5 \equiv 6x^2 \pmod{27}$. It follows that

(5) $x^2 \equiv 5 \pmod{9}$.

Since 5 is not a square residue mod 3 , (5) is impossible.
Therefore, if $n \equiv 2 \pmod{27}$, then s(n) is not a square.

By using some similarly elementary number theory methods, we can check that the congruence (4) does not hold for the remaining cases. The theorem is proved.

Reference:

1.H.Marimutha, "Smarandache concatenate type sequences", Bulletin of Pure and Applied Sciences, 16E (1997), No. 2, 225-226.

THE MODULE PERIODICITY OF SMARANDACHE CONCATENATED ODD SEQUENCE

Xigeng Chen
Maoming Educational College, Maoming, Guangdong, P.R.China

Maohua Le
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract

In this paper we prove that the residue sequence of Smarandache concatenated odd sequence mod 3 is periodical.

Let p be a prime. For any integer a, let $<a>_p$ denote the least nonnegative residue of a mod p. Furter, for an integer sequence

$A = \{a(n)\}_{n=1}^{\infty}$, the sequence $\{<a(n)>_p\}_{n=1}^{\infty}$ is called the residue sequence of A mod p, and denoted by $<A>_p$.

In [1], Marimutha defined the Smarandache concatenated odd

sequence $S=\{s(n)\}_{n=1}^{\infty}$, where

(1) s(1)=1, s(2)=13, s(3)=135 , s(4)=1357, ...

In this paper we discuss the periodicity of $<S>_p$. Clearly,
if p=2 or 5, then the residue sequence $<S>_p$ is periodical.
We now prove the following result:

Theorem. If p=3, then $<S>_p$ is periodical.

Prof. For ahy positive integer k, we have $10^k \equiv 1 \pmod 3$.
Hence, we see from (1) that

(2) $s(n) \equiv 1+3+5+...+(2n-1) = n^2 \pmod 3$.

Since

$$(3)\quad <n^2>_3 = \begin{cases} 0, \text{ if } n \equiv 0 \pmod 3; \\ 1, \text{ if } n \equiv 1 \text{ or } 2 \pmod 3, \end{cases}$$

we find from (2) and (3) that

$$(4) \qquad \langle s(n)\rangle_3 = \begin{cases} 0, \text{ if } n \equiv 0 \pmod 3; \\ 1, \text{ if } n \equiv 1 \text{ or } 2 \pmod 3, \end{cases}$$

Thus, by (4), the sequence $\langle S\rangle_3 = \{\langle s(n)_3\rangle\}_{n=1}^{\infty}$ is periodical.
The theorem is proved.

Finally, we pose the following

Question. Is the residue sequence $\langle S\rangle_p$ periodical for every odd prime p?

Reference:

1.H.Marimutha, "Smarandache concatenate type sequences", Bulletin of Pure and Applied Sciences, 16E (1997), No. 2, 225 - 226.

ON A CONJECTURE CONCERNING THE SMARANDACHE FUNCTION

I.Prodanescu
Lahovari College, Rm. Vâlcea, Romania

L.Tutescu
Vladimirescu Military College, Craiova, Romania

Let $S : Z^* \to N$, $S(n)$ is the smallest integer n such that n! is divisibil by m (Smarandache function), for any $m \in Z^*$.
Then the following Diophantine equation

$$S(x) = S(x+1), \text{ where } x > 0,$$

has no solution.

Some remarks:

$S(1) = 0$. Let $a \geq 2$, then $S(a) \neq 0$.

Anytime $S(a) \neq 1$, because $1! = 1 = 0!$ and $1 > 0$.

Lemma.

If $a \geq 2$ and $S(a) = b$, then $(a,b) \neq 1$.

Proof:

Let $a = p_1^{r_1} \ldots p_s^{r_s}$, with all p_i distinct prime numbers, its canonical factor decomposition.

Then $S(a) = \max \left\{ S\left(p_1^{r_1}\right), \ldots, S\left(p_s^{r_s}\right) \right\}$.

But $S\left(p_i^{r_i}\right)$ is a multiple of p_i, $\forall i \in \{1, \ldots, s\}$.

Therefore, $\exists q \in \{p_1, \ldots, p_s\}$ such that q divides S(a), but q divides a, too. Q.E.D.

These results do not solve the Conjecture 2068 proposed by Florentin Smarandache in 1986 (see [1]) and republished by Mike Mudge in 1992 as problem viii, a) (see [2]).

References:

[1] R.Muller, "Smarandache Function Journal", New York, Vol. 1., December 1990, 37.
[2] M.Mudge, "The Smarandache Function" in <Personal Computer Word>, London, July 1992, 420.

Remark:

Professor Lucian Tutescu considered that this conjecture may be extended for $S(\alpha x + \beta) = S(\gamma x + \delta)$ equations,
where $(\alpha x + \beta, \gamma x + \delta) = 1$ and $\alpha, \beta, \gamma, \delta \in Z$.

Erdős Conjecture I.

F. SAIDAK

ABSTRACT. An old conjecture of Paul Erdős [6] states that there exist only 7 integers $A = 4, 7, 15, 21, 45, 75$ and 105 such that the difference $A - 2^B$ is a prime for all B for which it is at least two. It is known that the conjecture is true for all $A < 2^{77}$, as Uchiyama and Yorinaga have verified in 1977 ([21]), and in this short paper I show how it is related to other famous unsolved problems in prime number theory. In order to do this, I formulate the main hypothetical result of this paper - a useful upper bound conjecture (Conjecture 3.), describing one aspect of the distribution of primes in various special forms, paying a brief attention to Fermat, Mersenne, Fibonacci, Lucas and Smarandache sequences, and I debate some side effects of the most surprising results it implies. At the end I also give connections of the questions discussed to other important areas of prime number theory, such as topics from the theory of distribution of primes in denser sequences, and along the way I mention some further conjectures of Erdős that have relevant applications there.

1. Sorbents.

Let me introduce the following notation:

DEFINITION 1. *Let f and g be two functions such that for every real number $k > 1$ there exists an integer constant x_0 such that for all $x > x_0$ we have $f^k(x) > g(x)$, then we will write $g \lll f$. If $f \lll g$ and $g \lll f$ at the same time, we'll say that f and g belong to the same sorbent, S say, and we'll write $f \equiv g$.*

Sometimes we might also write $f = s(g)$, s denoting the sorbent allocating map, or simply $f = g$ in cases when there is no possibility

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of confusion in notation. To sketch the use of sorbents I give a very elementary, but strikingly far reaching application.

Let f and g be two polynomials such that $f \equiv g$, then

$$s(\pi(f)) = \varsigma(f,g).s(\pi(g)), \tag{1}$$

where s is again the sorbent function, while $\varsigma(f,g)$ is either 0 or 1 according to whether f and g have the same irreducibility properties or not. One can write this equivalently as

CONJECTURE 2. *If A and B denote any one variable functions (polynomials in particular), then we have*

$$\rho(C = A.B) \equiv \varsigma(f,g).\rho(A).\rho(B), \tag{2}$$

where $\rho(H)$ is the density function corresponding to function H, and $\varsigma(f,g)$ is here again either 0 or 1 like before, depending only on whether C is trivial or not.

This also covers the case of the problem of occurences of primes of a given special form (A) as values of a given function (B), and happens to be in a close connection to the arithmetical functions theory, linking distribution of primes to things as diverse as odd perfect numbers.

At the same time one also immediately sees its direct relation to the results of the famous Bateman-Horn [**1**] quantitaive form of Schinzel's [**17**] Hypothesis H.

2. The Upper Bound Conjecture.

In order to develop possible implications of the Conjecture 2 into something more precise and useful, it is necessary to recall the basic property of the simple prime density function. The Prime Number Theorem ([**13**]) states that

$$\pi(x) \sim \mathfrak{Li}(x) = \lim_{\epsilon \to 0}\left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^{x}\right)\frac{dt}{\log t} \sim \mathfrak{Ls}(x) = \sum_{n=2}^{x}(\log x)^{-1},$$

the first estimation being due to Gauß, the second one due to Dirichlet. By definition, the local prime density, or equivalently the probability of primality of an integer in a small nbd of x, $\rho_1(x)$ can be recovered from this equation by (see [**11**]) differentiating the corresponding prime distribution function. In general hold

$$\rho_k(x) \equiv D_k'(x) \equiv (\log x)^{-k}, \tag{3}$$

and considering further a generalization to problems concerning the distribution of primes in "sparse" sequences and special forms, noticing

that $D(x) < x$ for all D, implying $0 \leq \rho < 1$, gives us the simple inequation

$$\rho(W = \{n_1, n_1, \cdots, n_k\}) \leq \prod_{i=1}^{k} \rho(n_i), \tag{4}$$

where the equality in (4) occures iff W is a stochastically independent set of integers. This is in turn equivalent to saying that if we let $\mathbf{C}$ be the set of all primality restrictions that are put upon a sequence $S(n)$, or conditions the sequence $S(n)$ must obey, then there exists a function f such that

$$D(S(n)) = \pi(x, S(n)) \sim f(C) \sum_{i=1}^{x} \rho(S(i)) \sim f(C) \int_1^x \rho(S(t))\, dt, \tag{5}$$

where f is *bounded* and depends only on the size and the structure of the condition set C. Furthermore f has a regular, approximable behaviour in all fixed, non-trivial cases, and its global properties can be deduced from arithmetical structure of integers in the particular sequence $S(n)$.

In terms of *sorbents* we in fact explicitely conjecture that $f \ll 1$, and that for a dense set of f we also have[1] $f \equiv 1$. The first, weak assumption gives us that for all sequences $S(n)$ defined in a closed arithmetical manner the corresponding distribution function $D(S)$ satisfies the inequality

$$\pi(x, S) \ll \int_1^{S^{-1}(x)} (\log S^{-1}(t))^{-1}\, dt, \tag{6}$$

S^{-1} denoting the inverse map of S, and it is plausible to conjecture that the "order" inequality in (6) could be replaced by the standard one for all sufficiently large x, and therefore hypothesize that we always have correctness of the following **UB** Conjecture, in the above notation written as

CONJECTURE 3. *For all $S(n)$, and all $x > X_0$ hold*

$$\pi(x, S) < \int_1^{S^{-1}(x)} (\log S^{-1}(t))^{-1}\, dt. \tag{7}$$

This sort of a thing is however justified only by establishing a deeper connection of the result (c.f. [**15**]) to a different "maximal prime density" conjecture of Erdős [**7**].

[1] the situation is a bit more delicate than one could pressume, as Graham demonstrated in [**9**]. His result can also be used to justify the density conjecture mentioned.

3. Mersenne and Fermat Primes.

As far as applications of the UBC are concerned, sparse sequences, such as functions of powers of integers, that are obviously arithmetically closed, are now evidently a very suitable point to start at, for if their local density $\rho^*(x)$ satisfies, say, the inequality

$$\rho^*(x) < e(x^{-1-\epsilon}) = e^{x^{-1-\epsilon}}$$

then

$$\int_2^x \log \rho^*(t)\, dt < \int_2^\infty \log \rho^*(t)\, dt < \int_2^\infty \frac{dt}{t^{1-\epsilon}} < K,$$

for some constant K, implying in conjunction with the Conjecture 3 that $D(S(n)) = \pi(x, S(n))$ will converge, i.e. will satisfy

$$D(S(n)) = 1. \tag{8}$$

This implies, for instance, that denoting F_n the n-th Fermat number, and $\mathbf{F}(x)$ the number of Fermat primes below x, the for all x we have

$$F(x) < 5 + \lim_{N\to\infty} \sum_{n=5}^{N} (\log 2^{2^n})^{-1} < 5 + \frac{1}{16 \log 2} < 5.1,$$

showing that it is rather unlikely that any new Fermat prime will ever be discovered[2].

Now, back to the old conjecture of P. Erdős [7], introduction of the ideas from the beginning of this paper shows that an analogy of a parallel between twins and Goldbach ([5]) can be obtained for the Erdős and Fermat problem here. There is nothing to it, really, by considering a generalization to an arbitrary function $f(x)$, the questions about distribution of primes in "sequences" $f(x)$ and $A - f(x)$ are complementary. Indeed, all we are interested in are integers A such that $f(x)$ is a prime for

$$x = 1, 2, 3, \cdots, f^{-1}(A).$$

Obviously, by the above, as long as $f(x)$ increases to infinity *sufficiently* slowly, the number of wanted As has to be finite. What does sufficiently mean here? Evidently as soon as $\log f = 1$ we should be alright.

This means that, for instance, the number of integers A such that the difference $A - n!$ is a prime for all n for which it is positive is going to be greater than what it was in the similar Erdős problem due to nothing but tendencies of growth of inverses of the functions concerned.

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[2] A conjecture of Selfridge [18].

For Mersenne primes the situation is different. By definition $S(i) = 2^i - 1$, giving by above something like

$$\int_{i=1}^{\infty} \frac{1}{\log S(t)}\, dt \sim \frac{1}{\log 2} \int_{i=1}^{\infty} \frac{1}{\log t}\, dt. \tag{9}$$

Now, as far as the conditions on primality of $2^n - 1$ are concerned, nothing explicit[3] can be said beyond the fact that n must be a prime itself. So, a uniformity property of potential divisors of n that give $S(n)$ prime is expected, and in fact can be show to strongly suggests a direct connection to the simple Eratosthenes sieve result. The only effect of this we care about is that except for the necessity of a factor 2, needed due to exclusion of even exponents n used in the sieve, everything stays unchanged, giving the conjecture $f(\mathbf{C}) = 2$.

Denoting $M(x)$ the number of Mersenne primes below x, one now immediately sees that

$$M(x) = \pi(x, 2^n - 1) \sim 2.\frac{e^{\gamma}}{2} \sum_{p<x} \frac{1}{\log p} \sim \frac{e^{\gamma}.\log\log x}{\log 2}, \tag{10}$$

This last asymptotic relation (10) is known as S. S. Wagstaff's conjecture [**22**], correcting the previous 1964 heuristical result of D. B. Gillies [**8**]. Therefore in sorbent theory notation we may conclude that

$$F(x) = 1,\ M(x) = \log\log x.$$

4. Fibonacci and Lucas Primes.

As far as Fibonacci (F_n^*) and Lucas (L_n) numbers are concerned, we have famous formulas like

$$F_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n, \tag{11}$$

where

$$|\frac{1-\sqrt{5}}{2}|^n < 1^n = 1,$$

so, denoting $[x]$ the integer part function, (11) gives

$$F_n = [c^n] + 1 + (-1)^n,$$

where c is the real number $(1+\sqrt{5})/2$, and an obvious connection to the Mersenne prime case is clearly visible. In fact, the existence of connection based strictly on the fashion of increase of the terms of these two sequences is what the main idea of sorbents, and their quantitative characteristics - Conjecture 3, is all about. Same thing happens for arbitrary Smarandache sequences (S_n) based on properties of digital

[3]this is again just a conjecture we know very little about ...

patterns of integers, although to treat exact behaviour of distribution functions of a particular sequence always needs an additional care. But in general

$$\log M_n = \log F_n^* = \log L_n = \log S_n = n, \tag{12}$$

and due to existence of bounds on conditional divisibility properties of terms of all these cases we also must have:

$$F(x) \ll \log\log x,\ L(x) \ll \log\log x,$$

and we can conjecture that

$$D(F^*(x)) = D(L(x)) = D(S(x)) = M(x) = \log\log N. \tag{13}$$

For certain special cases this could be made more precise through a discussion concerning the corresponding condition sets C, although we'll stay contempt with the illustration of this idea we gave in the case of Mersenne primes distribution.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, K7L 3N6.
E-mail address: saidak@mast.queensu.ca

All Solutions of the Equation S(n) + d(n) = n

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233 USA
e-mail 71603.522@compuserve.com

The number of divisors function d(n), is a classic function of number theory, having been defined centuries ago. In contrast, the Smarandache function S(n), was defined only a few decades ago. The purpose of this paper is to find all solutions to a simple equation involving both functions.

Theorem: The only solutions to the equation

$$S(n) + d(n) = n, \quad n > 0$$

are 1, 8 and 9.

Proof: Since S(1) = 0 and d(1) = 1 we have verified the special case of n = 1.

Furthermore, with S(p) = p for p a prime, it follows that any solution must be composite.

The following results are well-known.

a) $d(p_1^{a1} \ldots p_k^{ak}) = (a1 + 1) \ldots (ak + 1)$
b) $S(p^k) \leq kp$
c) $S(p_1^{a1} \ldots p_k^{ak}) = \max \{ S(p_1^{a1}) \ldots S(p_k^{ak}) \}$

Examining the first few powers of 2.

$S(2^2) = 4$, $d(2^2) = 3$
$S(2^3) = 4$ and $d(2^3) = 4$ which is a solution.
$S(2^4) = 6$, $d(2^4) = 5$

and in general

$$S(2^k) \leq 2k \quad \text{and} \quad d(2^k) = k + 1.$$

It is an easy matter to verify that

$$2k + k + 1 = 3k + 1 < 2^k$$

for $k > 4$.

Examining the first few powers of 3

$S(3^2) = 6$ and $d(3^2) = 3$, which is a solution.
$S(3^3) = 9$, $d(3^3) = 4$

and in general, $S(3^k) \leq 3k$ and $d(3^k) = k + 1$.

It is again an easy matter to verify that

$$3k + k + 1 < 3^k$$

for $k > 3$.

Consider $n = p^k$ where $p > 3$ is prime and $k > 1$. The expression becomes

$$S(p^k) + d(p^k) \leq kp + k + 1 = k(p+1) + 1.$$

Once again, it is easy to verify that this is less than p^k for $p \geq 5$.

Now, assume that $n = p_1^{a1} \ldots p_k^{ak}$, $k > 1$ is the unique prime factorization of n.

Case 1: $n = p_1 p_2$, where $p_2 > p_1$. Then $S(n) = p_2$ and $d(n) = 2 * 2 = 4$. Forming the sum,

$$p_2 + 4$$

we then examine the subcases.

Subcase 1: $p_1 = 2$. The first few cases are

$n = 2 * 3$, $S(n) + d(n) = 7$
$n = 2 * 5$, $S(n) + d(n) = 9$
$n = 2 * 7$, $S(n) + d(n) = 11$
$n = 2 * 11$, $S(n) + d(n) = 15$

and it is easy to verify that $S(n) + d(n) < n$, for p_2 a prime greater than 11.

Subcase 2: $p_1 = 3$. The first few cases are

$n = 3 * 5$, $S(n) + d(n) = 5 + 4$
$n = 3 * 7$, $S(n) + d(n) = 7 + 4$
$n = 3 * 11$, $S(n) + d(n) = 11 + 4$

and it is easy to verify that $S(n) + d(n) < n$ for p_2 a prime greater than 11.

Subcase 3: It is easy to verify that

$$p_2 + 4 < p_1p_2$$

for $p_1 \geq 5, p_2 > p_1$.

Therefore, there are no solutions for $n = p_1p_2$, $p_1 < p_2$.

Case 2: $n = p_1p_2^{a_2}$, where $a_2 > 1$ and $p_1 < p_2$. Then $S(n) \leq a_2p_2$ and $d(n) = 2(a_2 + 1)$.

$S(n) + d(n) \leq a_2p_2 + 2(a_2 + 1) = a_2p_2 + 2a_2 + 2$

We now induct on a_2 to prove the general inequality

$$a_2p_2 + 2a_2 + 2 < p_1p_2^{a_2}$$

Basis step: $a_2 = 2$. The formula becomes

$2p_2 + 4 + 2 = 2p_2 + 6$ on the left and

$p_1p_2p_2$ on the right. Since $p_2 \geq 3$, $2 + \frac{6}{p_2} \leq 4$ and $p_1p_2 \geq 6$. Therefore,

$$2 + \frac{6}{p_2} < p_1p_2$$

and if we multiply everything by p_2, we have

$$2p_2 + 6 < p_1p_2p_2.$$

Inductive step: Assume that the inequality is true for $k \geq 2$

$$kp_2 + 2k + 2 < p_1p_2^k.$$

and examine the case where the exponent is $k + 1$.

$(k + 1)p_2 + 2(k + 1) + 2 = kp_2 + p_2 + 2k + 2 + 2 = (kp_2 + 2k + 2) + p_2 + 2$

$< p_1p_2^k + p_2 + 2$ by the inductive hypothesis.

Since $p_1p_2^k$ when $k \geq 2$ is greater than $p_2 + 2$ is follows that

$$p_1p_2^k + p_2 + 2 < p_1p_2^{k+1}.$$

Therefore, $S(n) + d(n) < n$, where $n = p_1p_2^k$, $k \geq 2$.

Case 3: $n = p_1^{a_1} p_2$, where $a_1 > 1$.

We have two subcases for the value of S(n), depending on the circumstances

Subcase 1: $S(n) \leq a_1 p_1$.

Subcase 2: $S(n) = p_2$.

In all cases, $d(n) = 2(a_1 + 1)$.

Subcase 1: $S(n) + d(n) \leq a_1 p_1 + 2(a_1 + 1) = a_1 p_1 + 2a_1 + 2$.

Using an induction argument very similar to that applied in case 2, it is easy to prove that the inequality

$$a_1 p_1 + 2a_1 + 2 < p_1^{a_1} p_2.$$

is true for all $a_1 \geq 2$.

Subcase 2: $S(n) + d(n) = p_2 + 2(a_1 + 1) = p_2 + 2a_1 + 2$

It is again a simple matter to verify that the inequality

$$p_2 + 2a_1 + 2 < p_1^{a_1} p_2$$

is true for all $a_1 \geq 2$.

Case 4: $n = p_1^{a_1} p_2^{a_2}$, where $p_1 < p_2$ and $a_1, a_2 \geq 2$.

$d(n) = (a_1 + 1)(a_2 + 1)$

Subcase 1: $S(n) \leq a_1 p_1$

$$S(n) + d(n) \leq a_1 p_1 + (a_1 + 1)(a_2 + 1) < p_1^{a_1} + p_1^{a_1}(a_2 + 1) = p_1^{a_1}(a_2 + 2) < p_1^{a_1} p_2^{a_2}$$

Subcase 2: $S(n) \leq a_2 p_2$

$$S(n) + d(n) \leq a_2 p_2 + (a_1 + 1)(a_2 + 1) < p_2^{a_2} + p_2^{a_2}(a_1 + 1) = p_2^{a_2}(a_1 + 2) < p_1^{a_1} p_2^{a_2}$$

Case 5: $n = p_1^{a_1} \ldots p_k^{a_k}$, where $k \geq 2$.

The proof is by induction on k.

Basis step: Completed in the first four cases.

Inductive step: Assume that for $n_1 = p_1^{a_1} \ldots p_k^{a_k}$, $k \geq 2$

$$a_i p_i + (a_1 + 1) \ldots (a_k + 1) < n_1$$

where $S(n_1) \leq a_i p_i$. Which means that

$$S(n_1) + d(n_1) < n_1.$$

Consider $n_2 = p_1^{a_1} \ldots p_k^{a_k} p_{k+1}^{a_{k+1}}$.

Subcase 1: $S(n_2) = S(n_1)$. Since $p_{k+1} \geq 5$, it follows that $(a_{k+1} + 1) < p_{k+1}^{a_{k+1}}$ and we can this in combination with the inductive hypothesis to conclude

$$a_i p_i + (a_1 + 1) \ldots (a_k + 1)(a_{k+1} + 1) < n_1 p_{k+1}^{a_{k+1}},$$

which implies that $S(n_2) + d(n_2) < n_2$.

Subcase 2: $S(n_2) > S(n_1)$, which implies that $S(n_2) \leq a_{k+1} p_{k+1}$. Starting with the inductive hypotheses

$$a_i p_i + (a_1 + 1) \ldots (a_k + 1) < p_1^{a_1} \ldots p_k^{a_k}$$

and multply both sides by $a_{k+1} p_{k+1}$ to obtain the inequality

$$a_i p_i a_{k+1} p_{k+1} + a_{k+1} p_{k+1} (a_1 + 1) \ldots (a_k + 1) < p_1^{a_1} \ldots p_k^{a_k} a_{k+1} p_{k+1}$$

Since $p_{k+1} \geq 5$, it follows that

$$p_1^{a_1} \ldots p_k^{a_k} a_{k+1} p_{k+1} \leq p_1^{a_1} \ldots p_k^{a_k} p_{k+1}^{a_{k+1}}$$

and with $a_{k+1} p_{k+1} > (a_{k+1} + 1)$, we have

$$a_{k+1} p_{k+1} + (a_1 + 1) \ldots (a_k + 1)(a_{k+1} + 1) <$$

$$a_i p_i a_{k+1} p_{k+1} + a_{k+1} p_{k+1} (a_1 + 1) \ldots (a_k + 1).$$

Combining the inequalities, we have

$$a_{k+1} p_{k+1} + (a_1 + 1) \ldots (a_k + 1)(a_{k+1} + 1) < p_1^{a_1} \ldots p_k^{a_k} p_{k+1}^{a_{k+1}}$$

which implies

$$S(n_2) + d(n_2) < n.$$

Therefore, the only solutions to the equation

$$S(n) + d(n) = n$$

are 1, 8 and 9.

An inequality between prime powers dividing $n!$

Florian Luca

For any positive integer $n \geq 1$ and for any prime number p let $e_p(n)$ be the exponent at which the prime p appears in the prime factor decomposition of $n!$. In this note we prove the following:

Theorem.

Let $p < q$ be two prime numbers, and let $n > 1$ be a positive integer such that $pq \mid n$. Then,

$$p^{e_p(n)} > q^{e_q(n)}. \tag{1}$$

Inequality (1) was suggested by Balacenoiu at the First International Conference on Smarandache Notions in Number Theory (see [1]). In fact, in [1], Balacenoiu showed that (1) holds for $p = 2$. In what follows we assume that $p \geq 3$.

We begin with the following lemmas:

Lemma 1.

(i) The function

$$f(x) = \frac{x-1}{\log x} \tag{2}$$

is increasing for $x \geq e$.

(ii) Let $p \geq 3$ be a real number. Then,

$$x > (p-1)\log_p(x) \qquad \text{for } x \geq p. \tag{3}$$

(iii) Let $p \geq 3$ be a real number. The function

$$g_p(x) = \frac{x-2}{x-(p-1)\log_p(x)} \tag{4}$$

is positive and decreasing for $x \geq p(p+2)$.

(iv)

$$\frac{p+2}{p} > \frac{\log(p+4)}{\log p} \qquad \text{for } p > e^2. \tag{5}$$

(v)

$$\frac{p+1}{p} > \frac{\log(p+2)}{\log p} \qquad \text{for } p > e. \tag{6}$$

Proof. *(i)* Notice that

$$\frac{df}{dx} = \frac{1}{\log^2 x} \cdot \left(\log\left(\frac{x}{e}\right) + \left(\frac{1}{x}\right)\right) > 0 \qquad \text{for } x > e.$$

(ii) Suppose that $x \geq p \geq 3$. From *(i)* it follows that

$$\frac{x}{\log x} > \frac{x-1}{\log x} \geq \frac{p-1}{\log p}. \tag{7}$$

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Inequality (7) is clearly equivalent to

$$x > (p-1)\frac{\log x}{\log p} = (p-1)\log_p(x).$$

(*iii*) The fact that $g_p(x) > 0$ for $x \geq p \geq 3$ follows from (*ii*). Suppose that $x \geq p(p+2)$, and that $p \geq 3$. Then,

$$\frac{dg_p}{dx} = \frac{-\log(p)\big((p-1)x\log x - (2\log p + p - 1)x + 2(p-1)\big)}{x\big((p-1)\log x - x\log p\big)^2}. \tag{8}$$

From (8), it follows that in order to check that $dg_p/dx < 0$ it suffices to show that

$$(p-1)x\log x - (2\log p + p - 1)x > 0,$$

or that

$$\log x > \Big(2\frac{\log p}{p-1} + 1\Big) = \Big(\frac{2}{f(p)} + 1\Big). \tag{9}$$

The left hand side of (9) is increasing in x. By (*i*), the right hand side of (9) is decreasing in p. Thus, since $p \geq 3$, and $x \geq p(p+2) \geq 15$, it suffices to show that inequality (9) holds for $x = 15$ and $p = 3$. But this is straightforward.

(*iv*) Inequality (5) is equivalent to

$$p^{p+2} > (p+4)^p,$$

or

$$p^2 > \Big(1 + \frac{4}{p}\Big)^p = \Big[\Big(1 + \frac{4}{p}\Big)^{p/4}\Big]^4. \tag{10}$$

Since

$$e > (1+x)^{1/x} \qquad \text{for all } x > 0, \tag{11}$$

it follows, from inequality (11) with $x = 4/p$, that

$$e > \Big(1 + \frac{4}{p}\Big)^{p/4}. \tag{12}$$

From inequality (12) one can immediately see that (10) holds whenever $p > e^2$.

(*v*) Follows from arguments similar to the ones used at (*iv*).

For every prime number p and every positive integer n let $\tau_p(n)$ be the sum of the digits of n written in the base p.

Lemma 2.

Let $p < q$ be two prime numbers and let n be a positive integer. Assume that $pq \mid n$. Then,

(*i*) $\tau_q(n) \geq 2$.

(*ii*) $\tau_p(n) < (p-1)\log_p(n)$.

Proof. (*i*) Since $n > 0$ it follows that $\tau_q(n) \geq 1$. If $\tau_q(n) = 1$, it follows that n is a power of q which contradicts the fact that $p \mid n$. Hence, $\tau_q(n) \geq 2$.

(*ii*) Let $n = pql$ for some integer $l \geq 1$. Let

$$ql = a_0 + a_1 p + \ldots + a_s p^s,$$

where $0 \leq a_i \leq p-1$ for $1 \leq i \leq s$, and $a_s \neq 0$, be the representation of ql in the base p. Clearly,

$$s = [\log_p(ql)] < \log_p(ql).$$

Since

$$n = pql = a_0 p + a_1 p^2 + ... + a_s p^{s+1},$$

it follows that

$$\tau_p(n) = \sum_{i=0}^{s} a_i \leq (p-1)(s+1) < (p-1)\big(\log_p(ql)+1\big) = (p-1)\log_p(n).$$

The Proof of the Theorem. Suppose that $q > p \geq 3$ are prime numbers, and that $n > 1$ is such that $pq \mid n$. By applying logarithms in (1) it suffices to prove that

$$e_p(n)\log p > e_q(n)\log q. \tag{13}$$

Since

$$e_p(n) = \frac{n-\tau_p(n)}{p-1} \text{ and } e_q(n) = \frac{q-\tau_q(n)}{q-1},$$

it follows that (13) can be rewritten as

$$\frac{n-\tau_p(n)}{p-1} \cdot \log p > \frac{n-\tau_q(n)}{q-1} \cdot \log q,$$

or

$$\frac{(q-1)\log p}{(p-1)\log q} > \frac{n-\tau_q(n)}{n-\tau_p(n)}. \tag{14}$$

We distinguish two cases:

CASE 1. $q = p+2$. We distinguish two subcases:

CASE 1.1. $n = pq$. In this case, since $q = p+2$, and $p \geq 3$, it follows that $\tau_p(n) = \tau_p(p^2+2p) = 3$, and $\tau_q(n) = \tau_q(pq) = p$. Therefore inequality (14) becomes

$$\frac{(p+1)\log p}{(p-1)\log(p+2)} > \frac{p^2+2p-p}{p^2+2p-3} = \frac{p(p+1)}{p^2+2p-3}. \tag{15}$$

Inequality (15) is equivalent to

$$\frac{p^2+2p-3}{p(p-1)} > \frac{\log(p+2)}{\log p}. \tag{16}$$

By lemma 1 (v) we conclude that in order to prove inequality (16) it suffices to show that

$$\frac{p^2+2p-3}{p(p-1)} \geq \frac{p+1}{p}. \tag{17}$$

But (17) is equivalent to

$$\frac{p^2+2p-3}{p-1} \geq p+1, \tag{18}$$

or $p^2+2p-3 \geq p^2-1$, or $p \geq 1$ which is certainly true. This disposes of Case 1.1.

CASE 1.2. $n = pql$ where $l \geq 2$. In this case $n \geq 2p(p+2) > 2p^2$. By lemma 2 (i) and (ii), it follows that

$$\frac{n-2}{n-(p-1)\log_p(n)} > \frac{n-\tau_q(n)}{n-\tau_p(n)}. \tag{19}$$

Thus, in order to prove (14) it suffices to show that

$$\frac{(p+1)\log p}{(p-1)\log(p+2)} > \frac{n-2}{n-(p-1)\log_p(n)} = g_p(n). \tag{20}$$

Since $n > 2p^2 > p(p+2)$, and since $g_p(n)$ is decreasing for $n > p(p+2)$ (thanks to lemma 1 (iii)), it follows that in order to prove (20) it suffices to show that

$$\frac{(p+1)\log p}{(p-1)\log(p+2)} > g_p(2p^2) = \frac{2p^2-2}{2p^2-\log_p(2p^2)}. \tag{21}$$

Since $p \geq 3 > 2^{3/2}$, it follows that $p^{2/3} > 2$. Hence,

$$\log_p(2p^2) < \log_p(p^{2/3}p^2) = \frac{8}{3}.$$

We conclude that in order to prove (21) it suffices to show that

$$\frac{(p+1)\log p}{(p-1)\log(p+2)} > \frac{2p^2-2}{2p^2-\frac{8}{3}} = \frac{3(p-1)(p+1)}{3p^2-4}. \tag{22}$$

Inequality (22) is equivalent to

$$\frac{3p^2-4}{3(p-1)^2} > \frac{\log(p+2)}{\log p}. \tag{23}$$

Using inequality (6), it follows that in order to prove (23) it suffices to show that

$$\frac{3p^2-4}{3(p-1)^2} > \frac{p+1}{p}. \tag{24}$$

Notice now that (24) is equivalent to

$$3p^3-4p > 3(p-1)^2(p+1) = 3p^3-3p^2-3p+3,$$

or $3p^2 > p+3$ which is certainly true for $p \geq 3$. This disposes of Case 1.2.

CASE 2. $q \geq p+4$. Using inequality (19) it follows that in order to prove inequality (14) it suffices to show that

$$f(q)\cdot\frac{\log p}{p-1} = \frac{(q-1)\log p}{(p-1)\log q} > \frac{n-2}{n-(p-1)\log_p(n)} = g_p(n). \tag{25}$$

Since $f(q)$ is increasing for $q \geq 3$ (thanks to lemma 1 (i)), and since $g_p(n)$ is decreasing for $n \geq pq \geq p(p+4) > p(p+2)$, it follows that in order to prove (25)

it suffices to show that inequality (25) holds for $q = p + 4$, and $n = pq = p(p + 4)$. Hence, we have to show that

$$\frac{(p+3)\log p}{(p-1)\log(p+4)} > \frac{p^2+4p-2}{p^2+4p-(p-1)\log_p(p(p+4))}. \tag{26}$$

Inequality (26) is equivalent to

$$\frac{(p+3)}{(p-1)\log(p+4)} > \frac{p^2+4p-2}{(p^2+3p+1)\log p-(p-1)\log(p+4)},$$

or

$$\frac{(p+3)(p^2+3p+1)}{(p-1)(p^2+4p-2)+(p-1)(p+3)} > \frac{\log(p+4)}{\log p},$$

or

$$\frac{p^3+6p^2+10p+3}{p^3+4p^2-4p-1} > \frac{\log(p+4)}{\log p}. \tag{27}$$

One can easily check that (27) is true for $p = 3, 5, 7$. Suppose now that $p \geq 11 > e^2$. By lemma 1 (iv), it follows that in order to prove (27) it suffices to show that

$$\frac{p^3+6p^2+10p+3}{p^3+4p^2-4p-1} > \frac{p+2}{p}. \tag{28}$$

Notice that (28) is equivalent to

$$p^4+6p^3+10p^2+3p > (p+2)(p^3+4p^2-4p-1) = p^4+6p^3+4p^2-9p-2,$$

or $6p^2 + 11p + 2 > 0$, which is obvious. This disposes of the last case.

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Florian Luca
Department of Mathematics
Syracuse University
215 Carnegie Hall
Syracuse, New York
e-mail: fgluca@syr.edu

AN INEQUALITY CONCERNING THE SMARANDACHE FUNCTION

Maohua Le
Zhanjiang Normal College, Zhanjang, Guangdong, P.R.China

Abstract. For any positive integer n, let S(n) denote the Smarandache function of n. In this paper we prove that $S(mn) \le S(m)+S(n)$.

Let N be the set of all positive integers. For any positive integer n, let S(n) denote the Smarandache function of n. By [2], we have

(1) $S(n)= \min\{k \mid k\in N,\ n \mid k!\}$.

Recently, Jozsef[1] proved that

(2) $S(mn)\le mS(n)$, $m,n \in N$.

In this paper we give a considerable improvement for the upper bound (2). We prove the following result.

Theorem. For any positive integers m,n, we have

$$S(mn)\le S(m)+S(n).$$

Proof. Let a=S(m) and b=S(n). Then we have

(3) $$n \mid b!,$$

by (1). Let x be a positive integer with $x \ge a$, and let

(4) $$\binom{x}{a} = \frac{x(x-1)\cdots(x-a+1)}{a!}$$

be a binomial coeficient. It is a well known fact that $\binom{x}{a}$ is a positive integer. So we have

(5) $$a! \mid x(x-1)\cdots(x-a+1),$$

by (4). Further, since $m \mid a!$, we get from (5) that

(6) $$m \mid x(x-1)\cdots(x-a+1),$$

for any positive integer x with $x \ge a$. Put x=a+b. We see from (3) and (6) that

(7) $$mn \mid b!(b+1)\cdots(b+a)=(a+b)!.$$

Thus we get from (7) that $S(mn) \leq a+b=S(m)+S(n)$. The theorem is proved.

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THE SMARANDACHE FUNCTION AND THE DIOPHANTINE EQUATION $X!+a = y^2$

Maohua Le
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract. For any positive integer n, let S(n) denote the Smarandache function of n. In this paper we prove that if a is a nonsquare positive integer, then all positive integer solutions (x,y) of the equation $x!+a=y^2$ satisfy $x< 2S(a)$.

Let N be the set of all positive integers. For any positive integer n, let S(n) denote the Smarandache function of n. Let a be a fixed positive integer. Recently, Dabrowschi[1] proved that if a is not a square, then the equation

(1) $$x!+a = y^2,\ x,y \in N$$

has only finitely many solutions (x,y). In this paper we give an upper bound for the solutions of (1) as follows.

Theorem. If a not a square, then all solutions (x,y) of (1) satisfy $x<2S(a)$.

Proof. Since a is not a square, a has a prime factor p such that

(2) $$p^{2r+1} \mid a,$$

were r is a nonnegative integer. We now suppose that (x,y) is a solution of (1) with $x\geq 2S(a)$. By the result of [2], we have $S(mn)\leq S(m)+S(n)$ for any positive integers m, n. It implies that $2S(a)\geq S(a^2)$. So have

(3) $$a^2 \mid x!.$$

Therefore, we see from (1) and (3) that

(4) $$a \mid y^2.$$

Further, by (2) and (4), we get

(5) $$ap \mid y^2$$

Since p is a prime factor of a, we see from (3) that

(6) $a_p \mid x!$.

Thus, by (1), (5) and (6), we obtain $p \mid 1$, a contradiction. So we have $x<2S(a)$. The theorem is proved.

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ON SMARANDACHE CONCATENATED SEQUENCES I: PRIME POWER SEQUENCES

Maohua Le
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract. Let $A=\{p^n\}_{n=1}^{\infty}$, where p is a prime. Let $C(A)=\{c_n\}$ denote the Smarandache concatenated sequence of A. In this paper we prove that if $n>1$ and $p\neq 2$ or 5, then c_n does not belong to A.

Let $A=\{a_n\}_{n=1}^{\infty}$ be an infinite increasing sequence of positive integers. For any positive integer n, let c_n be the decimal integer such that

$$(1)\qquad c_n=\overline{a_1\,a_2\,\ldots a_n}\ .$$

Then sequence $C(A)=\{c_n\}_{n=1}^{\infty}$ is called the Smarandache concatenated sequence of A. In [1], Marimutha posed a generalquestions as follows:

Question. How many terms of C(A) belong to A?

In this serial paper, we shall consider some intersting cases for the above question. In this part we prove the following result.

Theorem. Let $A=\{p^n\}_{n=1}^{\infty}$,where p is a prime. If $n>1$ and $p\neq 2$ or 5, then c_n does not belong to A.

Proof. For any pisitive integer a, let d(a) denote the figure number of a in the decimal system.

If $A=\{p^n\}_{n=1}^{\infty}$, then from(1) we get

$$2)\quad c_n=p^n+p^{n-1}*10^{d(p^{n-1})}+\ldots+p^2*10^{d(p^2)+\ldots+d(p^3)}+p*10^{d(p)+\ldots+d(p^2)}$$

Further, if c_n belongs to A, then we have

$$(3)\qquad c_n=p^m\ ,$$

where m is a positive integer with $m\geq n$. It implies that

$$(4)\qquad p^2\mid c_n,$$

if $n>1$. Hoever, if $p\neq 2$ 0r 5, then $p\nmid 10^s$ for any positive

integer k . Therefore,by (2), we get

(5) $$p^2 \nmid c_n,$$

wich contradicts (4). Thus, c_n does not belong to A in this case. The theorem is proved.

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ON SMARANDACHE CONCATENETED SEQUENCES I: FACTORIAL SEQUENCE

Maohua Le
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract. let $A=\{n!\}_{n=1}^{\infty}$, and let $C(A)=\{c_n\}_{n=1}^{\infty}$ denote the Smarandache concatenated sequence of A. In this paper we prove that if n>1, then c_n does not belong to A.

Let $A=\{n!\}_{n=1}^{\infty}$, and let $C(A)=\{c_n\}_{n=1}^{\infty}$ denote the Smarandache concatenated sequence of A . In this part we prove the following result.

Theorem. If n>1, then c_n does not belong to A.

Proof. By the definition of the Smarandache concatenated sequence of A (see[1]), we have

$$(1) \qquad c_n = \overline{1!2!\ldots n!}$$

if n>1 and c_n belongs to A, then

$$(2) \qquad c_n = m!,$$

where m is a positive integer with m>n>1. Notice that c_n =12, 126, 12624, 12624120, 12624120720, 126241207205040 and 1262412072050404040320 for n=2,3,4,5,6,7 and 8, which are none factorial. We may assume that n≥9. Then we have m>9.

For any positive integer a, let d(a) denote the figure number of a in the decimal system. Since n≥9, we see from (1) that

$$(3)\ c_n = n!+(n-1)!10^{d(n!)}+\ldots+9!10^{d(n!)+\ldots+d(10!)}$$

$$+1262412072050404040320*10^{d(n!)+\ldots+d(10!)+d(9!)}.$$

Since $3^2 \mid 1262412072050404040320$ and $3^3 \mid k!$ for k≥9, we get from (3) that

$$(4) \qquad 3^2 \mid c_n, \quad n\geq 9.$$

Hoever, since m>n≥9, we obtain frpm (2) that $3^3 \mid c_n$, which contradicts (4). Thus, if n>1, then c_n does not belong to A. The Theorem is proved.

Reference

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ON THE INTERSECTED SMARANDACHE PRODUCT SEQUENCES

Maohua Le
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract. In this paper we discuss a question concerning the intersected Smarandache product sequences.

Let $U=\{U_n\}_{n=1}^{\infty}$ be an infinite increasing sequence of positive integers. For any positive integer n, let

$$(1) \qquad s_n = 1+u_1 u_2 \ldots u_n .$$

Then the the sequence $S(U)=\{s_n\}_{n=1}^{\infty}$ is called the Smarandache product sequence of U (see[1]). Further, if there exist infinitely many terms in U belonging to S(U), then S(U), is called intersected. In this paper we pose the following question:

Question. Wich of ordinary Smarandache product sequences are intersected?

We nou give some obvious examples as follows:

Example 1. If $U=\{n\}_{n=1}^{\infty}$, then S(U) is intersected. In this case, we see from (1) that $s_n = u_{n!+1}$ for any positive integer n.

Exemple 2. Let k be a positive integer with k>1. If $U=\{kn\}_{n=1}^{\infty}$, then S(U) is non-intersected, since $k \nmid s_n$ for any positive integer n.

Exemple 3. Let k be a positive integer with k>1. If $U=\{n^k\}_{n=1}^{\infty}$, then S(U) is non-intersected. In this case, we have $s_n = 1+1^k 2^k \ldots n^k = 1+(n!)^k$, which is not a k-th power.

Example 4. If $U=\{n!\}_{n=1}^{\infty}$, then S(U) is non-intersected. In this case, we have $s_n = 1+1!2!\ldots n!$, which is an odd integer if n>1. It implies that $u_n \in S(U)$ if and only if n=2.

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PRIMES IN THE SMARANDACHE SQUARE PRODUCT SEQUENCE

Maohua Le
Zhanjiang Normal College, Zhanjiang, Guangdong, P.R.China

Abstract. For any positive integer n, let a_n be the n-th square number, and let $s_n = 1+a_1 a_2 \ldots a_n$. In this paper we prove that if n>2, 2|n and 2n+1 is a prime, then s_n is not a prime.

For any positive integer n, let a_n be the n-th square number, and let $s_n = 1+a_1 a_2 \ldots a_n$. Then the sequence $S=\{s_n\}_{n=1}^{\infty}$ is called the Smarandache square product sequence. In[2], Iacobescu asked the following question.

Question. How many terms in S are primes?

In this paper we prove the following result:

Theorem. If n>2, 2|n and 2n+1 is a prime, then s_n is not a prime.

Proof. By the definition of s_n, we have

$$(1)\quad s_n = 1+a_1 a_2 \ldots a_n = 1+(n!)^2 .$$

Let p=2n+1. It is a well known fact that if 2|n and p is p is a prime, then we have

$$(2)\quad (n!)^2 \equiv -1 \pmod{p},$$

(see[1,p.88]). Therefore, by (1) and (2), we get

$$(3)\quad p|s_n .$$

Further, if n>2, then $s_n = 1+(n!)^2 > 2n+1 = p$. Thus, by (3), s_n is not a prime. The theorem is proved.

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On a characterization of the uniform repartition

Vasile Seleacu

An important role in the theory of the hi-square criterion is played by the following fact: *if* $x_1, x_2, \ldots, x_n$ are independent random variables with Gauss distribution $N(0, \delta^2)$, then the distribution of the central statistic hi-square $\chi^2 = \sum_{i=1}^{n} (x_i + a_i)^2$ depends on a_1, a_2, ..., a_n only by mean of the parameter $\sum_{i=1}^{n} a_i^2$. In the paper [1] one proves that this property is characteristic for the normal distribution of probability. The aim of this paper is to give a characterization of the uniform distribution of probability by mean of the hi-square statistic.

Theorem 1 *Let* x_1, x_2, ..., x_n *independent and equally distributed random variables, where* $n \geq 2$, *then the necessary and sufficient condition that the statistic distribution* $\chi^2 = \sum_{i=1}^{n} (x_i + a_i)^2$ *depend on* $\sum_{i=1}^{n} a_i^2$ *with* $a_i \in \mathbb{R}$ *is that* x_i *be uniformly distributed.*

Proof. We define the function:

$$\psi(a) = Ee^{-(x_i+a)^2}. \tag{1}$$

It is obvious that $\psi(a) > 0$ and ψ is derivable in every $a \in \mathbb{R}$.

Using the conditions of the theorem we have

$$Ee^{-\sum_{i=1}^{n}(x_i+a_i)^2} = \prod_{i=1}^{n} Ee^{-(x_i+a_i)^2} = \prod_{i=1}^{n} \psi(a_i) = \Phi\left(\sum_{i=1}^{n} a_i\right). \tag{2}$$

Let $h(a) = \log \psi(a)$ and $H(a) = \log \Phi(a)$. From (2) we obtain:

$$\sum_{i=1}^{n} h(a_i) = H\left(\sum_{i=1}^{n} a_i\right). \tag{3}$$

If we differentiate twice the both sides of (3) by a_1, then by a_2, we obtain for every $a_1, a_2, \ldots, a_n$:

$$H''\left(\sum_{i=1}^{n} a_i\right) = 0. \tag{4}$$

In this way

$$H(a) = C_1 a + C_2 . \tag{5}$$

From (1) and (3) we obtain:

$$\psi(a) = \int e^{-(x+a)^2} dF(x) = e^{C_1 a + C_3} \tag{6}$$

where $F(x) = P(x_i < x)$.

In the following step we consider the substitution:

$$e^{-x^2} dF(x) = dG. \tag{7}$$

In this case (6) can be written in the form:

$$\int e^{-2ax} dG(x) = e^{C_4 a + C_5}. \tag{8}$$

It follows, using the uniqueness theorem for the Laplace transformation, that $dG = C_5 \Delta(x - C_6)$ for every C_5 and C_6, where Δ is the Dirac function. Using again relation (7), it follows that F is the distribution function of the uniform random variable.

The sufficiency can be proved by a straightforward verification.

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Current address 13, "A.I.Cuza" st., Craiova, Romania

A NOTE ON THE SMARANDACHE PRIME PRODUCT SEQUENCE

A. A. K. MAJUMDAR

Department of Mathematics, Jahangirnagar University, Savar, Dhaka 1342, Bangladesh

ABSTRACT

This paper gives some properties of the Smarandache prime product sequence, (P_n), definded by

$$P_n = 1+p_1 p_2...p_n, n \geq 1,$$

where (p_n) is the sequence of primes in their natural order.

AMS (1991) Subject Classification: 11A41, 11A51.

1. INTRODUCTION

Let $(p_n) = (p_1, p_2,...)$ be the (infinite) sequence of primes in their natural numbers. The first few terms of the sequence are as follows:

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, p_8 = 19, p_9 = 23, p_{10} = 29.$$

Clearly, the sequence (p_n) is strictly increasing (in $n \geq 1$) with $p_n > p_1 p_2$ for all $n \geq 4$. Furthermore, $p_n > n$ for all $n \geq 1$.

The Smarandache prime product sequence, (p_n), is defined by (Smarandache [5])

$$P_n = 1+p_1 p_2...p_n, n \geq 1. \quad (1.1)$$

We note that the sequence (P_n) is strictly increasing (in $n \geq 1$), satisfying the following recursion formulas:

$$P_{n+1} = P_n+p_1 p_2... p_n(p_{n+1}-1), n \geq 1, \quad (1.2)$$

$$P_{n+1} = P_n p_{n+1}-(p_{n+1}-1), n \geq 1. \quad (1.3)$$

We also note that P_n is an odd (positive) integer for all $n \geq 1$; furthermore,

$$P_1 = 3, P_2 = 7, P_3 = 31, P_4 = 211, P_5 = 2311$$

are all primes, while the next five elements of the sequence (P_n) are all composites, since

$P_6 = 30031 = 59 \times 509,$

$P_7 = 510511 = 19 \times 97 \times 277,$

$P_8 = 9699691 = 347 \times 27953,$

$P_9 = 223092871 = 317 \times 703760,$

$P_{10} = 6469693231 = 331 \times 571 \times 34231.$

Some of the properties of the sequence (P_n) have been studied by Prakash [3], who conjectures that this sequence contains an infinite number of primes.

This note gives some properties of the sequence (P_n), some of which strengthens the corresponding result of Prakash [3]. This is done in §2 below, and show that for each $n \geq 1$, P_n is relatively prime to P_{n+1}. We conclude this paper with some remarks in the final §3.

2. MAIN RESULTS

We start with the following result which has been established by Majumdar [2] by induction on n (≥ 6), using the recurrence relationship (1.3).

Lemma 2.1: $P_n < (p_{n+1})^{n-2}$ for all $n \geq 6$.

Exploiting Lemma 2.1, Majumdar [2] has proved the following theorem which strengthens the corresponding result of Prakash [3].

Theorem 2.1: For each $n \geq 6$, P_n has at most n-3 prime factors (counting multiplicities).

Another property satisfied by the sequence (P_n) is given in Theorem 2.2. To prove the theorem, we would need the following results.

Lemma 2.2: For each $n \geq 1$, P_n is of the form $4k+3$ for some integer $k \geq 0$.

Proof: Since P_n is odd for all $n \geq 1$, it must be of the form $4k+1$ or $4k+3$ (see, for example, Shanks [4], pp. 4). But, P_n cannot be of the form $4k+1$, otherwise, from (1.1), we would have $p_1 p_2 \ldots p_n = 4k$,

that is, $4 | p_1 p_2 \ldots p_n$, which is absurd. Hence, P_n must be of the form $4k+3$. □

Lemma 2.3: (1) The product of two integers of the form $4k+1$ is an integer of the form $4k+1$, and in general, for any integer m > 0, $(4k+1)^m$ is again of the form $4k+1$,

(2) The product of two integers of the form $4k+3$ is an integer of the form $4k+1$, and the product of two integers, one of the form $4k+1$ and the other of the form $4k+3$, is integer of the form $4k+3$,

(3) For any integer $m > 0$, $(4k+3)^m$ is of the form $4k+1$ or $4k+3$ respectively according as m is even or odd.

Proof: Part (1) has been proved by Bolker ([1], Lemma 5.2, pp. 6). The proof of the remaining parts is similar. □

We now prove the following theorem.

Theorem 2.2: For each $n \geq 1$, P_n is never a square or higher power of any natural number (> 1).

Proof: If possible, let $P_n = N^2$ for some integer $N > 1$.

Now, since P_n is odd, N must be odd, and hence, N must be of the form $4k+1$ or $4k+3$ for some integer $k \geq 0$. But, in either case, by Lemma 2.3, $N^2 = P_n$ is of the form $4k+1$, contradicting Lemma 2.2. Hence, P_n cannot be a square of a natural number (> 1).

To prove the remaining part, let $P_n = N^l$ for some integers $N > 1$, $l \geq 3$. (*)

Without loss of generality, we may assume that l is a prime (if l is a composite number, let $l = rs$ where r is prime, and so $p_n = (N^s)^r$; setting $M = N^s$, we may proceed with this M in

place of N). By Theorem 2.1, $1 < n$, and hence, 1 must be one of the primes $p_2, p_3, \ldots, p_n$. By Fermat's Little Theorem (Bolker [1], Theorem 9.8, pp. 16),

$$p_1 p_2 \ldots p_n = N^1 - 1 \equiv N - 1 \equiv 0 \pmod 1.$$

Thus, $N = 1m+1$ for some integer $m > 0$,

and we get $p_1 p_2 \ldots p_n = (1m)^1 + \binom{1}{1}(1m)^{1-1} + \ldots + \binom{1}{1-1}(1m)$.

But the above expression shows that $1^2 | 1 p_1 p_2 \ldots p_n$, which is impossible.

Hence, the representation of P_n in the form $(*)$ is not possible, which we intend to prove. □

Some more properties related to the sequence (P_n) are given in the following two lemmas. Lemma 2.4: For each $n \geq 1$, $(P_n, P_{n+1}) = 1$.

Proof: Any prime factor p of P_{n+1} satisfies the inequality $p > p_{n+1}$.

Now, if $p | P_n$, then from (1.3), we see that $p | (p_{n+1} - 1)$, which is absurd. Hence, all the prime factors of P_{n+1} are different from each of the prime factors of P_n, which proves the lemma. □

Lemma 2.5: For each $n \geq 1$, P_n and P_{n+2} have at most one prime factor in common.

Proof: Since $P_{n+2} - P_n = p_1 p_2 \ldots p_n (p_{n+1} p_{n+2} - 1)$,

any prime factor common to both P_n and P_{n+2} must divide $p_{n+1} p_{n+2} - 1$. Now, any prime factor of P_{n+2} is greater than p_{n+2}. Hence, it follows that P_n and p_{n+2} can have at most one prime factor in common, since otherwise, the product of the prime factors is greater than $(p_{n+2})^2$, which cannot divide $p_{n+1} p_{n+2} - 1 < (p_{n+2})^2$. □

From the proof of the above lemma we see that, if all the prime factors of $p_{n+1} p_{n+2} - 1$ are less than p_{n+2}, then $(P_n, P_{n+2}) = 1$. And generalizing the lemma, we have the following result: For any $n \geq 1$, and $i \geq 1$, P_n and P_{n+i} can have at most $i-1$ number of prime factors in common.

3. SOME REMARKS

We conclude this paper with the following remarks.

(1) The sequence (P_n) is well known, it is used in elementary texts on the Theory of Numbers (see, for example, Bolker [1]and Shanks [4] to prove the infinititude of the primes. Some of the properties of the sequence (P_n) have been studied by Prakash [3]. Theorem 2.1 improves one of the results of Prakash [3], while our proof of Theorem 2.2 is much simpler than that followed by Prakash [3]. The expressions for P_6, P_7, P_8, P_9 and P_{10} show that Theorem 2.1 is satisfied with tighter bounds, but we could not improve it further.

(2) By Lemma 2.3 we see that, of all the prime factors of P_n (which is at most n-3 in number for $n \geq 6$, by Theorem 2.1), an odd number of these must be of the form $4k+3$. In this connection, we note that, in case of P_6, one of the prime factors (namely, 59) is of the form $4k+3$, while the other is of the form $4k+1$; and in case of P_7, all the three prime factors are of the form $4k+3$.

(3) The Conjecture that the sequence (P_n) contains infinitely many primes, still remains an open problem.

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Smarandache Lucky Math

by C. Ashbacher
C. Ashbacher Technologies
Hiawatha, Box 294
IA 52233, USA
E-mail: 71603.522@compuserve.com

The Smarandache Lucky Method/Algorithm/Operation/etc. is said to be any incorrect method or algorithm or operation etc. which leads to a correct result. The wrong calculation should be fun, somehow similarly to the students' common mistakes, or to produce confusions or paradoxes.

Can someone give an example of a Smarandache Lucky Derivation, or Integration, or Solution to a Differential Equation?

Reference:

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Problems

Edited by

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
119 Northwood Drive
Hiawatha, IA 52233 USA
71603.522@compuserve.com

Welcome to the latest installment of the problems section! Our goal as always is to present interesting and challenging problems in all areas and at all levels of difficulty with the only limits being good taste. Readers are encouraged to submit new problems and solutions to the editor at one of the addresses given above. All solvers will be acknowledged in a future issue. Please submit a solution along with your proposals if you have one. If there is no solution and the editor deems it appropriate, that problem may appear in the companion column of unsolved problems. Feel free to submit computer related problems and use computers in your work. Programs can also be submitted as part of the solution. While the editor is fluent in several programming languages, be cautious when submitting programs as solutions. Wading through several pages of an obtuse program to determine if the submitter has done it right is not the editors idea of a good time. Make sure you explain things in detail.

If no solution is currently available, the program will be flagged with an asterisk*. The deadline for submission of solutions will generally be six months after the date appearing on that issue. Regardless of deadline, no problem is ever officially closed in the sense that new insights or approaches are always welcome. If you submit problems or solutions and wish to guarantee a reply, please include a self-addressed stamped envelope or postcard with appropriate postage attached. Suggestions for improvement or modification are also welcome at any time. All proposals in this offering are by the editor.

Definition: Given any positive integer n, the value of the Smarandache function S(n) is the smallest integer m such that n divides m!.

Definition: Given any positive integer $n \geq 1$, the value of the Pseudo-Smarandache function Z(n) is the smallest integer m such that n divides $\sum_{k=1}^{m} k$. Note that this is equivalent to n divides $\frac{m(m+1)}{2}$.

New Problems

16. Prove that there are an infinite number of integers n such that S(n) = Z(n).

17. Prove that if n is an even perfect number, then S(n) and Z(n) are equal and prime.

18. The Smarandache Square-Partial-Digital Subsequence(SPDS) is the set of square numbers that can be partitioned into a set of square integers. For example, 101 = 1 | 0 | 1 and 1449169 = 144 | 9 | 169 are in SPDS. Widmer[1] closes his paper with the comment, "It is relatively easy to find two consecutive squares in SPDS. One example is $12^2 = 144$ and $13^2 = 169$. Does SPDS also contain a sequence of three or more consecutive integers?"

Find a sequence of three consecutive squares in SPDS.

19. Prove that if $k > 0$, then

$$Z(2^k * 3) = \begin{cases} 2^{k+1} - 1 & \text{if k is odd} \\ 2^{k+1} & \text{if k is even} \end{cases}$$

20. Prove that if $k > 0$, then $Z(2^k * 5) =$

a) 2^{k+2} if k is congruent to 0 modulo 4
b) 2^{k+1} if k is congruent to 1 modulo 4
c) $2^{k+2} - 1$ if k is congruent to 2 modulo 4
d) $2^{k+1} - 1$ if k is congruent to 3 modulo 4.

21. a) Prove that

$$S(Z(n)) - Z(S(n))$$

is positive infinitely often.

b) Prove that

$$S(Z(n)) - Z(S(n))$$

is negative infinitely often.

22. It is clear that if p is an odd prime,

$$Z(S(n)) = Z(n)$$

since S(p) = p. Prove that there are an infinite number of composite numbers that also satisfy the equation.

Reference

1. Lamarr Widmer, 'Construction of Elements of the Smarandache Square-Partial-Digital Subsequence', **Smarandache Notions Journal**, Vol. 8, No. 1-2-3, Fall, 1997.

Problem 23 (by Sabin Tabirca, England)

Prove the following equation $(\forall\, n>1)\ \sum_{i=1,(i,n)=1}^{n} i = \frac{n\cdot\varphi(n)}{2}$.

Proof

This proof is made based on the *Inclusion & Exclusion* principle.

Let $D_p = \{i = 1,2,\ldots,n \mid p|n\}$ be the set which contains the multiples of p.

This set satisfies

$$D_p = p\cdot\left\{1,2,\ldots,\frac{n}{p}\right\} \text{ and } \sum_{i\in D_p} i = p\cdot\sum_{i=1}^{\frac{n}{p}} i = p\cdot\frac{\frac{n}{p}\cdot\left(\frac{n}{p}+1\right)}{2} = \frac{n}{2}\cdot\left(\frac{n}{p}+1\right).$$

Let $n = p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots p_s^{k_s}$ be the prime number decomposition of n.

The following intersection of sets

$$D_{p_{j_1}}\cap D_{p_{j_2}}\cap\ldots\cap D_{p_{j_m}} = \{i = 1, 2,\ldots, n \mid p_{j_1}|n \wedge p_{j_2}|n\wedge\ldots\wedge p_{j_m}|n\}$$

is evaluated as follows

$$D_{p_{j_1}}\cap D_{p_{j_2}}\cap\ldots\cap D_{p_{j_m}} = \{i = 1, 2,\ldots, n \mid p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}|n\} = D_{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}$$

Therefore, the equation

$$\sum_{i\in D_{p_{j_1}}\cap D_{p_{j_2}}\cap\ldots\cap D_{p_{j_m}}} i = \sum_{i\in D_{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}} i = \frac{n}{2}\cdot\left(\frac{n}{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}+1\right) \qquad (1)$$

holds.

The *Inclusion & Exclusion* principle is applied based on

$$D = \{i = 1, 2,\ldots, n \mid (i,n) = 1\} = \{1, 2,\ldots, n\} - \bigcup_{j=1}^{s} D_{p_k}$$

and it gives

$$\sum_{i\le n,(i,n)=1} i = \sum_{i=1}^{n} i - \sum_{m=1}^{s}(-1)^{m-1}\cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \sum_{i\in D_{p_{j_1}}\cap D_{p_{j_2}}\cap\ldots\cap D_{p_{j_m}}} i \qquad (2)$$

Applying (1), the equation (2) becomes

$$\sum_{i\le n,(i,n)=1} i = \sum_{i=1}^{n} i + \sum_{m=1}^{s}(-1)^{m}\cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \frac{n}{2}\cdot\left(\frac{n}{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}+1\right). \qquad (3)$$

The right side of the equation (3) is simplified by reordering the terms as follows

$$\sum_{i\le n,(i,n)=1} i = \frac{n^2}{2}\cdot\left(1+\sum_{m=1}^{s}(-1)^{m}\cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \frac{1}{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}\right)+\frac{n}{2}\cdot\left(1+\sum_{m=1}^{s}(-1)^{m}\cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} 1\right)$$

$$\sum_{i\le n,(i,n)=1} i = \frac{n^2}{2}\cdot\prod_{m=1}^{s}\left(1-\frac{1}{p_{j_m}}\right)+\frac{n}{2}\cdot\left(1+\sum_{m=1}^{s}(-1)^{m}\cdot\binom{n}{m}\right)=\frac{n^2}{2}\cdot\prod_{m=1}^{s}\left(1-\frac{1}{p_{j_m}}\right)=\frac{n}{2}\cdot\varphi(n).$$

Therefore, the equation (14) holds.♣

Remark

Obviously, the equation does not hold for n=1 because $\sum_{i=1,(i,1)=1}^{1} i = 1$ and $\frac{n\cdot\varphi(n)}{2}=\frac{1}{2}$.

Problem 24 (by Sabin Tabirca, England)

Prove that there is no a magic square made with the numbers $S(1)$, $S(2)$,..., $S(n^2)$ where $n\in\{2, 3, 4, 5, 7, 8, 10\}$.

Proof

Let n be a number in the set $\{2, 3, 4, 5, 7, 8, 10\}$.

Let us suppose that there is a magic $x=\left(x_{i,j}\right)_{i,j=\overline{1,n}}$ square made with the number $S(1)$, $S(2)$,..., $S(n^2)$.

In this case, the following equations are true:

$$\left(\forall\, i=\overline{1,n}\right)\sum_{j=1}^{n} x_{i,j} = C \qquad (1)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j} = \sum_{i=1}^{n^2} S(i) = n\cdot C \qquad (2)$$

Therefore, the sum of the numbers $S(1)$, $S(2)$,..., $S(n^2)$ is divisible by n.

Let us denote $SS(n)=\sum_{i=1}^{n^2} S(i)$. In the cases $n\in\{2, 3, 4, 5, 7, 8, 10\}$, we have:

$n=2 \Rightarrow SS(2)=9$ is not divisible by 2.

$n=3 \Rightarrow SS(3)=34$ is not divisible by 3.

$n=4 \Rightarrow SS(4)=85$ is not divisible by 4.

$n=5 \Rightarrow SS(5)=187$ is not divisible by 5.

$n=7 \Rightarrow SS(7)=602$ is not divisible by 7.

$n=10 \Rightarrow SS(10)=2012$ is not divisible by 10.

A contradiction has been found for each case. Therefore, there is no a magic square with the elements $S(1), S(2),..., S(n^2)$.

Problem 25 (by Jose Castillo, Arizona)

The following number, which has 155 digits,

828180797877776...1110987654321

has been proved (Stephan [1]) with a computer to be a prime number called Smarandache Reverse Prime and it belongs to the sequence:

1,21,321,4321,54321,... .

What is the sum of the digits of this number?

Solution:

Write the number per groups:

```
                digit sum
      828180 ----------> 8*3+2+1+0              = 27
7978...727170 ----------> 7*10+(9+8+...+2+1+0) = 70+45
6968...626160 ----------> 6*10+(9+8+...+2+1+0) = 60+45
5958...525150 ----------> 5*10+(9+8+...+2+1+0) = 50+45
-------------------------------------------------
1918...121110 ----------> 1*10+(9+8+...+2+1+0) = 10+45
  98...    21 ----------> 0*10+(9+8+...+2+1+0) = 0+45
-------------------------------------------------
```

Total = 27+(70+60+50+...+10)+45*8 = 27+280+360 = 667

References:

[1] Stephan, Ralf W., "Factors and Primes in two Smarandache Sequences", URL: http://rws.home.pages.de, E-mail address: stephan@tmt.de .

[2] Sloane, N.J.A., "Enciclopedia of Integer Sequences", online, 1995-1998.

Solutions to Vol. 7, 1-2-3 Problems

1. The Euler phi function $\phi(n)$ is defined as the number of positive integers not exceeding n that are relatively prime to n.

a) Prove that there are no solutions to the equation

$$\phi(S(n)) = n$$

Proof: It is well-known that $S(n) \leq n$ and $\phi(n) < n$ for all $n > 0$.

b) Prove that there are no solutions to the equation

$$S(\phi(n)) = n$$

Proof: Use the same reasoning as in part (a).

c) Prove that there are an infinite number of solutions to the equation

$$n - \phi(S(n)) = 1$$

Proof: It is well-known that if p is an odd prime, $S(p) = p$ and $\phi(p) = p - 1$. Since there are an infinite number of odd primes, the result follows.

d) Prove that for every odd prime p, there is a number n such that

$$n - \phi(S(n)) = p + 1$$

Proof: It is well-known that if p is an odd prime, then $S(2p) = p$ and if p is an odd prime, $\phi(p) = p - 1$. Therefore,

$$\phi(S(2p)) = p - 1.$$

The result follows.

2) This problem was proposed in **Canadian Mathematical Bulletin** by P. Erdös and was listed as unsolved in the book **Index to Mathematical Problems 1980-1984,** edited by Stanley Rabinowitz and published by MathPro Press.

Prove that for infinitely many n

$$\phi(n) < \phi(n - \phi(n)).$$

Proof: It is easily verified that

$$\phi(30) = \phi(2)*\phi(3)*\phi(5) = 1*2*4 = 8 \text{ and}$$

$$\phi(30 - 8) = \phi(22) = \phi(2)*\phi(11) = 1*10 = 10$$

Now multiply 30 by any power of 2, 2^k. It is easy to verify using the well-known formula for the computation of the phi function

If $n = p_1^{a1} \ldots p_k^{ak}$ is the prime factorization of n, then

$$\phi(n) = n(1 - \tfrac{1}{p_1}) \ldots (1 - \tfrac{1}{p_k})$$

that

$$\phi(30 * 2^k) = 8 * 2^k \text{ and } \phi(30 *2^k - 8*2^k) = 10 * 2^k.$$

which creates the infinite set.

3) The following appeared as unsolved problem(21) in **Unsolved Problems Related to Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Are there m, n, k non-null positive integers, $m, n \neq 1$ for which

$$S(mn) = m^k * S(n)?$$

Find a solution.

Solution: m = n = 2 and k = 1 is a solution.

4) The following appeared as unsolved problem(22) in **Unsolved Problems Related to Smarandache Function**, edited by R. Muller and published by Number Theory Publishing Company.

Is it possible to find two distinct numbers k and n such that

$$\log_{(k^n)} S(n^k)$$

is an integer?

Find two integers n and k that satisfy these conditions.

Solution: For k = n = 2.

$$\log_{(2^2)} S(2^2) = \log_4 S(4) = \log_4 4 = 1$$

5) Solve the following doubly true Russian alphametic

ДВА	2
ДВА	2
ТРИ	3
-------	--
СЕМЬ	7

Solution:

There are many solutions, one is

```
  572
  572
  690
 -----
 1834
```

Solution:

There are many solutions, one is

```
  572
  572
  690
 -----
 1834
```

Unsolved Problems

Edited by

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233 USA
e-mail 71603.522@compuserve.com

Welcome to another installment of the unsolved problems column! In this section, problems are presented where the solution is either unknown or incomplete. This is meant to be an interactive endeavor, so input from readers is strongly encouraged. Always feel free to contact the editor at any of the addresses given above. It is hoped that we can work together to advance the flow of mathematics in some small way. There will be no deadlines here, and even if a problem is completely solved, new insights or more elegant proofs are always welcome. All correspondents who are the first to resolve any issue appearing here will have their efforts acknowledged in a future issue.

Definition of the Smarandache function, S(n).

S(n) = m where m is the smallest integer such that n divides m!.

Definition of the Pseudo-Smarandache function, Z(n).

Z(n) = m, where m is the smallest number such that n divides $\sum_{k=1}^{m} k$.

It is easy to verify that the expression

S(Z(n)) - Z(S(n))

is positive and negative an infinite number of times. It is also occasionally zero. A computer program was created to check the percentages. When run for $1 \leq n \leq 10{,}000$, the numbers were

Positive	4,744
Negative	5,227
Zero	29

This precentage was fairly constant for runs with smaller upper limits. Which leads to the question

Unsolved Question: What are the percentages of numbers for which the expression

$$S(Z(n)) - Z(S(n))$$

is positive, negative and zero?

It is possible to create polynomials with the variables the values of the Smarandache function. For example, the polynomial

$$S(n)^2 + S(n) = n$$

is such an expression. A computer search for all $n \geq 10{,}000$ yielded 23 values of n for which the expression is true.

A computer search for all values of $n \leq 10{,}000$ for which the expression

$$S(n)^2 + S(n) = 2n$$

is true yielded 33 solutions.

A computer search for all values of $n \leq 10{,}000$ for which the expression

$$S(n)^2 + S(n) = 3n$$

is true yielded 20 solutions.

A computer search for all values of $n \leq 10{,}000$ for which the expression

$$S(n)^2 + S(n) = 4n$$

is true yielded 24 solutions.

A computer search for all values of $n \leq 10{,}000$ for which the expression

$$S(n)^2 + S(n) = 5n$$

is true yielded 11 solutions.

A computer search for all values of $n \leq 10{,}000$ for which the expression

$$S(n)^2 + S(n) = 6n$$

is true yielded 26 solutions.

Unsolved Question: Is the number of solutions to each of the expressions above finite or infinite?

Unsolved Question: Is there a number k such that there is no number n for which

$$S(n)^2 + S(n) = kn?$$

Unsolved Question: Is there a largest number k for which there is some number n that satisfies the expression

$$S(n)^2 + S(n) = kn?$$

Unsolved Question: In examining the number of solutions for the runs for k = 1, 2, 3, 4, 5 and 6, it appears that there are more solutions when k is even than when k is odd. Is this true in general?

A computer search was performed for the expression

$$S(n)^3 + S(n)^2 + S(n) = n$$

for all $n \leq 10{,}000$ and no solutions were found.

Unsolved Question: What is the largest value of k such that there is a solution to the expression

$$S(n)^k + S(n)^{k-1} + \ldots + S(n) = n?$$

A computer search for solutions for all $n \leq 10{,}000$ was performed for the expression

$$S(n)^3 + S(n)^2 + S(n) = kn$$

for k=2, 3, 4, 5, and 6 and no solutions were found. However, two solutions were found for k=7.

Another computer search for all $n \leq 10{,}000$ for the expression

$$S(n)^4 + S(n)^3 + S(n)^2 + S(n) = kn$$

for k = 1, 2, 3, 4, 5, 6 and 7. One solution was found for k = 5.

Unsolved Question: Is there a largest value of m for which there are no values of n and k for which

$$S(n)^m + S(n)^{m-1} + \ldots + S(n) = kn?$$

There are several classic functions of number theory, and it is in some sense natural to examine problems with the Smarandache and Pseudo-Smarandache functions combined with the classic functions.

Definition: For $n \geq 1$, the divisors function d(n) is the number of integers m, where $1 \leq m \leq n$, such that m evenly divides n.

Unsolved Question: How many solutions are there to the equation

$$Z(n) = d(n)?$$

A computer search up through $n = 10{,}000$ yielded only the solutions n = 1, 3 and 10.

Unsolved Question: How many solutions are there to the equation

$$Z(n) + d(n) = n?$$

A computer search up through n = 10,000 yielded only the solution n = 56, as d(56) = 8 and Z(56) = 48.

Unsolved Question: How many solutions are there to the equation

$$S(n) = d(n)?$$

A computer search up through n = 10,000 yielded 12 solutions, 10 of which were less than 5,000 and the last two were n = 5,000 and n = 8750. Given the obvious thinning of the solutions as n gets larger, it may be that there are very few solutions.

Definition: For $n \geq 1$, the Euler phi function $\phi(n)$ is the number of integers k, $1 \leq k \leq n$ that are relatively prime to n.

Using the Euler phi function, we can create an additional problem.

Unsolved Problem: How many solutions are there to the expression

$$S(n) + d(n) + \phi(n) = n?$$

A computer search for all n up through 10,000 yielded only the trivial solutions n =1.

C O N T E N T S