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CONTENTS

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On the Geometry of the General Solution for the Vacuum Field of the Point-Mass

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The black hole, which arises solely from an incorrect analysis of the Hilbert solution, is based upon a misunderstanding of the significance of the coordinate radius r. This quantity is neither a coordinate nor a radius in the gravitational field and cannot of itself be used directly to determine features of the field from its metric. The appropriate quantities on the metric for the gravitational field are the proper radius and the curvature radius, both of which are functions of r. The variable r is actually a Euclidean parameter which is mapped to non-Euclidean quantities describing the gravitational field, namely, the proper radius and the curvature radius.

1 Introduction

The variable r has given rise to much confusion. In the conventional analysis, based upon the Hilbert metric, which is almost invariably and incorrectly called the "Schwarzschild" solution, r is taken both as a coordinate and a radius in the spacetime manifold of the point-mass. In my previous papers [1, 2] on the general solution for the vacuum field, I proved that r is neither a radius nor a coordinate in the gravitational field (M_g, g_g), as Stavroulakis [3, 4, 5] has also noted. In the context of (M_g, g_g) r is a Euclidean parameter in the flat spacetime manifold (M_s, g_s) of Special Relativity. Insofar as the point-mass is concerned, r specifies positions on the real number line, the radial line in (M_s, g_s), not in the spacetime manifold of the gravitational field, (M_g, g_g). The gravitational field gives rise to a mapping of the *distance* $D = |r - r_0|$ between two *points* $r,\ r_0 \in \Re$ into (M_g, g_g). Thus, r becomes a *parameter* for the spacetime manifold associated with the gravitational field. If $R_p \in (M_g, g_g)$ is the proper radius, then the gravitational field gives rise to a mapping ψ,

$$\psi : |r - r_0| \in (\Re - \Re^-) \rightarrow R_p \in (M_g, g_g)\,, \qquad (A)$$

where $0 \leqslant R_p < \infty$ in the gravitational field, on account of R_p being a *distance* from the point-mass located at the point $R_p(r_0) \equiv 0$.

The mapping ψ must be obtained from the geometrical properties of the metric tensor of the solution to the vacuum field. The r-parameter location of the point-mass does not have to be at $r_0 = 0$. The point-mass can be located at any point $r_0 \in \Re$. A test particle can be located at any point $r \in \Re$. The point-mass and the test particle are located at the end points of an interval along the *real line* through r_0 and r. The distance between these points is $D = |r - r_0|$. In (M_s, g_s), r_0 and r may be thought of as describing 2-spheres about an origin $r_c = 0$, but only the distance between these 2-spheres enters into consideration. Therefore, if two test particles are located, one at any point on the 2-sphere $r_0 \neq 0$ and one at a point on the 2-sphere $r \neq r_0$ on the radial line through r_0 and r, the distance between them is the length of the radial interval between the 2-spheres, $D = |r - r_0|$. Consequently, the domain of both r_0 and r is the real number line. In this sense, (M_s, g_s) may be thought of as a parameter space for (M_g, g_g), because ψ maps the *Euclidean distance* $D = |r - r_0| \in (M_s, g_s)$ into the *non-Euclidean proper distance* $R_p \in (M_g, g_g)$: the radial line in (M_s, g_s) is precisely the real number line. Therefore, the required mapping is appropriately written as,

$$\psi : |r - r_0| \in (M_s, g_s) \rightarrow R_p \in (M_g, g_g)\,. \qquad (B)$$

In the pseudo-Euclidean (M_s, g_s) the polar coordinates are $r,\ \theta, \varphi$, but in the pseudo-Riemannian manifold (M_g, g_g) of the point-mass and point-charge, r is *not* the radial coordinate. Conventionally there is the persistent misconception that what are polar coordinates in Minkowski space must also be polar coordinates in Einstein space. This however, does not follow in any rigorous way. In (M_g, g_g) the variable r is nothing more than a real-valued parameter, of no physical significance, for the true radial quantities in (M_g, g_g). The parameter r *never* enters into (M_g, g_g) directly. Only in Minkowski space does r have a direct physical meaning, as mapping (B) indicates, where it is a radial coordinate. Henceforth, when I refer to the radial coordinate or r-parameter I always mean $r \in (M_s, g_s)$.

The solution for the gravitational field of the simple configurations of matter and charge requires the determination of the mapping ψ. The orthodox analysis has completely failed to understand this and has consequently failed to solve the problem.

The conventional analysis simply looks at the Hilbert metric and makes the following unjustified assumptions, tacitly or otherwise;

(a) *The variable r is a radius and/or coordinate of some kind in the gravitational field.*

(b) *The regions $0 < r < 2m$ and $2m < r < \infty$ are both valid.*

(c) *A singularity in the gravitational field must occur only where the Riemann tensor scalar curvature invariant (Kretschmann scalar) $f = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded.*

The orthodox analysis has never proved these assumptions, but nonetheless simply takes them as given, finds for itself a curvature singularity at $r = 0$ in terms of f, and with legerdemain reaches it by means of an *ad hoc* extension in the ludicrous Kruskal-Szekeres formulation. However, the standard assumptions are incorrect, which I shall demonstrate with the required mathematical rigour.

Contrary to the usual practise, one *cannot* talk about extensions into the region $0 < r < 2m$ or division into R and T regions until it has been rigorously established that the said regions are valid to begin with. One *cannot* treat the r-parameter as a radius or coordinate of any sort in the gravitational field without first demonstrating that it is such. Similarly, one *cannot* claim that the scalar curvature must be unbounded at a singularity in the gravitational field until it has been demonstrated that this is truly required by Einstein's theory. Mere *assumption* is *not* permissible.

2 The basic geometry of the simple point-mass

The usual metric g_s of the spacetime manifold (M_s, g_s) of Special Relativity is,

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \quad (1)$$

The foregoing metric can be statically generalised for the simple (i. e. non-rotating) point-mass as follows,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \quad (2a)$$

$$A, B, C > 0 ,$$

where A, B, C are analytic functions. I emphatically remark that *the geometric relations between the components of the metric tensor of (2a) are precisely the same as those of (1).*

The standard analysis writes (2a) as,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \quad (2b)$$

and claims it the most general, which is incorrect. The form of $C(r)$ cannot be pre-empted, and must in fact be rigorously determined from the general solution to (2a). The physical features of (M_g, g_g) must be determined exclusively by means of the resulting $g_{\mu\mu} \in (M_g, g_g)$, *not* by foisting upon (M_g, g_g) the interpretation of elements of (M_s, g_s) in the misguided fashion of the orthodox relativists who, having written (2b), incorrectly treat r in (M_g, g_g) precisely as the r in (M_s, g_s).

With respect to (2a) I identify the coordinate radius, the r-parameter, the radius of curvature, and the proper radius as follows:

(a) The coordinate radius is $D = |r - r_0|$.

(b) The r-parameter is the variable r.

(c) The radius of curvature is $R_c = \sqrt{C(r)}$.

(d) The proper radius is $R_p = \int \sqrt{B(r)}dr$.

The orthodox motivation to equation (2b) is to evidently obtain the circumference χ of a great circle, $\chi \in (M_g, g_g)$ as,

$$\chi = 2\pi r ,$$

to satisfy its unproven assumptions about r. But this equation is only formally the same as the equation of a circle in the Euclidean plane, because in (M_g, g_g) it describes a non-Euclidean great circle and therefore does not have the same meaning as the equation for the ordinary circle in the Euclidean plane. The orthodox assumptions distort the fact that r is only a real parameter in the gravitational field and therefore that (2b) is not a general, but a particular expression, in which case the form of $C(r)$ has been fixed to $C(r) = r^2$. Thus, the solution to (2b) can only produce a particular solution, not a general solution in terms of $C(r)$, for the gravitational field. Coupled with its invalid assumptions, the orthodox relativists obtain the Hilbert solution, a correct *particular form* for the metric tensor of the gravitational field, but interpret it incorrectly with such a great thoroughness that it defies rational belief.

Obviously, the spatial component of (1) describes a sphere of radius r, centred at the point $r_0 = 0$. On this metric $r \geqslant r_0$ is usually assumed. Now in (1) the distance D between two points on a radial line is given by,

$$D = |r_2 - r_1| = r_2 - r_1 . \quad (3)$$

Furthermore, owing to the "origin" being usually fixed at $r_1 = r_0 = 0$, there is no distinction between D and r. Hence r is both a coordinate *and* a radius (distance). However, the correct description of points by the spatial part of (1) must still be given in terms of *distance*. Any point in any direction is specified by its *distance* from the "origin". It is this distance which is the important quantity, not the coordinate. It is simply the case that on (1), in the usual sense, the distance and the coordinate are identical. Nonetheless, the distance from the designated "origin" is still the important quantity, not the coordinate. It is therefore clear that the designation of an origin is arbitrary and one can select *any* $r_0 \in \Re$ as the origin of coordinates. Thus, (1) is a special case of a general expression in which the origin of coordinates is arbitrary and the distance from the origin to another point does not take the same value as the coordinate designating it. The "origin" $r_0 = 0$ has *no intrinsic meaning*. The relativists and the mathematicians have evidently failed to understand this elementary geometrical fact. Consequently, they have managed to attribute to $r_0 = 0$ miraculous qualities of which it is not worthy, one of which is the formation of the black hole.

Equations (1) and (2a) are not sufficiently general and so their forms suppress their true geometrical characteristics. Consider two points P_1 and P_2 on a radial line in Euclidean 3-space. With the usual Cartesian coordinates let P_1 and P_2 have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. The distance between these points is,

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \\ = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2 + |z_1 - z_2|^2} \geqslant 0\,. \tag{4}$$

If $x_1 = y_1 = z_1 = 0$, D is usually called a radius and so written $D \equiv r$. However, one may take P_1 or P_2 as an origin for a sphere of radius D as given in (4). Clearly, a general description of 3-space must rightly take this feature into account. Therefore, the most general line-element for the gravitational field in quasi-Cartesian coordinates is,

$$ds^2 = F dt^2 - G\left(dx^2 + dy^2 + dz^2\right) - \\ - H\left(|x - x_0|dx + |y - y_0|dy + |z - z_0|dz\right)^2 , \tag{5}$$

where $F, G, H > 0$ are functions of

$$D = \sqrt{|x - x_0|^2 + |y - y_0|^2 + |z - z_0|^2} = |r - r_0|\,,$$

and $P_0\,(x_0, y_0, z_0)$ is an arbitrary origin of coordinates for a sphere of radius D centred on P_0.

Transforming to spherical-polar coordinates, equation (5) becomes,

$$ds^2 = -H|r - r_0|^2 dr^2 + F dt^2 - \\ - G\left(dr^2 + |r - r_0|^2 d\theta^2 + |r - r_0|^2 \sin^2\theta d\varphi^2\right) = \\ = A(D)dt^2 - B(D)dr^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{6}$$

where $A, B, C > 0$ are functions of $D = |r - r_0|$. Equation (6) is just equation (2a), but equation (2a) has suppressed the significance of distance and the arbitrary origin and is therefore invariably taken with $D \equiv r \geqslant 0$, $r_0 = 0$.

In view of (6) the most general expression for (1) for a sphere of radius $D = |r - r_0|$, centred at some $r_0 \in \Re$, is therefore,

$$ds^2 = dt^2 - dr^2 - \left(r - r_0\right)^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) = \tag{7a}$$

$$= dt^2 - \frac{\left(r - r_0\right)^2}{|r - r_0|^2} dr^2 - \left|r - r_0\right|^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) = \tag{7b}$$

$$= dt^2 - \left(d|r - r_0|\right)^2 - \left|r - r_0\right|^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{7c}$$

The spatial part of (7) describes a sphere of radius $D = |r - r_0|$, centred at the arbitrary point r_0 and reaching to some point $r \in \Re$. Indeed, the curvature radius R_c of (7) is,

$$R_c = \sqrt{\left(r - r_0\right)^2} = \left|r - r_0\right|\,, \tag{8}$$

and the circumference χ of a great circle centred at r_0 and reaching to r is,

$$\chi = 2\pi\left|r - r_0\right|\,. \tag{9}$$

The proper radius (distance) R_p from r_0 to r on (7) is,

$$R_p = \int_0^{|r - r_0|} d\left|r - r_0\right| = \int_{r_0}^{r} \left[\frac{r - r_0}{\left|r - r_0\right|}\right] dr = \left|r - r_0\right|\,. \tag{10}$$

Thus $R_p \equiv R_c \equiv D$ on (7), owing to its pseudo-Euclidean nature.

It is evident by similar calculation that $r \equiv R_c \equiv R_p$ in (1). Indeed, (1) is obtained from (7) when $r_0 = 0$ and $r \geqslant r_0$ (although the absolute value is suppressed in (1) and (7a)). The geometrical relations between the components of the metric tensor are *inviolable*. Therefore, in the case of (1), the following obtain,

$$\begin{gathered} D = |r| = r\,, \\ R_c = \sqrt{|r|^2} = \sqrt{r^2} = r\,, \\ \chi = 2\pi|r| = 2\pi r\,, \\ R_p = \int_0^{|r|} d\,|r| = \int_0^{r} dr = r\,. \end{gathered} \tag{11}$$

However, equation (1) hides the true arbitrary nature of the origin r_0. Therefore, the correct geometrical relations have gone unrecognized by the orthodox analysis. I note, for instance, that G. Szekeres [6], in his well-known paper of 1960, considered the line-element,

$$ds^2 = dr^2 + r^2 d\omega^2\,, \tag{12}$$

and proposed the transformation $\overline{r} = r - 2m$, to allegedly carry (12) into,

$$ds^2 = d\overline{r}^2 + \left(\overline{r} - 2m\right)^2 d\omega^2\,. \tag{13}$$

The transformation to (13) by $\overline{r} = r - 2m$ is incorrect: by it Szekeres should have obtained,

$$ds^2 = d\overline{r}^2 + \left(\overline{r} + 2m\right)^2 d\omega^2\,. \tag{14}$$

If one sets $r = \overline{r} - 2m$, then (13) obtains from (12). Szekeres then claims on (13),

> "*Here we have an apparent singularity on the sphere $\overline{r} = 2m$, due to a spreading out of the origin over a sphere of radius $2m$. Since the exterior region $\overline{r} > 2m$ represents the whole of Euclidean space (except the origin), the interior $\overline{r} < 2m$ is entirely disconnected from it and represents a distinct manifold.*"

His claims about (13) are completely false. He has made an incorrect assumption about the origin. His equation (12) describes a sphere of radius r centred at $r = 0$, being identical to the spatial component of (1). His equation (13) is precisely the spatial component of equation (7) with $r_0 = 2m$ and $r \geqslant r_0$, and therefore actually describes a sphere of radius $D = \overline{r} - 2m$ centred at $\overline{r}_0 = 2m$. His claim that $\overline{r} = 2m$ describes a sphere is due to his invalid assumption that $\overline{r} = 0$ has some intrinsic meaning. It did not come from his transformation. The claim is false. Consequently there is no interior region at all and no distinct manifold anywhere. All Szekeres did unwittingly was to move the origin for a sphere from the coordinate value $r_0 = 0$ to the coordinate value $r_0 = 2m$. In fact, he effectively repeated the same error committed by Hilbert [8] in 1916, an error, which in one guise or another, has been repeated relentlessly by the orthodox theorists.

It is now plain that r is neither a radius nor a coordinate in the metric (6), but instead gives rise to a *parameterization* of the relevant radii R_c and R_p on (6).

Consider (7) and introduce a test particle at each of the *points* r_0 and r. Let the particle located at r_0 acquire mass. The coordinates r_0 and r do not change, however in the gravitational field (M_g, g_g) the distance between the point-mass and the test particle, and the radius of curvature of a great circle, centred at r_0 and reaching to r in the parameter space (M_s, g_s), will no longer be given by (11).

The solution of (6) for the vacuum field of a point-mass will yield a mapping of the Euclidean *distance* $D = |r - r_0|$ into a non-Euclidean proper radius $R_P(r)$ in the pseudo-Riemannian manifold (M_g, g_g), locally generated by the presence of matter at the r-parameter $r_0 \in (M_s, g_s)$, i. e. at the invariant point $R_p(r_0) \equiv 0$ in (M_g, g_g).

Transform (6) by setting,

$$R_c = \sqrt{C(D(r))} = \frac{\chi}{2\pi}\,, \qquad (15)$$

$$D = |r - r_0|\,.$$

Then (6) becomes,

$$ds^2 = A^*(R_c)dt^2 - B^*(R_c)dR_c^2 - R_c^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \qquad (16)$$

In the usual way one obtains the solution to (16) as,

$$ds^2 = \left(\frac{R_c - \alpha}{R_c}\right)dt^2 - \left(\frac{R_c}{R_c - \alpha}\right)dR_c^2 - R_c^2(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

$$\alpha = 2m\,,$$

which by using (15) becomes,

$$ds^2 = \left(\frac{\sqrt{C} - \alpha}{\sqrt{C}}\right)dt^2 - \left(\frac{\sqrt{C}}{\sqrt{C} - \alpha}\right)\frac{C'^2}{4C}\left[\frac{r - r_0}{|r - r_0|}\right]^2 dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

that is,

$$ds^2 = \left(\frac{\sqrt{C} - \alpha}{\sqrt{C}}\right)dt^2 - \left(\frac{\sqrt{C}}{\sqrt{C} - \alpha}\right)\frac{C'^2}{4C}dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,, \qquad (17)$$

which is the line-element derived by Abrams [7] by a different method. Alternatively one could set $r = R_c$ in (6), as Hilbert in his work [8] effectively did, to obtain the familiar Droste/Weyl/(Hilbert) line-element,

$$ds^2 = \left(\frac{r - \alpha}{r}\right)dt^2 - \left(\frac{r}{r - \alpha}\right)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \qquad (18)$$

and then noting, as did J. Droste [9] and A. Eddington [10], that r^2 can be replaced by a general analytic function of r without destroying the spherical symmetry of (18). Let that function be $C(D(r))$, $D = |r - r_0|$, and so equation (17) is again obtained. Equation (18) taken literally is an *incomplete* particular solution since the boundary on the r-parameter has not yet been rigorously established, but equation (17) provides a way by which the form of $C(D(r))$ might be determined to obtain a means by which all particular solutions, in terms of an infinite sequence, may be constructed, according to the general prescription of Eddington. Clearly, the correct form of $C(D(r))$ must naturally yield the Droste/Weyl/(Hilbert) solution, as well as the true Schwarzschild solution [11], and the Brillouin solution [12], amongst the infinitude of particular solutions that the field equations admit. (Fiziev [13] has also shown that there exists an infinite number of solutions for the point-mass and that the Hilbert black hole is not consistent with general relativity.)

In the gravitational field only the circumference χ of a great circle is a measurable quantity, from which R_c and R_p

are calculated. To obtain the metric for the field in terms of χ, use (15) in (17) to yield,

$$ds^2 = \left(1 - \frac{2\pi\alpha}{\chi}\right) dt^2 - \left(1 - \frac{2\pi\alpha}{\chi}\right)^{-1} \frac{d\chi^2}{4\pi^2} - \\ - \frac{\chi^2}{4\pi^2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad (19)$$

$$\alpha = 2m\,.$$

Equation (19) is independent of the r-parameter entirely. Since only χ is a measurable quantity in the gravitational field, (19) constitutes the correct solution for the gravitational field of the simple point-mass. In this way (19) is truely the *only* solution to Einstein's field equations for the simple point-mass.

The only assumptions about r that I make are that the point-mass is to be located somewhere, and that somewhere is r_0 in parameter space (M_s, g_s), the value of which must be obtained rigorously from the geometry of equation (17), and that a test particle is located at some $r \neq r_0$ in parameter space, where $r,\ r_0 \in \Re$.

The geometrical relationships between the components of the metric tensor of (1) must be precisely the same in (6), (17), (18), and (19). Therefore, the circumference χ of a great circle on (17) is given by,

$$\chi = 2\pi\sqrt{C(D(r))}\,,$$

and the proper distance (proper radius) $R_p(r)$ on (6) is,

$$R_p(r) = \int \sqrt{B(D(r))}dr\,.$$

Taking $B(D(r))$ from (17) gives,

$$R_p(D) = \int \sqrt{\frac{\sqrt{C}}{\sqrt{C}-\alpha}}\,\frac{C'}{2\sqrt{C}}\,dr = \\ = \sqrt{\sqrt{C(D)}\left(\sqrt{C(D)}-\alpha\right)} + \\ + \alpha\ln\left|\frac{\sqrt{\sqrt{C(D)}}+\sqrt{\sqrt{C(D)}-\alpha}}{K}\right|, \qquad (20)$$

$$D = |r - r_0|\,,$$

$$K = \text{const.}$$

The relationship between r and R_p is,

$$as\ r \to r_0^{\pm},\ R_p(r) \to 0^+\,,$$

or equivalently,

$$as\ D \to 0^+,\ R_p(r) \to 0^+\,,$$

where r_0 is the parameter space location of the point-mass. Clearly $0 \leqslant R_p < \infty$ always and the point-mass is invariantly located at $R_p(r_0) \equiv 0$ in (M_g, g_g), a manifold with boundary.

From (20),

$$R_p(r_0) \equiv 0 = \sqrt{\sqrt{C(r_0)}\left(\sqrt{C(r_0)}-\alpha\right)} + \\ + \alpha\ln\left|\frac{\sqrt{\sqrt{C(r_0)}}+\sqrt{\sqrt{C(r_0)}-\alpha}}{K}\right|,$$

and so,

$$\sqrt{C(r_0)} \equiv \alpha,\ K = \sqrt{\alpha}.$$

Therefore (20) becomes

$$R_p(r) = \sqrt{\sqrt{C(|r-r_0|)}\left(\sqrt{C(|r-r_0|)}-\alpha\right)} + \\ + \alpha\ln\left|\frac{\sqrt{\sqrt{C(|r-r_0|)}}+\sqrt{\sqrt{C(|r-r_0|)}-\alpha}}{\sqrt{\alpha}}\right|, \qquad (21)$$

$$r, r_0 \in \Re\,,$$

and consequently for (19),

$$2\pi\alpha < \chi < \infty\,.$$

Equation (21) is the required mapping. One can see that r_0 cannot be determined: in other words, r_0 is entirely arbitrary. One also notes that (17) is consequently singular only when $r = r_0$ in which case $g_{00} = 0$, $\sqrt{C_n(r_0)} \equiv \alpha$, and $R_p(r_0) \equiv 0$. There is no value of r that makes $g_{11} = 0$. One therefore sees that the condition for singularity in the gravitational field is $g_{00} = 0$; indeed $g_{00}(r_0) \equiv 0$.

Clearly, contrary to the orthodox claims, r does not determine the geometry of the gravitational field directly. It is not a radius in the gravitational field. The quantity $R_p(r)$ is the non-Euclidean radial coordinate in the pseudo-Riemannian manifold of the gravitational field around the *point* $R_p = 0$, which corresponds to the parameter *point* r_0.

Now in addition to the established fact that, in the case of the simple (i .e . non-rotating) point-mass, the lower bound on the radius of curvature $\sqrt{C(D(r_0))} \equiv \alpha$, $C(D(r))$ must also satisfy the no matter condition so that when $\alpha = 0$, $C(D(r))$ must reduce to,

$$C(D(r)) \equiv \left|r - r_0\right|^2 = (r - r_0)^2\,; \qquad (22)$$

and it must also satisfy the far-field condition (spatially asympotically flat),

$$\lim_{r\to\pm\infty}\frac{C(D(r))}{(r - r_0)^2} \to 1. \qquad (23)$$

When $r_0 = 0$ equation (22) reduces to,

$$C(|r|) \equiv r^2 ,$$

and equation (23) reduces to,

$$\lim_{r\to\pm\infty} \frac{C(|r|)}{r^2} \to 1.$$

Furthermore, $C(r)$ must be a strictly monotonically increasing function of r to satisfy (15) and (21), and $C'(r) \neq 0$ $\forall\ r \neq r_0$ to satisfy (17) from (2a). The only general form for $C(D(r))$ satisfying all the required conditions (the Metric Conditions of Abrams [7]), from which an infinite sequence of particular solutions can be obtained [1] is,

$$C_n(D(r)) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}} , \tag{24}$$

$$n \in \Re^+,\ \ r \in \Re,\ r_0 \in \Re ,$$

where n and r_0 are arbitrary. Then clearly, when $\alpha = 0$, equations (7) are recovered from equation (17) with (24), and when $r_0 = 0$ and $\alpha = 0$, equation (1) is recovered.

According to (24), when $r_0 = 0$ and $r \geqslant r_0$, and n is taken in integers, the following infinite sequence of particular solutions obtains,

$C_1(r) = (r + \alpha)^2$ (Brillouin's solution [12])
$C_2(r) = r^2 + \alpha^2$
$C_3(r) = (r^3 + \alpha^3)^{\frac{2}{3}}$ (Schwarzschild's solution [11])
$C_4(r) = (r^4 + \alpha^4)^{\frac{1}{2}}$, etc.

When $r_0 = \alpha$ and $r \in \Re^+$, and n is taken in integers, the following infinite sequence of particular solutions is obtained,

$C_1(r) = r^2$ (Droste/Weyl/(Hilbert) [9, 14, 8])
$C_2(r) = (r - \alpha)^2 + \alpha^2$
$C_3(r) = [(r - \alpha)^3 + \alpha^3]^{\frac{2}{3}}$
$C_4(r) = [(r - \alpha)^4 + \alpha^4]^{\frac{1}{2}}$, etc.

The Schwarzschild forms obtained from (24) satisfy Eddington's prescription for a general solution.

By (17) and (24) the circumference χ of a great circle in the gravitational field is,

$$\chi = 2\pi\sqrt{C_n(r)} = 2\pi \left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}} , \tag{25}$$

and the proper radius $R_p(r)$ is, from (21),

$$R_p(r) = \sqrt{\left(|r-r_0|^n+\alpha^n\right)^{\frac{1}{n}}\left[\left(|r-r_0|^n+\alpha^n\right)^{\frac{1}{n}} - \alpha\right]} + \alpha \ln\left|\frac{\left(|r-r_0|^n+\alpha^n\right)^{\frac{1}{2n}} + \sqrt{\left(|r-r_0|^n+\alpha^n\right)^{\frac{1}{n}} - \alpha}}{\sqrt{\alpha}}\right| . \tag{26}$$

According to (24), $\sqrt{C_n(D(r_0))} \equiv \alpha$ is a scalar invariant, being independent of the value of r_0. Nevertheless the field is singular at the point-mass. By (21),

$$\lim_{r\to\pm\infty} \frac{R_p^2}{|r - r_0|^2} = 1 ,$$

and so,

$$\lim_{r\to\pm\infty} \frac{R_p^2}{C_n(D(r))} = \lim_{r\to\pm\infty} \frac{\frac{R_p^2}{|r-r_0|^2}}{\frac{C_n(D(r))}{|r-r_0|^2}} = 1 .$$

Now the ratio $\frac{\chi}{R_p} > 2\pi$ for *all* finite R_p, and

$$\lim_{r\to\pm\infty} \frac{\chi}{R_p} = 2\pi ,$$

$$\lim_{r\to r_0^\pm} \frac{\chi}{R_p} = \infty ,$$

so $R_p(r_0) \equiv 0$ is a quasiregular singularity and cannot be extended. The singularity occurs when parameter $r = r_0$, irrespective of the values of n and r_0. Thus, there is no sense in the orthodox notion that the region $0 < r < \alpha$ is an *interior* region on the Hilbert metric, since $r_0 \neq 0$ on that metric. Indeed, by (21) and (24) $r_0 = \alpha$ on the Hilbert metric. Equation (26) amplifies the fact that it is the *distance* $D = |r - r_0|$ that is mapped from parameter space into the proper radius (distance) in the gravitational field, and a distance *must* be $\geqslant 0$.

Consequently, strictly speaking, r_0 is not a singular point in the gravitational field because r is merely a parameter for the radial quantities in (M_g, g_g); r is neither a radius nor a coordinate in the gravitational field. No value of r can really be a singular point in the gravitational field. However, r_0 is mapped invariantly to $R_p = 0$, so $r = r_0$ *always* gives rise to a quasiregular singularity in the gravitational field, at $R_p(r_0) \equiv 0$, reflecting the fact that r_0 is the boundary on the r-parameter. Only in this sense should r_0 be considered a singular point.

The Kretschmann scalar $f = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ for equation (17) with equation (24) is,

$$f = \frac{12\alpha^2}{[C_n(D(r))]^3} = \frac{12\alpha^2}{\left(|r - r_0|^n + \alpha^n\right)^{\frac{6}{n}}} . \tag{27}$$

Taking the near-field limit on (27),

$$\lim_{r\to r_0^\pm} f = \frac{12}{\alpha^4} ,$$

so $f(r_0) \equiv \frac{12}{\alpha^4}$ is a scalar invariant, irrespective of the values of n and r_0, invalidating the orthodox assumption that the singularity must occur where the curvature is unbounded. Indeed, no curvature singularity can arise in the gravitational field. The orthodox analysis claims an unbounded curvature singularity at $r_0 = 0$ in (18) purely and simply by its invalid initial assumptions, *not* by mathematical imperative. It incorrectly assumes $\sqrt{C_n(r)} \equiv R_p(r) \equiv r$, then with its additional invalid assumption that $0 < r < \alpha$ is valid on the Hilbert metric, finds from (27),

$$\lim_{r \to 0^+} f(r) = \infty\,,$$

thereby satisfying its third invalid assumption, by *ad hoc* construction, that a singularity occurs only where the curvature invariant is unbounded.

The Kruskal-Szekeres form has no meaning since the r-parameter is not the radial coordinate in the gravitational field at all. Furthermore, the value of r_0 being entirely arbitrary, $r_0 = 0$ has no particular significance, in contrast to the mainstream claims on (18).

The value of the r-parameter of a certain spacetime event depends upon the coordinate system chosen. However, the proper radius $R_p(D(r))$ and the curvature radius $\sqrt{C_n(D(r))}$ of that event are independent of the coordinate system. This is easily seen as follows. Consider a great circle centred at the point-mass and passing through a spacetime event. Its circumference is measured at χ. Dividing χ by 2π gives,

$$\frac{\chi}{2\pi} = \sqrt{C_n(D(r))}\,.$$

Putting $\frac{\chi}{2\pi} = \sqrt{C_n(D(r))}$ into (21) gives the proper radius of the spacetime event,

$$R_p(r) = \sqrt{\frac{\chi}{2\pi}\left(\frac{\chi}{2\pi} - \alpha\right)} + \alpha \ln \left| \frac{\sqrt{\frac{\chi}{2\pi}} + \sqrt{\frac{\chi}{2\pi} - \alpha}}{\sqrt{\alpha}} \right|,$$

$$2\pi\alpha \leqslant \chi < \infty,$$

which is independent of the coordinate system chosen. To find the r-parameter in terms of a particular coordinate system set,

$$\frac{\chi}{2\pi} = \sqrt{C_n(D(r))} = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{1}{n}},$$

so

$$|r - r_0| = \left[\left(\frac{\chi}{2\pi}\right)^n - \alpha^n\right]^{\frac{1}{n}}.$$

Thus r for any particular spacetime event depends upon the arbitrary values n and r_0, which establish a coordinate system. Then when $r = r_0$, $R_p = 0$, and the great circumference $\chi = 2\pi\alpha$, irrespective of the values of n and r_0. A truly coordinate independent description of spacetime events has been attained.

The mainstream insistence, on the Hilbert solution (18), without proof, that the r-parameter is a radius of sorts in the gravitational field, the insistence that its r can, without proof, go down to zero, and the insistence, without proof, that a singularity in the field must occur only where the curvature is unbounded, have produced the irrational notion of the black hole. The fact is, the radius *always* does go down to zero in the gravitational field, but that radius is the *proper radius* R_p ($R_p = 0$ corresponding to a coordinate radius $D = 0$), not the curvature radius R_c, and certainly not the r-parameter.

There is no escaping the fact that $r_0 = \alpha \neq 0$ in (18). Indeed, if $\alpha = 0$, (18) *must* give,

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

the metric of Special Relativity when $r_0 = 0$. One cannot set the lower bound $r_0 = \alpha = 0$ in (18) and simultaneously keep $\alpha \neq 0$ in the components of the metric tensor, which is effectively what the orthodox analysis has done to obtain the black hole. The result is unmitigated nonsense. The correct form of the metric (18) is obtained from the associated Schwarzschild form (24): $C(r) = r^2$, $r_0 = \alpha$. Furthermore, the proper radius of (18) is,

$$R_p(r) = \int_{\alpha}^{r} \sqrt{\frac{r}{r - \alpha}}\,dr\,,$$

and so

$$R_p(r) = \sqrt{r\,(r - \alpha)} + \alpha \ln \left| \frac{\sqrt{r} + \sqrt{r - \alpha}}{\sqrt{\alpha}} \right|.$$

Then,

$$r \to \alpha^+ \Rightarrow D = |r - \alpha| = (r - \alpha) \to 0\,,$$

and in (M_g, g_g),

$$r^2 \equiv C(r) \to C(\alpha) = \alpha^2 \Rightarrow R_p(r) \to R_p(\alpha) = 0\,.$$

Thus, the r-parameter is mapped to the radius of curvature $\sqrt{C(r)} = \frac{\chi}{2\pi}$ by ψ_1, and the radius of curvature is mapped to the proper radius R_p by ψ_2. With the mappings established the r-parameter can be mapped directly to R_p by $\psi(r) = \psi_2 \circ \psi_1(r)$. In the case of the simple point-mass the mapping ψ_1 is just equation (24), and the mapping ψ_2 is given by (21).

The local acceleration of a test particle approaching the point-mass along a radial geodesic has been determined by N. Doughty [15] at,

$$a = \frac{\sqrt{-g_{rr}}\,(-g^{rr})\,|g_{tt,r}|}{2g_{tt}}\,. \tag{28}$$

For (17) the acceleration is,

$$a = \frac{\alpha}{2C_n^{\frac{3}{4}}\left(C_n^{\frac{1}{2}} - \alpha\right)^{\frac{1}{2}}}.$$

Then,

$$\lim_{r \to r_0^{\pm}} a = \infty,$$

since $C_n(r_0) \equiv \alpha^2$; thereby confirming that matter is indeed present at the point $R_p(r_0) \equiv 0$.

In the case of (18), where $r \in \Re^+$,

$$a = \frac{\alpha}{2r^{\frac{3}{2}} (r - \alpha)^{\frac{1}{2}}},$$

and $r_0 = \alpha$ by (24), so,

$$\lim_{r \to \alpha^+} a = \infty\,.$$

Y. Hagihara [16] has shown that all those geodesics which do not run into the boundary at $r = \alpha$ on (18) are complete. Now (18) with $\alpha < r < \infty$ is a particular solution by (24), and $r_0 = \alpha$ is an arbitrary point at which the point-mass is located in parameter space, therefore all those geodesics in (M_g, g_g) not running into the point $R_p(r_0) \equiv 0$ are complete, irrespective of the value of r_0.

Modern relativists do not interpret the Hilbert solution over $0 < r < \infty$ as Hilbert did, instead making an arbitrary distinction between $0 < r < \alpha$ and $\alpha < r < \infty$. The modern relativist maintains that one is entitled to just "choose" a region. However, as I have shown, this claim is inadmissible. J. L. Synge [17] made the same unjustified assumptions on the Hilbert line-element. He remarks,

> "*This line-element is usually regarded as having a singularity at $r = \alpha$, and appears to be valid only for $r > \alpha$. This limitation is not commonly regarded as serious, and certainly is not so if the general theory of relativity is thought of solely as a macroscopic theory to be applied to astronomical problems, for then the singularity $r = \alpha$ is buried inside the body, i. e. outside the domain of the field equations $R_{mn} = 0$. But if we accord to these equations an importance comparable to that which we attach to Laplace's equation, we can hardly remain satisfied by an appeal to the known sizes of astronomical bodies. We have a right to ask whether the general theory of relativity actually denies the existence of a gravitating particle, or whether the form (1.1) may not in fact lead to the field of a particle in spite of the apparent singularity at $r = \alpha$.*"

M. Kruskal [18] remarks on his proposed extension of the Hilbert solution into $0 < r < 2m$,

> "*That this extension is possible was already indicated by the fact that the curvature invariants of the Schwarzschild metric are perfectly finite and well behaved at $r = 2m$∗.*"

which betrays the very same unproven assumptions.

G. Szekeres [6] says of the Hilbert line-element,

> "*. . . it consists of two disjoint regions, $0 < r < 2m$, and $r > 2m$, separated by the singular hypercylinder $r = 2m$.*"

which again betrays the same unproven assumptions.

I now draw attention to the following additional problems with the Kruskal-Szekeres form.

(a) Applying Doughty's acceleration formula (28) to the Kruskal-Szekeres form, it is easily found that,

$$\lim_{r \to 2m^-} a = \infty.$$

But according to Kruskal-Szekeres there is *no matter* at $r = 2m$. Contra-hype.

(b) As $r \to 0$, $u^2 - v^2 \to -1$. These loci are spacelike, and therefore *cannot* describe *any* configuration of matter or energy.

Both of these anomalies have also been noted by Abrams in his work [7]. Either of these features alone proves the Kruskal-Szekeres form inadmissible.

The correct geometrical analysis excludes the interior Hilbert region on the grounds that it is not a region at all, and invalidates the assumption that the r-parameter is some kind of radius and/or coordinate in the gravitational field. Consequently, the Kruskal-Szekeres formulation is meaningless, both physically and mathematically. In addition, the so-called "Schwarzschild radius" (*not* due to Schwarzschild) is also a meaningless concept - it is not a radius in the gravitational field. Hilbert's $r = 2m$ is indeed a point, i. e. the "Schwarzschild radius" is a point, in both parameter space and the gravitational field: by (21), $R_p(2m) = 0$.

The *form* of the Hilbert line-element is given by Karl Schwarzschild in his 1916 paper, where it occurs there in the equation he numbers (14), in terms of his "auxiliary parameter" R. However Schwarzschild also includes there the equation $R = \left(r^3 + \alpha^3\right)^{\frac{1}{3}}$, having previously established the range $0 < r < \infty$. Consequently, Schwarzschild's auxiliary parameter R (which is actually a curvature radius) has the lower bound $R_0 = \alpha = 2m$. Schwarzschild's R^2 and Hilbert's r^2 can be replaced with any appropriate analytic function $C_n(r)$ as given by (24), so the range and the boundary on r will depend upon the function chosen. In the case of Schwarzschild's particular solution the range is $0 < r < \infty$ (since $r_0 = 0$, $C_3(r) = \left(r^3 + \alpha^3\right)^{\frac{2}{3}}$) and in Hilbert's particular solution the range is $2m < r < \infty$ (since $r_0 = 2m$, $C_1(r) = r^2$).

The geometry and the invariants are the important properties, but the conventional analysis has shockingly erred in its geometrical analysis and identification of the invariants, as a direct consequence of its initial invalidated assumptions about the r-parameter, and clings irrationally to these assumptions to preserve the now sacrosanct, but nonetheless ridiculous, black hole.

The only reason that the Hilbert solution conventionally breaks down at $r = \alpha$ is because of the initial arbitrary and incorrect assumptions made about the parameter r. There is no *pathology of coordinates* at $r = \alpha$. If there is anything pathological about the Hilbert metric it has nothing to do with coordinates: the etiology of a pathology must therefore be found elewhere.

There is no doubt that the Kruskal-Szekeres form is a solution of the Einstein vacuum field equations, however that does not guarantee that it is a solution to the problem. There exists an infinite number of solutions to the vacuum field equations which do not yield a solution for the gravitational field of the point-mass. Satisfaction of the field equations is a necessary but insufficient condition for a potential solution to the problem. It is evident that the conventional conditions (see [19]) that must be met are inadequate, viz.,

1. *be analytic*;
2. *be Lorentz signature*;
3. *be a solution to Einstein's free-space field equations*;
4. *be invariant under time translations*;
5. *be invariant under spatial rotations*;
6. *be (spatially) asymptotically flat*;
7. *be inextendible to a wordline L*;
8. *be invariant under spatial reflections*;
9. *be invariant under time reflection*;
10. *have a global time coordinate*.

This list must be augmented by a boundary condition at the location of the point-mass, which is, in my formulation of the solution, $r \to r_0^{\pm} \Rightarrow R_p(r) \to 0$. Schwarzschild actually applied a form of this boundary condition in his analysis. Marcel Brillouin [12] also pointed out the necessity of such a boundary condition in 1923, as did Abrams [7] in more recent years, who stated it equivalently as, $r \to r_0 \Rightarrow C(r) \to \alpha^2$. The condition has been disregarded or gone unrecognised by the mainstream authorities. Oddly, the orthodox analysis violates its own stipulated condition for a global time coordinate, but quietly disregards this inconsistency as well.

Any constants appearing in a valid solution must appear in an invariant derived from the solution. The solution I obtain meets this condition in the invariance, at $r = r_0$, of the circumference of a great circle, of Kepler's 3rd Law [1, 2], of the Kretschmann scalar, of the radius of curvature $C(r_0) = \alpha^2$, of $R_p(r_0) \equiv 0$, and not only in the case of the point-mass, but also in all the relevant configurations, with or without charge.

The fact that the circumference of a great circle approaches the finite value $2\pi\alpha$ is no more odd than the conventional oddity of the change in the arrow of time in the "interior" Hilbert region. Indeed, the latter is an even more violent oddity: inconsistent with Einstein's theory. The finite limit of the said circumference is consistent with the geometry resulting from Einstein's gravitational tensor. The variations of θ and φ displace the proper radius vector, $R_p(r_0) \equiv 0$, over the spherical surface of finite area $4\pi\alpha^2$, as noted by Brillouin. Einstein's theory admits nothing more pointlike.

Objections to Einstein's formulation of the gravitational tensor were raised as long ago as 1917, by T. Levi-Civita [20], on the grounds that, from the mathematical standpoint, it lacks the invariant character actually required of General Relativity, and further, produces an unacceptable consequence concerning gravitational waves (i.e they carry neither energy nor momentum), a solution for which Einstein vaguely appealed *ad hoc* to quantum theory, a last resort obviated by Levi-Civita's reformulation of the gravitational tensor (which extinguishes the gravitational wave), of which the conventional analysis is evidently completely ignorant: but it is not pertinent to the issue of whether or not the black hole is consistent with the theory as it currently stands on Einstein's gravitational tensor.

3 The geometry of the simple point-charge

The fundamental geometry developed in section **2** is the same for all the configurations of the point-mass and the point-charge. The general solution for the simple point-charge [2] is,

$$ds^2 = \left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right) dt^2 - \left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)^{-1} \times \\ \times \frac{C_n'^{\,2}}{4C_n} dr^2 - C_n (d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{29}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}},$$

$$\beta = m + \sqrt{m^2 - q^2}\,, \quad q^2 < m^2\,,$$

$$n \in \Re^+, \quad r, r_0 \in \Re\,.$$

where n and r_0 are arbitrary.

From (29), the radius of curvature is given by,

$$R_c = \sqrt{C_n(r)} = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{1}{n}},$$

which gives for the near-field limit,

$$\lim_{r \to r_0^{\pm}} \sqrt{C_n(r)} = \sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - q^2}\,.$$

The expression for the proper radius is,

$$R_p(r) = \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2} + \\ + m \ln \left| \frac{\sqrt{C(r)} - m + \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2}}{\sqrt{m^2 - q^2}} \right|.$$

Then

$$\lim_{r \to r_0^\pm} R_p(r) = R_p(r_0) \equiv 0\,.$$

The ratio $\frac{\chi}{R_p} > 2\pi$ for all finite R_p, and

$$\lim_{r \to \pm\infty} \frac{\chi}{R_p(r)} = 2\pi\,,$$

$$\lim_{r \to r_0^\pm} \frac{\chi}{R_p(r)} = \infty\,,$$

so $R_p(r_0) \equiv 0$ is a quasiregular singularity and cannot be extended.

Now, since the circumference χ of a great circle is the only measurable quantity in the gravitational field, the unique solution for the field of the simple point-charge is,

$$\begin{aligned} ds^2 &= \left(1 - \frac{2\pi\alpha}{\chi} + \frac{4\pi^2 q^2}{\chi^2}\right) dt^2 - \\ &- \left(1 - \frac{2\pi\alpha}{\chi} + \frac{4\pi^2 q^2}{\chi^2}\right)^{-1} \frac{d\chi^2}{4\pi^2} - \\ &- \frac{\chi^2}{4\pi^2}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \end{aligned} \tag{30}$$

$$2\pi\left(m + \sqrt{m^2 - q^2}\right) < \chi < \infty\,.$$

Equation (30) is entirely independent of the r-parameter.

In terms of equation (29), the Kretschmann scalar takes the form [21],

$$f(r) = \frac{8\left[6\left(m\sqrt{C_n(r)} - q^2\right)^2 + q^4\right]}{C_n^4(r)}\,, \tag{31}$$

so

$$\lim_{r \to r_0^\pm} f(r) = f(r_0) = \frac{8\left[6\left(m\beta - q^2\right)^2 + q^4\right]}{\beta^8} = \frac{8\left[6\left(m^2 + m\sqrt{m^2 - q^2} - q^2\right)^2 + q^4\right]}{\left(m + \sqrt{m^2 - q^2}\right)^8}\,,$$

which is a scalar invariant. Thus, no curvature singularity can arise in the gravitational field of the simple point-charge.

The standard analysis incorrectly takes $\sqrt{C_n(r)} \equiv R_p(r) \equiv r$, then with this assumption, and the additional invalid assumption that $0 < r < \infty$ is true on the Reissner-Nordstrom solution, obtains from equation (31) a curvature singularity at $r = 0$, satisfying, by an *ad hoc* construction, its third invalid assumption that a singularity can only arise at a point where the curvature invariant is unbounded.

Equation (29) is singular *only* when $g_{00} = 0$; indeed $g_{00}(r_0) \equiv 0$. Hence, $0 \leqslant g_{00} \leqslant 1$.

Applying Doughty's acceleration formula (28) to equation (29) gives,

$$a = \frac{\left|m\sqrt{C_n(r)} - q^2\right|}{C_n(r)\sqrt{C_n(r) - \alpha\sqrt{C_n(r)} + q^2}}\,.$$

Then,

$$\lim_{r \to r_0^\pm} a = \frac{\left|m\beta - q^2\right|}{\beta^2\sqrt{\beta^2 - \alpha\beta + q^2}} = \infty\,,$$

confirming that matter is indeed present at $R_p(r_0) \equiv 0$.

4 The geometry of the rotating point-charge

The usual expression for the Kerr-Newman solution is, in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \end{aligned} \tag{32}$$

$$a = \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta\,,$$

$$\Delta = r^2 - r\alpha + a^2 + q^2, \quad 0 < r < \infty\,.$$

This metric is alleged to have an event horizon r_h and a static limit r_b, obtained by setting $\Delta = 0$ and $g_{00} = 0$ respectively, to yield,

$$r_h = m \pm \sqrt{m^2 - a^2 - q^2}$$

$$r_b = m \pm \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,.$$

These expressions are conventionally quite arbitrarily taken to be,

$$r_h = m + \sqrt{m^2 - a^2 - q^2}$$

$$r_b = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,,$$

apparently because no-one has been able to explain away the meaning of the the "inner" horizon and the "inner" static limit. This in itself is rather disquieting, but nonetheless accepted with furtive whispers by the orthodox theorists. It is conventionally alleged that the "region" between r_h and r_b is an ergosphere, in which spacetime is dragged in the direction of the of rotation of the point-charge.

The conventional taking of the r-parameter for a radius in the gravitational field is manifest. However, as I have shown, the r-parameter is neither a coordinate nor a radius in the gravitational field. Consequently, the standard analysis is erroneous.

I have already derived elsewhere [2] the general solution for the rotating point-charge, which I write in most general form as,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 -$$
$$-\frac{\sin^2\theta}{\rho^2}\left[\left(C_n + a^2\right)d\varphi - adt\right]^2 - \frac{\rho^2}{\Delta}\frac{C_n'^{\,2}}{4C_n}dr^2 - \rho^2 d\theta^2\,,$$
$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad n \in \Re^+\,, \tag{33}$$
$$r,\ r_0 \in \Re, \quad \beta = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,,$$
$$a^2 + q^2 < m^2, \quad a = \frac{L}{m}, \quad \rho^2 = C_n + a^2\cos^2\theta\,,$$
$$\Delta = C_n - \alpha\sqrt{C_n} + q^2 + a^2\,,$$

where n and r_0 are arbitrary.

Once again, since only the circumference of a great circle is a measurable quantity in the gravitational field, the unique general solution for all configurations of the point-mass is,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 -$$
$$-\frac{\sin^2\theta}{\rho^2}\left[\left(\frac{\chi^2}{4\pi^2} + a^2\right)d\varphi - adt\right]^2 - \frac{\rho^2}{\Delta}\frac{d\chi^2}{4\pi^2} - \rho^2 d\theta^2\,,$$
$$a^2 + q^2 < m^2, \quad a = \frac{L}{m}, \quad \rho^2 = \frac{\chi^2}{4\pi^2} + a^2\cos^2\theta\,, \tag{34}$$
$$\Delta = \frac{\chi^2}{4\pi^2} - \frac{\alpha\chi}{2\pi} + q^2 + a^2\,,$$
$$2\pi\left(m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\right) < \chi < \infty\,.$$

Equation (34) is entirely independent of the r-parameter.

Equation (34) emphasizes the fact that the concept of a point in pseudo-Euclidean Minkowski space is not attainable in the pseudo-Riemannian gravitational field. A point-mass (or point-charge) is characterised by a proper radius of zero and a finite, non-zero radius of curvature. Einstein's universe admits of nothing more pointlike. The relativists have assumed that, insofar as the point-mass is concerned, the Minkowski point can be achieved in Einstein space, which is not correct.

The radius of curvature of (33) is,

$$\sqrt{C_n(r)} = \left(|r - r_0|^n + \beta^n\right)^{\frac{1}{n}}\,, \tag{35}$$

which goes down to the limit,

$$\lim_{r\to r_0^{\pm}}\sqrt{C_n(r)} = \sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,, \tag{36}$$

where the proper radius $R_p(r_0) \equiv 0$. The standard analysis incorrectly takes (36) for the "radius" of its static limit.

It is evident from (35) and (36) that the radius of curvature depends upon the direction of radial approach. Therefore, the spacetime is not isotropic. Only when $a = 0$ is spacetime isotropic. The point-charge is always located at $R_p(r_0) \equiv 0$ in $(M_g,\, g_g)$, irrespective of the value of n, and irrespective of the value of r_0. The conventional analysis has failed to realise that its r_b is actually a varying radius of curvature, and so incorrectly takes it as a measurable radius in the gravitational field. It has also failed to realise that the location of the point-mass in the gravitational field is not uniquely specified by the r-coordinate at all. The point-mass is *always* located just where $R_p = 0$ in (M_g, g_g) and its "position" in (M_g, g_g) is otherwise meaningless. The test particle has already encountered the source of the gravitational field when the radius of curvature has the value $C_n(r_0) = \beta$. The so-called ergosphere also arises from the aforesaid misconceptions.

When $\theta = 0$ the limiting radius of curvature is,

$$\sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - q^2 - a^2}\,, \tag{37}$$

and when $\theta = \frac{\pi}{2}$, the limiting radius of curvature is,

$$\sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - q^2}\,,$$

which is the limiting radius of curvature for the simple point-charge (i. e. no rotation) [2].

The standard analysis incorrectly takes (37) as the "radius" of its event horizon.

If $q = 0$, then the limiting radius of curvature when $\theta = 0$ is,

$$\sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - a^2}\,, \tag{38}$$

and the limiting radius of curvature when $\theta = \frac{\pi}{2}$ is,

$$\sqrt{C_n(r_0)} = \beta = 2m = \alpha\,,$$

which is the radius of curvature for the simple point-mass.

The radii of curvature at intermediate azimuth are given generally by (36). In all cases the near-field limits of the radii of curvature give $R_p(r_0) \equiv 0$.

Clearly, the limiting radius of curvature is minimum at the poles and maximum at the equator. At the equator the effects of rotation are not present. A test particle approaching the rotating point-charge or the rotating point-mass equatorially experiences the effects only of the non-rotating situation of each configuration respectively. The effects of the rotation manifest only in the values of azimuth other than $\frac{\pi}{2}$. There is no rotational drag on spacetime, no ergosphere and no event horizon, i. e. no black hole.

The effects of rotation on the radius of curvature will necessarily manifest in the associated form of Kepler's 3rd Law, and the Kretschmann scalar [22].

I finally remark that the fact that a singularity arises in the gravitational field of the point-mass is an indication that a material body cannot collapse to a point, and therefore such a model is inadequate. A more realistic model must be sought in terms of a non-singular metric, of which I treat elsewhere [23].

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

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A Theory of Gravity Like Electrodynamics

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This study looks at the field of inhomogeneities of time coordinates. Equations of motion, expressed through the field tensor, show that particles move along time lines because of rotation of the space itself. Maxwell-like equations of the field display its sources, which are derived from gravitation, rotations, and inhomogeneity of the space. The energy-momentum tensor of the field sets up an inhomogeneous viscous media, which is in the state of an ultrarelativistic gas. Waves of the field are transverse, and the wave pressure is derived from mainly sub-atomic processes — excitation/relaxation of atoms produces the positive/negative wave pressures, which leads to a test of the whole theory.

Contents

1 Inhomogeneity of observable time. Defining the field

The meaning of Einstein's General Theory of Relativity consists of his idea that all properties of the world are derived from the geometrical structure of space-time, from the world-geometry, in other words. This is a way to geometrize physics. The introduction of his artificial postulates became only of historical concern subsequent to his setting up of the meaning of the theory — all the postulates are naturally contained in the geometry of a four-dimensional pseudo-Riemannian space with the sign-alternating signature $(-+++)$ or $(+---)$ he assigned to the basic space-time of the theory.

Verification of the theory by experiments has shown that the four-dimensional pseudo-Riemannian space satisfies our observable world in most cases. In general we can say that all that everything we can obtain theoretically in this space geometry must have a physical interpretation.

Here we take a pseudo-Riemannian space with the signature $(+---)$, where time is real and spatial coordinates are imaginary, because the observable projection of a four-dimensional impulse on the spatial section of any given observer is positive in this case. We also assign to space-time Greek indices, while spatial indices are Latin*.

As it is well-known [1], $dS = m_0 c\,ds$ is an elementary action to displace a free mass-bearing particle of rest-mass m_0 through a four-dimensional interval of length ds. What happens to matter during this action? To answer this question let us substitute the square of the interval $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$ into the action. As a result we see that

$$dS = m_0 c\,ds = m_0 c \sqrt{g_{\alpha\beta}\,dx^\alpha dx^\beta}\,, \tag{1}$$

so the particle moves in space-time along geodesic lines (free motion), because the field carries the fundamental metric tensor $g_{\alpha\beta}$. At the same time Einstein's equations link the metric tensor $g_{\alpha\beta}$ to the energy-momentum tensor of matter through the four-dimensional curvature of space-time. This implies that the gravitational field is linked to the field of the space-time metric in the frames of the General Theory of Relativity. For this reason one regularly concludes that the action (1) displacing free mass-bearing particles is produced by the gravitational field.

Let us find which field will manifest by the action (1) as a source of free motion, if the space-time interval ds therein is written with quantities which would be observable by a real observer located in the four-dimensional pseudo-Riemannian space.

A formal basis here is the mathematical apparatus of physically observable quantities (the theory of chronometric invariants), developed by Zelmanov in the 1940's [2, 3]. Its essence is that if an observer accompanies his reference body,

*Alternatively, Landau and Lifshitz in their *The Classical Theory of Fields* [1] use the space signature $(-+++)$, which gives an advantage in certain cases. They also use other notations for tensor indices: in their book space-time indices are Latin, while spatial indices are Greek.

his observable quantities are projections of four-dimensional quantities on his time line and the spatial section — *chronometrically invariant quantities*, made by projecting operators $b^{\alpha} = \frac{dx^{\alpha}}{ds}$ and $h_{\alpha\beta} = -g_{\alpha\beta} + b_{\alpha} b_{\beta}$ which fully define his real reference space (here b^{α} is his velocity with respect to his real references). Thus, the chr.inv.-projections of a world-vector Q^{α} are $b_{\alpha} Q^{\alpha} = \frac{Q_0}{\sqrt{g_{00}}}$ and $h^i_{\alpha} Q^{\alpha} = Q^i$, while chr.inv.-projections of a world-tensor of the 2nd rank $Q^{\alpha\beta}$ are $b^{\alpha} b^{\beta} Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}$, $h^{i\alpha} b^{\beta} Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}$, $h^i_{\alpha} h^k_{\beta} Q^{\alpha\beta} = Q^{ik}$. Physically observable properties of the space are derived from the fact that chr.inv.-differential operators $\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}} \frac{\partial}{\partial t}$ and $\frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2} v_i \frac{^*\partial}{\partial t}$ are non-commutative $\frac{^*\partial^2}{\partial x^i \partial t} - \frac{^*\partial^2}{\partial t\, \partial x^i} = \frac{1}{c^2} F_i \frac{^*\partial}{\partial t}$ and $\frac{^*\partial^2}{\partial x^i \partial x^k} - \frac{^*\partial^2}{\partial x^k \partial x^i} = \frac{2}{c^2} A_{ik} \frac{^*\partial}{\partial t}$, and also from the fact that the chr.inv.-metric tensor h_{ik} may not be stationary. The observable characteristics are the chr.inv.-vector of gravitational inertial force F_i, the chr.inv.-tensor of angular velocities of the space rotation A_{ik}, and the chr.inv.-tensor of rates of the space deformations D_{ik}, namely

$$F_i = \frac{1}{\sqrt{g_{00}}} \left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t} \right), \quad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} \tag{2}$$

$$A_{ik} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k} \right) + \frac{1}{2c^2} \left(F_i v_k - F_k v_i \right), \tag{3}$$

$$D_{ik} = \frac{1}{2} \frac{^*\partial h_{ik}}{\partial t}, \quad D^{ik} = -\frac{1}{2} \frac{^*\partial h^{ik}}{\partial t}, \quad D^k_k = \frac{^*\partial \ln \sqrt{h}}{\partial t}, \tag{4}$$

where w is gravitational potential, $v_i = -c \frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation, $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ is the chr.inv.-metric tensor, and also $h = \det \|h_{ik}\|$, $h g_{00} = -g$, $g = \det \|g_{\alpha\beta}\|$. Observable inhomogeneity of the space is set up by the chr.inv.-Christoffel symbols $\Delta^i_{jk} = h^{im} \Delta_{jk,m}$, which are built just like Christoffel's usual symbols $\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\sigma} \Gamma_{\mu\nu,\sigma}$ using h_{ik} instead of $g_{\alpha\beta}$.

A four-dimensional generalization of the main chr.inv.-quantities F_i, A_{ik}, and D_{ik} (by Zelmanov, the 1960's [4]) is: $F_{\alpha} = -2c^2 b^{\beta} a_{\beta\alpha}$, $A_{\alpha\beta} = c h^{\mu}_{\alpha} h^{\nu}_{\beta} a_{\mu\nu}$, $D_{\alpha\beta} = c h^{\mu}_{\alpha} h^{\nu}_{\beta} d_{\mu\nu}$, where $a_{\alpha\beta} = \frac{1}{2} (\nabla_{\alpha} b_{\beta} - \nabla_{\beta} b_{\alpha})$, $d_{\alpha\beta} = \frac{1}{2} (\nabla_{\alpha} b_{\beta} + \nabla_{\beta} b_{\alpha})$.

In this way, for any equations obtained using general covariant methods, we can calculate their physically observable projections on the time line and the spatial section of any particular reference body and formulate the projections in terms of their real physically observable properties, from which we obtain equations containing only quantities measurable in practice.

Expressing $ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$ through the observable time interval

$$d\tau = \frac{1}{c}\, b_{\alpha} dx^{\alpha} = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i \tag{5}$$

and also the observable spatial interval $d\sigma^2 = h_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = h_{ik}\, dx^i dx^k$ (note, $b^i = 0$ for an observer who accompanies his reference body), we come to the formula

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{6}$$

Using this formula, we can write down the action (1) to displace a free mass-bearing particle in the form

$$dS = m_0 c \sqrt{b_{\alpha} b_{\beta}\, dx^{\alpha} dx^{\beta} - h_{\alpha\beta}\, dx^{\alpha} dx^{\beta}}\,. \tag{7}$$

If the particle is at rest with respect to the observer's reference body, then its observable displacement along his spatial section is $dx^i = 0$, so its observable chr.inv.-velocity vector equals zero; $\mathrm{v}^i = \frac{dx^i}{d\tau} = 0$. Such a particle moves only along time lines. In this case, in the accompanying reference frame, we have $h_{\alpha\beta}\, dx^{\alpha} dx^{\beta} = h_{ik}\, dx^i dx^k = 0$ hence the action is

$$dS = m_0 c\, b_{\alpha} dx^{\alpha}, \tag{8}$$

so the mass-bearing particle moves freely along time lines because it is carried solely by the vector field b^{α}.

What is the physical meaning of this field? The vector b^{α} is the operator of projection on time lines (non-uniform, in general case) of a real observer, who accompanies his reference body. This implies that the vector field b^{α} defines the geometrical structure of the real space-time along time lines. Projecting an interval of four-dimensional coordinates dx^{α} onto the time line of a real observer in his accompanying reference frame, we obtain the interval of real physical time $d\tau = \frac{1}{c} b_{\alpha} dx^{\alpha} = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2} v_i dx^i$ he observes. For his measurements in the same spatial point, in other words, along the same time line, $d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt$. This formula and the previous one lead us to the conclusion that the components of the observer's vector b^{α} define a "density" of physically observable time in his accompanying reference frame. As it is easy to see, the observable time density depends on the gravitational potential and, in the general case, on the rotation of the space. Hence, the vector field b^{α} in the accompanying reference frame is the field of inhomogeneity of observable time references. For this reason we will call it the *field of density of observable time*.

In the same way, a field of the tensor $h_{\alpha\beta} = -g_{\alpha\beta} + b_{\alpha} b_{\beta}$ projecting four-dimensional quantities on the observer's spatial section is the *field of density of the spatial section*.

From the geometric viewpoint, we can illustrate the conclusions in this way. The vector field b^{α} and the tensor field $h_{\alpha\beta}$ of the accompanying reference frame of an observer, located in a four-dimensional pseudo-Riemannian space, "split" the space into time lines and a spatial section, properties of which (such as inhomogeneity, anisotropy, curvature, etc.) depend on the physical properties of the observer's reference body. Owing to this "splitting" process, the field of the fundamental metric tensor $g_{\alpha\beta}$, containing the geometrical structure of this space, "splits" as well (7). Its "transverse component" is the time density field, a four-dimensional potential of which is the monad vector b^α. The "longitudinal component" of this splitting is the field of density of the spatial section.

2 The field tensor. Its observable components: gravitational inertial force and the space rotation tensor

Chr.inv.-projections of the four-dimensional vector potential b^α of a time density field are, respectively

$$\varphi = \frac{b_0}{\sqrt{g_{00}}} = 1\,, \qquad q^i = b^i = 0\,. \tag{9}$$

Emulating the way that Maxwell's electromagnetic field tensor is introduced, we introduce the *tensor of a time density field* as the rotor of its four-dimensional vector potential

$$F_{\alpha\beta} = \nabla_\alpha\, b_\beta - \nabla_\beta\, b_\alpha = \frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\,. \tag{10}$$

Taking into account that $F_{00} = F^{00} = 0$, as for any antisymmetric tensor of the 2nd rank, after some algebra we obtain the other components of the field tensor $F_{\alpha\beta}$

$$F_{0i} = \frac{1}{c^2}\sqrt{g_{00}}\,F_i\,, \qquad F_{ik} = \frac{1}{c}\left(\frac{\partial v_i}{\partial x^k} - \frac{\partial v_k}{\partial x^i}\right), \tag{11}$$

$$F_{0\cdot}^{\cdot 0} = -\frac{1}{c^3}\,v_k F^k, \qquad F_{0\cdot}^{\cdot i} = -\frac{1}{c^2}\sqrt{g_{00}}\,F^i, \tag{12}$$

$$F_{k\cdot}^{\cdot 0} = -\frac{1}{\sqrt{g_{00}}}\left(\frac{1}{c^2}\,F_k + \frac{2}{c^2}\,v^m A_{mk} - \frac{1}{c^4}\,v_k v_m F^m\right), \tag{13}$$

$$F_{k\cdot}^{\cdot i} = \frac{1}{c^3}\,v_k F^i + \frac{2}{c}\,A_{k\cdot}^{\cdot i}\,, \qquad F^{ik} = -\frac{2}{c}\,A^{ik}, \tag{14}$$

$$F^{0k} = -\frac{1}{\sqrt{g_{00}}}\left(\frac{1}{c^2}\,F^k + \frac{2}{c^2}\,v_m A^{mk}\right). \tag{15}$$

We denote chr.inv.-projections of the field tensor just like the chr.inv.-projections of the Maxwell tensor [5], to display their physical sense. We will refer to the time projection

$$E^i = \frac{F_{0\cdot}^{\cdot i}}{\sqrt{g_{00}}} = -\frac{1}{c^2}\,F^i, \qquad E_i = h_{ik}E^k = -\frac{1}{c^2}\,F_i \tag{16}$$

of the field tensor $F_{\alpha\beta}$ as "electric". The spatial projection

$$H^{ik} = F^{ik} = -\frac{2}{c}\,A^{ik}, \quad H_{ik} = h_{im}h_{kn}F^{mn} = -\frac{2}{c}\,A_{ik} \tag{17}$$

of the field tensor will be referred to as "magnetic". So, we arrive at physical definitions of the components:

> The "electric" observable component of a time density field manifests as the gravitational inertial force F_i. The "magnetic" observable component of a time density field manifests as the angular velocity A_{ik} of the space rotation.

In accordance with the above, two particular cases of time density fields are possible. These are:

1. If a time density field has $H_{ik} = 0$ and $E^i \neq 0$, then the field is strictly of the "electric" kind. This particular case corresponds to a holonomic (non-rotating) space filled with gravitational force fields;
2. A time density field is of the "magnetic" kind, if therein $E^i = 0$ and $H_{ik} \neq 0$. This is a non-holonomic space, where fields of gravitational inertial forces are homogeneous or absent. This case is possible also if, according to the chr.inv.-definition of the force

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{18}$$

where the first term — a force of gravity would be reduced by the second term — is a centrifugal force of inertia.

In addition to the field tensor $F_{\alpha\beta}$, we introduce the field pseudotensor $F^{*\alpha\beta}$ dual and in the usual way [1]

$$F^{*\alpha\beta} = \frac{1}{2}\,E^{\alpha\beta\mu\nu}F_{\mu\nu}\,, \qquad F_{*\alpha\beta} = \frac{1}{2}\,E_{\alpha\beta\mu\nu}F^{\mu\nu}, \tag{19}$$

where the four-dimensional completely antisymmetric discriminant tensors $E^{\alpha\beta\mu\nu} = \frac{e^{\alpha\beta\mu\nu}}{\sqrt{-g}}$ and $E_{\alpha\beta\mu\nu} = e_{\alpha\beta\mu\nu}\sqrt{-g}$, transforming regular tensors into pseudotensors in inhomogeneous anisotropic pseudo-Riemannian spaces, are not physically observable quantities. The completely antisymmetric unit tensor $e^{\alpha\beta\mu\nu}$, being defined in a Galilean reference frame in Minkowski space [1], does not have this quality either. Therefore we employ Zelmanov's chr.inv.-discriminant tensors $\varepsilon^{\alpha\beta\gamma} = b_\sigma E^{\sigma\alpha\beta\gamma}$ and $\varepsilon_{\alpha\beta\gamma} = b^\sigma E_{\sigma\alpha\beta\gamma}$ [2], which in the accompanying reference frame are

$$\varepsilon^{ikm} = \frac{e^{ikm}}{\sqrt{h}}\,, \qquad \varepsilon_{ikm} = e_{ikm}\sqrt{h}\,. \tag{20}$$

Using components of the field tensor $F_{\alpha\beta}$, we obtain chr.inv.-projections of the field pseudotensor, which are

$$H^{*i} = \frac{F_{0\cdot}^{*\cdot i}}{\sqrt{g_{00}}} = -\frac{1}{c}\,\varepsilon^{ikm}A_{km} = -\frac{2}{c}\,\Omega^{*i}, \tag{21}$$

$$E^{*ik} = F^{*ik} = \frac{1}{c^2}\,\varepsilon^{ikm}F_m\,, \tag{22}$$

where $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{ikm}A_{km}$ is the chr.inv.-pseudovector of angular velocities of the space rotation. Their relations to the field tensor chr.inv.-projections express themselves just like any chr.inv.-pseudotensors [5, 6], by the formulae

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}H_{mn}\,, \qquad H_{*i} = \frac{1}{2}\,\varepsilon_{imn}H^{mn}, \tag{23}$$

$$\varepsilon^{ipq}H_{*i} = \frac{1}{2}\,\varepsilon^{ipq}\varepsilon_{imn}H^{mn} = H^{pq}, \tag{24}$$

$$\varepsilon_{ikp}H^{*p} = E_{ik}\,, \qquad E^{*ik} = -\varepsilon^{ikm}E_m\,, \tag{25}$$

where $\varepsilon^{ipq}\varepsilon_{imn} = \delta^p_m\delta^q_n - \delta^q_m\delta^p_n$, see [1, 5, 6] for details.

We introduce the invariants $J_1 = F_{\alpha\beta}F^{\alpha\beta}$ and $J_2 = F_{\alpha\beta}F^{*\alpha\beta}$ for a time density field. Their formulae are

$$J_1 = F_{\alpha\beta}F^{\alpha\beta} = \frac{4}{c^2}A_{ik}A^{ik} - \frac{2}{c^4}F_iF^i, \qquad (26)$$

$$J_2 = F_{\alpha\beta}F^{*\alpha\beta} = -\frac{8}{c^3}F_i\Omega^{*i}, \qquad (27)$$

so the time density field can be *spatially isotropic* (one of the invariants becomes zero) under the conditions:

- the invariant $A_{ik}A^{ik}$ of the space rotation field and the invariant F_iF^i of the gravitational inertial force field are proportional one to another $A_{ik}A^{ik} = \frac{1}{2c^2}F_iF^i$;
- $F_i\Omega^{*i} = 0$, so the acting gravitational inertial force F_i is orthogonal to the space rotation pseudovector Ω^{*i};
- both of the conditions are realized together.

3 Equations of free motion. Putting the acting force into a form like Lorentz's force

Time lines are geodesics by definition. In accordance with the *least action principle*, an action replacing a particle along a geodesic line is minimum. Actually, the least action principle implies that geodesic lines are also lines of the least action. This is the physical viewpoint.

We are going to consider first a free mass-bearing particle, which is at rest with respect to an observer and his reference body. Such a particle moves only along a time line, so it moves solely because of the action of the inhomogeneity of time coordinates along the time line — a time density field.

The action that a time density field expends in displacing a free mass-bearing particle of rest-mass m_0 at dx^α has the value $dS = m_0 c\, b_\alpha dx^\alpha$ (8). Because of the least action, variation of the action integral along geodesic lines equals zero

$$\delta\int_a^b dS = 0\,, \qquad (28)$$

which, after substituting $dS = m_0 c b_\alpha dx^\alpha$, becomes $\delta\int_a^b dS = m_0 c\,\delta\int_a^b b_\alpha dx^\alpha = m_0 c\int_a^b \delta b_\alpha dx^\alpha + m_0 c\int_a^b b_\alpha d\delta x^\alpha$ where $\int_a^b b_\alpha d\delta x^\alpha = b_\alpha\delta x^\alpha\big|_a^b - \int_a^b db_\alpha\delta x^\alpha = -\int_a^b db_\alpha\delta x^\alpha$. Because $\delta b_\alpha = \frac{\partial b_\alpha}{\partial x^\beta}\delta x^\beta$ and $db_\alpha = \frac{\partial b_\alpha}{\partial x^\beta}dx^\beta$,

$$\delta\int_a^b dS = m_0 c\,\delta\int_a^b\left(\frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\right)dx^\beta\delta x^\alpha. \qquad (29)$$

This variation is zero, so along time lines we have

$$m_0 c\left(\frac{\partial b_\beta}{\partial x^\alpha} - \frac{\partial b_\alpha}{\partial x^\beta}\right)dx^\beta = 0\,. \qquad (30)$$

This condition, being divided by the interval ds, gives general covariant equations of motion of the particle

$$m_0 c\,F_{\alpha\beta}U^\beta = 0\,, \qquad (31)$$

wherein $F_{\alpha\beta}$ is the time density field tensor and U^β is the particle's four-dimensional velocity*.

Taking chr.inv.-projections of (31) multiplied by c^2, we obtain chr.inv.-equations of motion of the particle

$$m_0 c^3\frac{F_{0\sigma}U^\sigma}{\sqrt{g_{00}}} = 0\,, \qquad m_0 c^2 F^{i}_{\cdot\sigma}U^\sigma = 0\,, \qquad (32)$$

where the scalar equation gives the work to displace the particle, and the vector equations its observable acceleration.

It is interesting to note that the left side of the equations, which is the acting force, both in the general covariant form and its chr.inv.-projections we have obtained, has the same form as Lorentz's force, which displaces charged particles in electromagnetic fields [5]. From the mathematical viewpoint this fact implies that the time density field acts on mass-bearing particles as the electromagnetic field moves electric charge.

Taking $ds^2 = c^2d\tau^2 - d\sigma^2 = c^2d\tau^2\left(1 - \frac{\mathrm{v}^2}{c^2}\right)$, that is formula (4) into account, we obtain

$$U^\alpha = \frac{dx^\alpha}{ds} = \frac{1}{c\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\frac{dx^\alpha}{d\tau}\,, \qquad (33)$$

$$U^0 = \frac{\frac{1}{c^2}v_k\mathrm{v}^k + 1}{\sqrt{g_{00}}\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,, \qquad U^i = \frac{1}{c\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,\mathrm{v}^i. \qquad (34)$$

Using the components obtained for the field tensor $F_{\alpha\beta}$ (11–15) and taking into account that the observable velocity of the particle we are considering is $\mathrm{v}^i = 0$, we transform the chr.inv.-equations of motion (32) into the final form. The scalar equation becomes zero, while the vector equations become $m_0F^i = 0$ or, substituting $E^i = -\frac{1}{c^2}F^i$ (16),

$$m_0 c^2 E^i = 0\,, \qquad (35)$$

leading us to the following conclusions:

1. The "electric" and the "magnetic" components of a time density field do not produce work to displace a free mass-bearing particle along time lines. Such a particle falls freely along its own time line under the time density field;
2. In this case $E^i = 0$, so the particle falls freely along its own time line, being carryied solely by the "magnetic" component $H_{ik} = -\frac{2}{c}A_{ik} \neq 0$ of the time density field;
3. Inhomogeneity of the spatial section (the chr.inv.-Christoffel symbols Δ^i_{jk}) or its deformations (the chr.inv.-deformation rate tensor D_{ik}) do not have an effect on free motion along time lines.

*Do not confound this vector $U^\alpha = \frac{dx^\alpha}{ds}$ with the vector $b^\alpha = \frac{dx^\alpha}{ds}$: they are built on different dx^α. The vector b^α contains displacement of the observer with respect to his reference body, while the vector U^α contains displacement of the particle.

In other words, the "magnetic" component $H_{ik} = -\frac{2}{c} A_{ik}$ of a time density field "screws" particles into time lines (a very rough analogy). There are no other sources which could cause particles to move along time lines, because observable particles with the whole spatial section move from past into future, hence $H_{ik} \neq 0$ everywhere in our real world. So, our real space is strictly non-holonomic, $A_{ik} \neq 0$.

This purely mathematical result brings us to the very important conclusion that under any conditions a real space is non-holonomic at the "start", that is, a "primordial non-orthogonality" of the real spatial section to time lines. Conditions such as three-dimensional rotations of the reference body, are only additions, intensifying or reducing this start-rotation of the space, depending on their relative directions*.

We are now going to consider the second case of free motion — the general case, where a free mass-bearing particle moves freely not only along time lines, but also along the spatial section with respect to the observer and his reference body. Chr.inv.-equations of motion in this general case had been deduced by Zelmanov [2]. They have the form

$$\begin{aligned} &\frac{dE}{d\tau} - mF_i \mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = 0, \qquad E = mc^2 \\ &\frac{d(m\mathrm{v}^i)}{d\tau} - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0. \end{aligned} \tag{36}$$

Let us express the equations through the "electric" and the "magnetic" observable components of the acting field of time density. Substituting $E^i = -\frac{1}{c^2} F^i$ and $H_{ik} = -\frac{2}{c} A_{ik}$ into the Zelmanov equations (36), we obtain

$$\begin{aligned} &\frac{dE}{d\tau} + mc^2 E_i \mathrm{v}^i + mD_{ik}\mathrm{v}^i\mathrm{v}^k = 0, \\ &\frac{d(m\mathrm{v}^i)}{d\tau} + mc^2\left(E^i + \frac{1}{c} H^{ik}\mathrm{v}_k\right) + \\ &\qquad + 2mD^i_k\mathrm{v}^k + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0. \end{aligned} \tag{37}$$

From this we see that a free mass-bearing particle moves freely along the spatial section because of the factors:

1. The particle is carried with a time density field by its "electric" $E^i \neq 0$ and "magnetic" $H_{ik} \neq 0$ components;
2. The particle is also moved by forces which manifest as an effect of inhomogeneity Δ^i_{nk} and deformations D_{ik} of the spatial section. As we can see from the scalar equation, the field of the space inhomogeneities does not produce any work to displace free mass-bearing particles, only the space deformation field produces the work.

In particular, a mass-bearing particle can be moved freely along the spatial section, solely because of the field of time density. As it easy to see from equations (37), this is possible under the following conditions

$$D_{ik}\mathrm{v}^i\mathrm{v}^k = 0\,, \qquad D^i_k = -\frac{1}{2}\,\Delta^i_{nk}\mathrm{v}^n, \tag{38}$$

so it is possible in the following particular cases:

- if the spatial section has no deformations, $D_{ik} = 0$;
- if, besides the absence of the deformations ($D_{ik} = 0$), the spatial section is homogeneous, $\Delta^i_{nk} = 0$.†

The scalar equations of motion (37) also show that, under the particular conditions (38), the energy dE to displace the particle at dx^i equals the work

$$dE = -mc^2 E_i dx^i \tag{39}$$

the "electric" field component E_i expends for this displacement. The vector equations of motion in this particular case show that the "electric" and the "magnetic" components of the acting field of time density accelerate the particle just like external forces‡

$$\frac{dp^i}{d\tau} = -mc^2\left(E^i + \frac{1}{c} H^{ik}\mathrm{v}_k\right). \tag{40}$$

Looking at the right sides of equations (39, 40), we see that they have a form identical to the right sides of the chr.inv.-equations of motion of a charged particle in the electromagnetic field [5]. This implies also that the field of time density acts on mass-bearing particles as an electromagnetic field moves electric charge.

4 The field equations like electrodynamics

As is well-known, the theory of the electromagnetic field, in a pseudo-Riemannian space, characterizes the field by a system of equations known also as the *field equations*:

- Lorentz's condition stipulates that the four-dimensional vector potential A^α of the field remains unchanged just like any four-dimensional vector in a four-dimensional pseudo-Riemannian space

$$\nabla_\sigma A^\sigma = 0\,; \tag{41}$$

- the charge conservation law (the continuity equation) shows that the field-inducing charge cannot be destroyed, but merely re-distributed in the space

$$\nabla_\sigma j^\sigma = 0\,, \tag{42}$$

*A similar conclusion had also been given by the astronomer Kozyrev [7], from his studies of the interior of stars. In particular, besides the "start" self-rotation of the space, he had come to the conclusion that additional rotations will produce an inhomogeneity of observable time around rotating bulky bodies like stars or planets. The consequences should be more pronounced in the interaction of the components of bulky double stars [8]. He was the first to used the term "time density field". It is interesting that his arguments, derived from a purely phenomenological analysis of astronomical observations, did not link to Riemannian geometry and the mathematical apparatus of the General Theory of Relativity.

†However the first condition $D_{ik} = 0$ would be sufficient.

‡Here the chr.inv.-vector $p^i = m\mathrm{v}^i$ is the particle's observable impulse.

where j^α is the four-dimensional current vector; its observable projections are the chr.inv.-charge density scalar $\rho=\frac{1}{c\sqrt{g_{00}}}\,j_0$ and the chr.inv.-current density vector j^i, which are sources inducing the field;

- Maxwell's equations show properties of the field, expressed by components of the field tensor $F_{\alpha\beta}$ and its dual pseudotensor $F^{*\alpha\beta}$. The first group of the Maxwell equations contains the field sources ρ and j^i, the second group does not contain the sources

$$\nabla_\sigma F^{\alpha\sigma}=\frac{4\pi}{c}\,j^\alpha\,,\qquad \nabla_\sigma F^{*\alpha\sigma}=0\,. \tag{43}$$

We can put all the equations into chr.inv.-form, employing Zelmanov's formula [2] for the divergence of a vector Q^α, where he expressed the divergence through chr.inv.-projections $\varphi=\frac{Q_0}{\sqrt{g_{00}}}$ and $q^i=Q^i$ of this vector

$$\nabla_\sigma Q^\sigma=\frac{1}{c}\left(\frac{^*\partial\varphi}{\partial t}+\varphi D\right)+{}^*\nabla_i\,q^i-\frac{1}{c^2}F_i q^i\,, \tag{44}$$

where we use his notation for *chr.inv.-divergence*

$${}^*\nabla_i\,q^i=\frac{^*\partial q^i}{\partial x^i}+q^i\,\frac{^*\partial\ln\sqrt{h}}{\partial x^i}=\frac{^*\partial q^i}{\partial x^i}+q^i\Delta^j_{ji}\,. \tag{45}$$

In particular, the chr.inv.-Maxwell equations, which are chr.inv.-projections of the Maxwell general covariant equations (43), had first been obtained for an arbitrary field potential by del Prado and Pavlov [9], Zelmanov's students, at Zelmanov's request. The equations are

$$\left.\begin{aligned}&{}^*\nabla_i\,E^i-\frac{1}{c}\,H^{ik}A_{ik}=4\pi\rho\\&{}^*\nabla_k\,H^{ik}-\frac{1}{c^2}F_kH^{ik}-\frac{1}{c}\left(\frac{^*\partial E^i}{\partial t}+E^iD\right)=\frac{4\pi}{c}\,j^i\end{aligned}\right\}\text{I}, \tag{46}$$

$$\left.\begin{aligned}&{}^*\nabla_i\,H^{*i}-\frac{1}{c}\,E^{*ik}A_{ik}=0\\&{}^*\nabla_k\,E^{*ik}-\frac{1}{c^2}F_kE^{*ik}-\frac{1}{c}\left(\frac{^*\partial H^{*i}}{\partial t}+H^{*i}D\right)=0\end{aligned}\right\}\text{II}, \tag{47}$$

From the mathematical viewpoint, equations of the field are a system of 10 equations in 10 unknowns (the Lorentz condition, the charge conservation law, and two groups of the Maxwell equations), which define the given vector field A^α and its inducing sources in a pseudo-Riemannian space. Actually, equations like these should exist for any four-dimensional vector field, a time density field included. The only difference should be that the equations should be changed according to a formula for the specific vector potential.

We are going to deduce such equations for the field b^α we are considering — *equations of a time density field*.

Because $\varphi=1$ and $q^i=0$ are chr.inv.-projections of the potential b^α of a time density field, the *Lorentz condition* $\nabla_\sigma\,b^\sigma=0$ for a time density field b^α becomes the equality

$$D=0\,, \tag{48}$$

where $D=h^{ik}D_{ik}$, being the spur of the deformation rate tensor, is the rate of expansion of an elementary volume. Actually, the obtained Lorentz condition (48) implies that the value of an elementary volume filled with a time density field remains unchanged under its deformations.

We now collect chr.inv.-projections of the tensor of a time density field $F_{\alpha\beta}$ and of the field pseudotensor $F^{*\alpha\beta}$ together: $E_i=-\frac{1}{c^2}\,F_i$, $H^{ik}=-\frac{2}{c}\,A^{ik}$, $H^{*i}=-\frac{2}{c}\,\Omega^{*i}$, $E^{*ik}=$ $=\frac{1}{c^2}\,\varepsilon^{ikm}F_m$. We also take Zelmanov's identities for the chr.inv.-discriminant tensors [2] into account

$$\frac{^*\partial\varepsilon_{imn}}{\partial t}=\varepsilon_{imn}\,D\,,\qquad \frac{^*\partial\varepsilon^{imn}}{\partial t}=-\varepsilon^{imn}D\,, \tag{49}$$

$${}^*\nabla_k\,\varepsilon_{imn}=0\,,\qquad {}^*\nabla_k\,\varepsilon^{imn}=0\,. \tag{50}$$

Substituting the chr.inv.-projections into (46, 47) along with the obtained Lorentz condition $D=0$ (48), we arrive at *Maxwell-like chr.inv.-equations* for a time density field

$$\left.\begin{aligned}&\frac{1}{c^2}\,{}^*\nabla_i\,F^i-\frac{2}{c^2}A_{ik}A^{ik}=-\,4\pi\rho\\&\frac{2}{c}\,{}^*\nabla_k\,A^{ik}-\frac{2}{c^3}F_kA^{ik}-\frac{1}{c^3}\frac{^*\partial F^i}{\partial t}=-\frac{4\pi}{c}\,j^i\end{aligned}\right\}\text{I}, \tag{51}$$

$$\left.\begin{aligned}&{}^*\nabla_i\,\Omega^{*i}+\frac{1}{c^2}\,F_i\,\Omega^{*i}=0\\&{}^*\nabla_k\left(\varepsilon^{ikm}F_m\right)-\frac{1}{c^2}\,\varepsilon^{ikm}F_kF_m+2\,\frac{^*\partial\Omega^{*i}}{\partial t}=0\end{aligned}\right\}\text{II}, \tag{52}$$

so that the field-inducing sources ρ and j^i are

$$\rho=-\frac{1}{4\pi c^2}\left({}^*\nabla_i\,F^i-2A_{ik}A^{ik}\right), \tag{53}$$

$$j^i=-\frac{1}{2\pi}\,{}^*\nabla_k\,A^{ik}-\frac{1}{2\pi c^2}\,F_kA^{ik}-\frac{1}{4\pi c^2}\frac{^*\partial F^i}{\partial t}\,. \tag{54}$$

The "*charge*" *conservation law* $\nabla_\sigma\,j^\sigma=0$ (the continuity equation), after substituting chr.inv.-projections $\varphi=c\rho$ and $q^i=j^i$ of the "current" vector j^α, takes the chr.inv.-form

$$\begin{aligned}&\frac{1}{c^2}\frac{^*\partial}{\partial t}\left(A_{ik}A^{ik}\right)+\frac{1}{c^2}F_i\,\frac{^*\partial A^{ik}}{\partial x^k}-\frac{^*\partial^2A^{ik}}{\partial x^i\partial x^k}+\\&\quad+\left(\frac{1}{c^2}F_i\Delta^j_{jk}+\frac{^*\partial\Delta^j_{jk}}{\partial x^i}+\Delta^j_{ji}\Delta^l_{lk}\right)A^{ik}-\\&\quad-\frac{1}{2c^2}\,F^i\,\frac{^*\partial\Delta^j_{ji}}{\partial t}-\frac{1}{c^4}\,F_iF_kA^{ik}=0\,,\end{aligned} \tag{55}$$

The Lorentz condition (48), the Maxwell-like equations (51, 52), and the continuity equation (55) we have obtained are *chr.inv.-equations of a time density field*.

5 Waves of the field

Let us turn now to d'Alembert's equations. We are going to obtain the equations for a time density field.

d'Alembert's operator $\square = g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta$, being applied to a field, may or may not be zero. The second case is known as the d'Alembert equations with field-inducing sources, while the first case is known as the d'Alembert equations without sources. If the field has no sources, then the field is free. This is a wave. So, the d'Alembert equations without sources are equations of propagation of waves of the field.

From this reason, the d'Alembert equations for the vector potential b^α of a time density field without the sources

$$\square b^\alpha = 0 \qquad (56)$$

are the equations of propagation of waves of the time density field. Chr.inv.-projections of the equations are

$$b_\sigma \square b^\sigma = 0\,, \qquad h^i_\sigma \square b^\sigma = 0\,. \qquad (57)$$

We substitute chr.inv.-projections $\varphi = 1$ and $q^i = 0$ of the field potential b^α into this. Then, taking into account that the Lorentz condition for the field b^α is $D = 0$ (48), after some algebra we obtain the *chr.inv.-d'Alembert equations* for the time density field without sources

$$\begin{gathered} \frac{1}{c^2}\,F_iF^i - D_{ik}D^{ik} = 0\,, \\ \frac{1}{c^2}\frac{{}^*\partial F^i}{\partial t} + h^{km}\left\{\frac{{}^*\partial D^i_m}{\partial x^k} + \frac{{}^*\partial A_{m\cdot}^{\cdot i}}{\partial x^k} + \right. \\ \left. + \,\Delta^i_{kn}\left(D^n_m - A_{m\cdot}^{\cdot n}\right) - \Delta^n_{km}\left(D^i_n - A_{n\cdot}^{\cdot i}\right)\right\} = 0\,. \end{gathered} \qquad (58)$$

Unfortunately, a term like $\frac{1}{a^2}\frac{\partial^2 q^i}{\partial t^2}$ containing the linear speed a of the waves is not present, because of $q^i = 0$. For this reason we have no possibility of saying anything about the speed of waves traveling in time density fields. At the same time the obtained equations (58) display numerous specific peculiarities of a space filled with the waves:

1. The rate of deformations of a surface element in waves of a time density field is powered by the value of the acting gravitational inertial force F_i. If $F_i = 0$, the observable spatial metric h_{ik} is stationary;
2. If a space, filled with waves of a time density field, is homogeneous $\Delta^i_{kn} = 0$ and also the acting force field is stationary $F_i = \text{const}$, the spatial structure of the space deformations is the same as that of the space rotation field.

6 Energy-momentum tensor of the field

Proceeding from the general covariant equations of motion along only time lines, we are going to deduce the energy-momentum tensor for time density fields. It is possible to do this in the following way.

The aforementioned equations $m_0 c\,F_{\alpha\beta}\,U^\beta = 0$ (31), being taken in contravariant (upper-index) form, are

$$m_0 c F_{\cdot\sigma}^{\alpha\cdot}\,U^\sigma = 0\,, \qquad (59)$$

where U^σ is the four-dimensional velocity of the particle. The left side of the equations has the dimensions $[\text{gramme}/\text{sec}]$ as well as a four-dimensional force. Because of motion along only time lines, such particle moves solely under the action of a time density field whose tensor is $F_{\alpha\beta}$.

If this free-moving particle is not a point-mass, then it can be represented by a current j^α of the time density field. On the other hand, such currents are defined by the 1st group $\nabla_\sigma F^{\alpha\sigma} = \frac{4\pi}{c}\,j^\alpha$ of the Maxwell-like equations of the field. In this case equations of motion (59), drawing an analogy with an electromagnetic field current, take the form

$$\mu F_{\cdot\sigma}^{\alpha\cdot}\,j^\sigma = 0\,. \qquad (60)$$

The numerical coefficient μ here is a new fundamental constant. This new constant having the dimension $[\text{gramme}/\text{sec}]$ gives the dimensions $[\text{gramme}/\text{cm}^2\times\text{sec}^2]$ to the left side of the equations, making the left side a current of the acting four-dimensional force (59) through 1 cm^2 per 1 second. The numerical value of this constant μ can be found from measurements of the wave pressure of a time density field, see formula (101) below. However it does not exclude that future studies of the problem will yield an analytic formula for μ, linking it to other fundamental constants.

Chr.inv.-projections of the equations (60)

$$\frac{\mu F_{0\sigma}\,j^\sigma}{\sqrt{g_{00}}} = 0\,, \qquad \mu F_{\cdot\sigma}^{i\cdot}\,j^\sigma = 0\,, \qquad (61)$$

after substituting the $F_{\alpha\beta}$ components (11–15) take the form

$$\mu E_k\,j^k = 0\,, \qquad \mu c\left(\rho E^i - \frac{1}{c}\,H_{\cdot k}^{i\cdot}\,j^k\right) = 0\,, \qquad (62)$$

where E^i is the "electric" observable component and H_{ik} is the "magnetic" observable component of the time density field. Sources ρ and j^i inducing the field are defined by the 1st group of the Maxwell-like chr.inv.-equations (51).

Actually, the term*

$$f^\alpha = \mu F_{\cdot\sigma}^{\alpha\cdot}\,j^\sigma \qquad (63)$$

on the left side of the general covariant equations of motion (60) can be transformed with the 1st Maxwell-like group $\nabla_\beta F^{\sigma\beta} = \frac{4\pi}{c}\,j^\sigma$ to the form $f_\alpha = \frac{\mu c}{4\pi}\,F_{\alpha\sigma}\nabla_\beta F^{\sigma\beta}$ which is

$$f_\alpha = \frac{\mu c}{4\pi}\left[\nabla_\beta\left(F_{\alpha\sigma}F^{\sigma\beta}\right) - F^{\sigma\beta}\,\nabla_\beta F_{\alpha\sigma}\right], \qquad (64)$$

where we express the second term in the form $F^{\sigma\beta}\nabla_\beta F_{\alpha\sigma} = \frac{1}{2}\,F^{\sigma\beta}\left(\nabla_\beta F_{\alpha\sigma} + \nabla_\sigma F_{\beta\alpha}\right) = -\frac{1}{2}\,F^{\sigma\beta}\left(\nabla_\beta F_{\sigma\alpha} + \nabla_\sigma F_{\alpha\beta}\right) = -\frac{1}{2}\,F^{\sigma\beta}\nabla_\sigma F_{\alpha\beta} = \frac{1}{2}\,F^{\sigma\beta}\nabla_\alpha F_{\sigma\beta}$. Using this formula, we transform the current f^α (63) to the form

$$f_\alpha = \frac{\mu c}{4\pi}\,\nabla_\beta\left(-F_{\alpha\sigma}F^{\beta\sigma} + \frac{1}{4}\,\delta^\beta_\alpha F_{pq}F^{pq}\right), \qquad (65)$$

*From the physical viewpoint, this term is a current of the acting four-dimensional force, produced by the time density field.

so we write the current f^α in the form

$$f^\alpha = \nabla_\beta T^{\alpha\beta} \qquad (66)$$

just as electrodynamics does to deduce the energy-momentum tensor $T^{\alpha\beta}$ of electromagnetic fields. In this way, we obtain the *energy-momentum tensor* of a time density field, which is

$$T^{\alpha\beta} = \frac{\mu c}{4\pi}\left(-F^{\alpha\cdot}_{\cdot\sigma} F^{\beta\sigma} + \frac{1}{4}\, g^{\alpha\beta} F_{pq} F^{pq}\right), \qquad (67)$$

the form of which is the same as the energy-momentum tensor of electromagnetic fields [1, 5] to within the coefficient of its dimension. It is easy to see that the tensor is symmetric, so its spur is zero, $T^\sigma_\sigma = g_{\alpha\beta} T^{\alpha\beta} = 0$.

So forth we deduce the chr.inv.-projections of the energy-momentum tensor of a time density field

$$q = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\,T^i_0}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik}. \qquad (68)$$

After substituting the required components of the field tensor $F_{\alpha\beta}$ (11–15), we obtain

$$q = \frac{\mu}{4\pi c}\left(A_{ik}A^{ik} + \frac{1}{2c^2} F_k F^k\right), \qquad (69)$$

$$J^i = -\frac{\mu}{2\pi c} F_k A^{ik}, \qquad (70)$$

$$U^{ik} = -\frac{\mu c}{4\pi}\Bigg(4A^{i\cdot}_{\cdot m} A^{mk} + \frac{1}{c^2} F^i F^k + A_{pq}A^{pq} h^{ik} - \frac{1}{2c^2} F_p F^p h^{ik}\Bigg). \qquad (71)$$

In accordance with dimensions, the chr.inv.-projections have the following physical meanings:

- the time observable projection q [gramme/cm×sec²] is the energy [gm×cm²/sec²] this time density field contains in 1 cm³. Actually, the chr.inv.-scalar q is the *observable density of the field*;
- the mixed observable projection J^i [gramme/sec³] is the energy the time density field transfers through 1 cm² per second, in other words, this is the *observable density of the field momentum*;
- the spatial observable projection U^{ik} [gm×cm/sec⁴] is the tensor of the field momentum flux observable density, in other words, the *field strength tensor*.

7 Physical properties of the field

It has been proven by Zelmanov [10], that the chr.inv.-field strength tensor U^{ik}, can be written in covariant (lower index) form as follows

$$U_{ik} = p_0 h_{ik} - \alpha_{ik} = p h_{ik} - \beta_{ik}\,, \qquad (72)$$

where $\alpha_{ik} = \beta_{ik} + \frac{1}{3}\,\alpha h_{ik}$ is the viscous strength tensor of the field. Zelmanov called α_{ik} the *viscosity of the 2nd kind* (here $\alpha = h^{ik}\alpha_{ik} = \alpha^n_n$ is its spur). Its anisotropic part β_{ik}, called the *viscosity of the 1st kind*, manifests as anisotropic deformations of the space. The quantity p_0 is that pressure inside the medium, which equalizes its density in the absence of viscosity, p is the true pressure of the medium*. It is easy to see that the viscous strength tensors α_{ik} and β_{ik} are chr.inv.-quantities by their definitions.

By extracting the viscous strength tensors α_{ik} and β_{ik} from the formula of the strength tensor U_{ik} of a time density field, we are going to deduce the equation of state of the field.

Transforming U^{ik} (71) into covariant form and also keeping the formula for q (69) in the mind, we write

$$U_{ik} = -qc^2 h_{ik} - \frac{\mu c}{4\pi}\left(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2} F_i F_k - \frac{1}{c^2} F_m F^m h_{ik}\right), \qquad (73)$$

which, after equating to $U_{ik} = p_0 h_{ik} - \alpha_{ik}$ (72), gives the equilibrium pressure in the field

$$p_0 = -qc^2, \qquad (74)$$

while the *viscous strength tensor* of the field is

$$\alpha_{ik} = \frac{\mu c}{4\pi}\left(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2} F_i F_k - \frac{1}{c^2} F_m F^m h_{ik}\right). \qquad (75)$$

Because the spur of this tensor α_{ik}, as it is easy to see, is not zero, $\alpha = h^{ik}\alpha_{ik} = -\frac{\mu c}{\pi}\left(A_{ik}A^{ik} + \frac{1}{2c^2} F_k F^k\right) \neq 0$, the tensor $\alpha_{ik} = \beta_{ik} + \frac{1}{3}\,\alpha h_{ik}$ has the non-zero anisotropic part

$$\beta_{ik} = \frac{\mu c}{4\pi}\Bigg(4A_{im}A^{m\cdot}_{\cdot k} + \frac{1}{c^2} F_i F_k - \frac{1}{3c^2} F_m F^m h_{ik} + \frac{4}{3} A_{mn}A^{mn} h_{ik}\Bigg), \qquad (76)$$

so viscous strengths of time density fields are anisotropic. It is also easy to see that this anisotropy increases with the value $A_{pq}A^{pq}$ of the space rotation.

Because the viscous strengths α_{ik} are anisotropic, the equilibrium pressure $p_0 = -qc^2$ and the true pressure p inside the medium are different. The true pressure is

$$p = \frac{\mu c}{12\pi}\left(A_{ik}A^{ik} + \frac{1}{2c^2} F_k F^k\right), \qquad (77)$$

*The equation of state of a medium is the relation between the pressure p inside the medium and its density q. In a non-viscous medium or where the viscous strengths are isotropic, the true pressure p is the same as the equilibrium pressure p_0. The equation of state of a dust medium has the form $p = 0$. Ultra-relativistic gases have the equation of state $p = \frac{1}{3}qc^2$. The equation of state of matter inside atomic nuclei is $p = qc^2$. Vacuum and μ-vacuum have the equation of state $p = -qc^2$, see [5].

which gives the *equation of state* for time density fields

$$p = \frac{1}{3}\,qc^2. \qquad (78)$$

Finally, we write the strength tensor $U_{ik} = ph_{ik} - \beta_{ik}$ of a time density field in the form

$$U_{ik} = \frac{1}{3}\,qc^2 h_{ik} - \beta_{ik}\,. \qquad (79)$$

So, we can conclude for the physical properties of time density fields:

1. In general, a time density field is a non-stationary distributed medium, because its density may be $q \neq \text{const}$. The field becomes stationary $q = \text{const}$ under stationary space rotation $A_{ik} = \text{const}$, and stationary gravitational inertial force $F_i = \text{const}$;
2. A time density field bears momentum, because $J^i = -\frac{\mu}{2\pi c}\,F_k A^{ik} \neq 0$. So, the field can transfer impulse. The field does not transfer impulse $J^i = 0$, if the space does not rotate $A_{ik} = 0$. The absence of gravitation does not affect the field's transfer of impulse, because the "inertial" part of the force F_i remains unchanged even in the absence of gravitational fields;
3. A time density field is an emitting medium $J^i \neq 0$ in a non-holonomic (rotating) space. In a holonomic (non-rotating) space the field does not produce radiations;
4. A time density field is a viscous medium. The viscosity α_{ik} (75), derived from non-zero rotation of the space or from gravitational inertial force, is anisotropic. The anisotropy β_{ik} increases with the space rotation speed. The field is viscous anisotropic anyhow, because its viscous strengths would be $\alpha_{ik} = 0$ and $\beta_{ik} = 0$ only if both $A_{ik} = 0$ and $F_i = 0$. But in this case the field density would be $q = 0$, so the field itself is not there;
5. Therefore the equilibrium pressure p_0 does not possess a physical sense for time density fields; only the true pressure is real $p = p_0 - \frac{1}{3}\alpha$;
6. The equation of state for time density fields is $p = \frac{1}{3}qc^2$ (78) indicating that such fields are in the *state of an ultrarelativistic gas* — at positive density of the medium its inner pressure becomes positive, the medium is compressed.

8 Action of the field without sources

According to §27 of *The Classical Theory of Fields* [1], an elementary action for a whole system consisting of an electromagnetic field and a single charged particle, which are located in a pseudo-Riemannian space, contains three parts*

$$dS = dS_{\text{m}} + dS_{\text{mf}} + dS_{\text{f}} = \\ = m_0 c\, ds + \frac{e}{c}\,\mathcal{A}_\alpha dx^\alpha + a\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} dV\,dt\,, \qquad (80)$$

where $\mathcal{A}^\alpha$ is the four-dimensional electromagnetic field potential, $\mathcal{F}_{\alpha\beta} = \nabla_\alpha \mathcal{A}_\beta - \nabla_\beta \mathcal{A}_\alpha$ is the electromagnetic field tensor, $dV = dx\,dy\,dz$ is an elementary thee-dimensional volume filled with this field.

The first term S_{m} is "that part of the action which depends only on the properties of the particles, that is, just the action for free particles. . . . The quantity S_{mf} is that part of the action which depends on the interaction between the particles and the field. . . . Finally S_{f} is that part of the action which depends only on the properties of the field itself, that is, S_{f} is the action for a field in the absence of charges".

Because the action S_{f} must depend only on the field properties, the action must be taken over the space volume, filled with the field. The action must be scalar: only the 1st field invariant $J_1 = \mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta}$ has this property. The 2nd field invariant $J_2 = \mathcal{F}_{\alpha\beta}\,\mathcal{F}^{*\alpha\beta}$ is pseudoscalar, not scalar, leading to the detailed discussion in Landau and Lifshitz.

"The numerical value of a depends on the choice of units for measurement of the field. . . . From now on we shall use the Gaussian system of units; in this system a is a dimensionless quantity equal to $\frac{1}{16\pi}$".

According to §27 of *The Classical Theory of Fields* we have $dS_{\text{f}} = a\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} dV\,dt = \frac{1}{16\pi c}\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} d\Omega$, where $d\Omega = c\,dt\,dV = c\,dt\,dx\,dy\,dz$ is an elementary space (four-dimensional) volume. So the action (80) takes the final form

$$dS = m_0 c\, ds + \frac{e}{c}\,\mathcal{A}_\alpha dx^\alpha + \frac{1}{16\pi c}\,\mathcal{F}_{\alpha\beta}\,\mathcal{F}^{\alpha\beta} d\Omega\,. \qquad (81)$$

According to this consideration, we write an elementary action for the whole system consisting of a time density field and a single mass-bearing particle, which falls freely along time lines in a pseudo-Riemannian space, as follows

$$dS = dS_{\text{m}} + dS_{\text{mt}} = m_0 c\, ds + a_{\text{mt}}\,F_{\alpha\beta}\,F^{\alpha\beta} d\Omega = \\ = m_0 c\, b_\alpha dx^\alpha + a_{\text{mt}}\,F_{\alpha\beta}\,F^{\alpha\beta} d\Omega\,, \qquad (82)$$

where $F_{\alpha\beta}$ is the time density field tensor, a_{mt} is a constant consisting of other fundamental constants.

The first term S_{m} is that part of the action for the interaction between the particle and the time density field carrying it into motion along time lines. The second term

*In accordance with the least action principle, this action must have a minimum, so the integral of the action between a pair of world-points and the action itself must be positive. A negative action could give rise to a quantity with arbitrarily "large" negative values, which cannot have a minimum. Because in *The Classical Theory of Fields* Landau and Lifshitz take a pseudo-Riemannian space with the signature $(-+++)$, they write in §3 that ". . . the clock at rest always indicates a greater time interval than the moving one". Therefore they put "minus" before the action. To the contrary, we stick to a pseudo-Riemannian space with Zelmanov's signature $(+---)$, because in this case three-dimensional observable impulse is positive. In a space with such a signature, a regular observer takes his own flow of observable time positive always, $d\tau > 0$. Any particle, moving from past into future, has also a positive count of its own time coordinate $dt > 0$ with respect to the observer's clock. Therefore we put "plus" before the action.

$S_{\rm mt}$, depending only on the field properties, is the action for the field in the absence of its sources. In the absence of time density fields the second term $S_{\rm mt}$ is zero, so only $S_{\rm m}=m_0cds$ remains here. A time density field is absent if the space is free of rotation $A_{ik}=0$ and gravitational inertial forces $F_i=0$, therefore if the conditions $g_{0i}=0$ and $g_{00}=1$ are true. This situation is possible in a pseudo-Riemannian space with a unit diagonal metric, which is the Minkowski space of the Special Theory of Relativity, where there is no gravitational field and no rotation. But in considering real space, we are forced to take a time density field into account. So we need to consider the terms $S_{\rm m}$ and $S_{\rm mt}$ together.

The constant $a_{\rm mt}$, according to its dimension, is the same as the constant μ in the energy-momentum tensor of time density fields, taken with the numerical coefficient $a=\frac{1}{16\pi}$, in the Gaussian system of units.

As a result, we obtain the action (82) in the final form

$$dS = dS_{\rm m}+dS_{\rm mt} = m_0c\,b_\alpha dx^\alpha + \frac{\mu}{16\pi}F_{\alpha\beta}F^{\alpha\beta}d\Omega\,. \quad (83)$$

Because an action for a system is expressed through Lagrange's function L of the system as $dS=Ldt$, we take the action $dS_{\rm mt}$ in the form $dS_{\rm mt}=\frac{\mu c}{16\pi}F_{\alpha\beta}F^{\alpha\beta}dV dt=Ldt$, for the Lagrangian of an elementary volume $dV=dxdydz$ of the field. We therefore obtain the *Lagrangian density* in time density fields

$$\Lambda = \frac{\mu c}{16\pi}F_{\alpha\beta}F^{\alpha\beta} = \frac{\mu}{4\pi c}\left(A_{ik}A^{ik}-\frac{1}{2c^2}F_iF^i\right). \quad (84)$$

The term $A_{ik}A^{ik}$ here, being expressed through the space rotation angular velocity pseudovector Ω^{*i}, is

$$A_{km}A^{km} = \varepsilon_{kmn}\Omega^{*n}A^{km} = 2\Omega_{*n}\Omega^{*n}, \quad (85)$$

because $\varepsilon_{nkm}\Omega^{*n}=\frac{1}{2}\varepsilon^{npq}\varepsilon_{nkm}A_{pq}=\frac{1}{2}\left(\delta^p_k\delta^q_m-\delta^q_k\delta^p_m\right)A_{pq}= =A_{km}$ and $\Omega_{*n}=\frac{1}{2}\varepsilon_{nkm}A^{km}$. So the space rotation plays the first violin, defining the Lagrangian density in time density fields. Rotation velocities in macro-processes are incommensurably small in comparison with rotations of atoms and particles. For instance, in the 1st Bohr orbit in an atom of hydrogen, measuring the value of Λ in the units of the energy-momentum constant μ, we have $\Lambda\simeq 9.1\times10^{21}\mu$. On the Earth's surface near the equator the value is $\Lambda\simeq 2.8\times10^{-20}\mu$, so it is in order of 10^{42} less than in atoms. Therefore, because the Lagrangian of a system is the difference between its kinetic and potential energies, we conclude that time density fields produce their main energy flux in atoms and sub-atomic interactions, while the energy flux produced by the fields of macro-processes is negligible.

9 Plane waves of the field under gravitation is neglected. The wave pressure

In general, because the electric and the magnetic strengths of a time density field are $E_i=-\frac{1}{c^2}F_i$ and $H^{ik}=-\frac{1}{c}A^{ik}$, the chr.inv.-vector of its momentum density J^i (70) can be written as follows

$$J^i = -\frac{\mu}{2\pi c}F_kA^{ik} = -\frac{\mu c}{4\pi}E_kH^{ik}. \quad (86)$$

We are going to consider a particular case, where the field depends on only one coordinate. Waves of such a field traveling in one direction are known as *plane waves*.

We assume the field depends only on the axis $x^1=x$, so only the component $J^1=-\frac{\mu}{2\pi c}F_kA^{1k}$ of the field's chr.inv.-momentum density vector is non-zero. Then a plane wave of the field travels along the axis $x^1=x$. Assuming the space rotating in xy plane (only the components $A^{12}=-A^{21}$ are non-zeroes) and replacing the tensor A^{ik} with the space rotation angular velocity pseudovector Ω_{*m} in the form $\varepsilon^{mik}\Omega_{*m}=\frac{1}{2}\varepsilon^{mik}\varepsilon_{mpq}A^{pq}=\frac{1}{2}\left(\delta^i_p\delta^k_q-\delta^k_p\delta^i_q\right)A^{pq}=A^{ik}$, we obtain

$$J^1 = -\frac{\mu}{2\pi c}F_2A^{12} = -\frac{\mu}{2\pi c}F_2\,\varepsilon^{123}\Omega_{*3}\,. \quad (87)$$

It is easy to see that while a plane wave of the field travels along the axis $x^1=x$, the field's "electric" and "magnetic" strengths are directed along the axes $x^2=y$ and $x^3=z$, i. e. orthogonal to the direction the wave travels. Therefore waves travelling in time density fields are *transverse waves*.

Following the arguments of Landau and Lifshitz in §47 of *The Classical Theory of Fields* [1], we define the *wave pressure* of a field as the total flux of the field energy-momentum, passing through a unit area of a wall. So the pressure $\mathfrak{F}_i$ is the sum

$$\mathfrak{F}_i = T_{ik}n^k + T'_{ik}n^k \quad (88)$$

of the spatial components of the energy-momentum tensor $T_{\alpha\beta}$ in a wave, falling on the wall, and of the energy-momentum tensor $T'_{\alpha\beta}$ in the reflected wave, projected onto the unit spatial vector $\vec{n}_{(k)}$ orthogonal to the wall surface.

Because the chr.inv.-strength tensor of a field is $U_{ik}= =c^2h_{i\alpha}h_{k\beta}T^{\alpha\beta}=c^2T_{ik}$ [2], we obtain

$$\mathfrak{F}_i = \frac{1}{c^2}\left(U_{ik}n^k + U'_{ik}n^k\right), \quad (89)$$

where $U_{ik}=c^2T_{ik}$ and $U'_{ik}=c^2T'_{ik}$ are the chr.inv.-strength tensors in the falling wave and in the reflected wave. So the three-dimensional wave pressure vector $\mathfrak{F}_i$ has the property of chronometric invariance.

Using our formulae for the density q (68) and the strength tensor U_{ik} (71) obtained for time density fields, we are going to find the pressure a wave of such field exerts on a wall.

We consider the problem in a weak gravitational field, assuming its potential w and the attracting force of gravity negligible. We can do this because formulae (68) and (71) contain gravitation in only higher order terms. So the space rotation plays the first violin in the wave pressure $\mathfrak{F}_i$ in time density fields, gravitational inertial forces act there only because of their inertial part.

A plane wave travels along a single spatial direction: we assume axis $x^1 = x$. In this case the chr.inv.-field strength tensor U_{ik} has the sole non-zero component U_{11}. All the other components of the strength tensor U_{ik} are zero, which simplifies this consideration.

We assume the space rotating around the axis $x^3 = z$ (the rotation is in the xy-plane) at a constant angular velocity Ω. In this case $A_{12} = -A_{21} = -\Omega$, $A_{13} = 0$, $A_{23} = 0$, so the components of the rotation linear velocity $v_i = A_{ik}x^k$ are $v_1 = -\Omega y$, $v_2 = \Omega x$, $v_3 = 0$. Then the components of the acting gravitational inertial force will be $F_1 = -\frac{\partial v_1}{\partial t} = \Omega \frac{\partial y}{\partial t} = \Omega v_2 = \Omega^2 x$, $F_2 = -\frac{\partial v_2}{\partial t} = \Omega^2 y$, $F_3 = 0$. Because in this case $A_{ik}A^{ik} = 2A_{12}A^{12} = 2\Omega^2$ and $A_{1m}A_{\cdot 1}^{m\cdot} = A_{1m}A^{mn}h_{1n} = A_{12}A^{21}h_{11} = -\Omega^2 h_{11}$, we obtain

$$q = \frac{\mu}{4\pi c}\left[2\Omega^2 + \frac{1}{2c^2}\,\Omega^4\left(x^2+y^2\right)\right], \tag{90}$$

$$U_{11} = \frac{\mu c}{4\pi}\left[2\Omega^2 h_{11} - \frac{1}{c^2}\,\Omega^4 x^2 + \frac{1}{2c^2}\,\Omega^4\left(x^2+y^2\right)h_{11}\right]. \tag{91}$$

We assume a coefficient of the reflection $\mathfrak{R}$ as the ratio between the density of the field energy q' in the reflected wave to the energy density q in the falling wave. Actually, because $q' = \mathfrak{R}q$, the reflection coefficient $\mathfrak{R}$ is the energy loss of the field after the reflection.

We assume $x = x_0 = 0$ at the reflection point on the surface of the wall. Then we have $U_{11} = qc^2 h_{11}$, which, after substituting into (89), gives the pressure

$$\mathfrak{F}_1 = (1+\mathfrak{R})\,q\,h_{11}n^1 \tag{92}$$

that a plane wave of a time density field exerts on the wall.

To bring this formula into final form in a Riemannian space becomes a problem, because the coordinate axes are curved there, and inhomogeneous. For this reason we cannot define the angles between directions in a Riemannian space itself, the angle of incidence and the angle of reflection of a wave for instance. At the same time, to consider this problem in the Minkowski space of the Special Theory of Relativity, as done by Landau and Lifshitz for the pressure of plane electromagnetic waves [1], would be senseless — because in Minkowski space we have $g_{00} = 1$ and $g_{0i} = 0$, then $F_i = 0$ and $A_{ik} = 0$, which implies no time density fields there.

To solve this problem correctly for a Riemannian space, let us introduce a *locally geodesic reference frame*, following Zelmanov. We therefore introduce a locally geodesic reference frame at the point of reflection of a wave on the surface of a wall. Within infinitesimal vicinities of any point of such a reference frame the fundamental metric tensor is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{\alpha\beta}}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \ldots, \tag{93}$$

i. e. its components at a point, located in the vicinities, are different from those at the point of reflection to within only the higher order terms, values of which can be neglected. Therefore, at any point of a locally geodesic reference frame the fundamental metric tensor can be considered constant, while the first derivatives of the metric (the Christoffel symbols) are zero.

As a matter of fact, within infinitesimal vicinities of any point located in a Riemannian space, a locally geodesic reference frame can be set up. At the same time, at any point of this locally geodesic reference frame a tangential flat Euclidean space can be set up so that this reference frame, being locally geodesic for the Riemannian space, is the global geodesic for that tangential flat space.

The fundamental metric tensor of a flat Euclidean space is constant, so the values of $\tilde{g}_{\mu\nu}$, taken in the vicinities of a point of the Riemannian space, converge to the values of the tensor $g_{\mu\nu}$ in the flat space tangential at this point. Actually, this means that we can build a system of basis vectors $\vec{e}_{(\alpha)}$, located in this flat space, tangential to curved coordinate lines of the Riemannian space.

In general, coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other (if the space is non-holonomic). So the lengths of the basis vectors may sometimes be very different from unity.

We denote a four-dimensional vector of infinitesimal displacement by $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$, so that $d\vec{r} = \vec{e}_{(\alpha)}dx^\alpha$, where components of the basis vectors $\vec{e}_{(\alpha)}$ tangential to the coordinate lines are $\vec{e}_{(0)} = \{e^0_{(0)}, 0, 0, 0\}$, $\vec{e}_{(1)} = \{0, e^1_{(1)}, 0, 0\}$, $\vec{e}_{(2)} = \{0, 0, e^2_{(2)}, 0\}$, $\vec{e}_{(3)} = \{0, 0, 0, e^2_{(3)}\}$. The scalar product of the vector $d\vec{r}$ with itself is $d\vec{r}d\vec{r} = ds^2$. On the other hand, the same quantity is $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$. As a result we have $g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)}e_{(\beta)}\cos\left(x^\alpha; x^\beta\right)$. So we obtain

$$g_{00} = e^2_{(0)}\,, \qquad g_{0i} = e_{(0)}e_{(i)}\cos\left(x^0; x^i\right), \tag{94}$$

$$g_{ik} = e_{(i)}e_{(k)}\cos\left(x^i; x^k\right). \tag{95}$$

The gravitational potential is $\mathrm{w} = c^2(1-\sqrt{g_{00}})$. So the time basis vector $\vec{e}_{(0)}$ tangential to the time line $x^0 = ct$, having the length $e_{(0)} = \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}$, is smaller than unity the greater is the gravitational potential w.

The space rotation linear velocity v_i and, according to it, the chr.inv.-metric tensor h_{ik} are

$$v_i = -c\,e_{(i)}\cos\left(x^0; x^i\right), \tag{96}$$

$$h_{ik} = e_{(i)}e_{(k)}\Big[\cos(x^0; x^i)\cos(x^0; x^k) - \cos(x^i; x^k)\Big]. \tag{97}$$

Harking back to the formula for the pressure $\mathfrak{F}_1$ (92), that a plane wave of a time density field traveling along the axis $x^1 = x$ exerts on a wall, we have

$$\mathfrak{F}_1 = (1+\mathfrak{R})\,q\Big[\cos^2(x^0; x^1) + 1\Big]\,n_{(1)}e^2_{(1)}\cos(x^1; n^1), \tag{98}$$

because according to the signature $(+---)$, the spatial coordinate axes in the pseudo-Riemannian space are directed opposite to the same axes x^i in the tangential flat Euclidean space.

We denote $\cos(x^1; n^1) = \cos\theta$, where θ is the angle of reflection. Assuming $e_{(1)} = 1$, $n_{(1)} = 1$, $v_{(1)} = v$ we obtain the field density $q = \frac{\mu}{2\pi c}\Omega^2\left(1+\frac{v^2}{4c^2}\right)$, so that the wave pressure $\mathfrak{F}_{\text{N}} = \mathfrak{F}_1 \cos\theta$ normal to the wall surface is

$$\mathfrak{F}_{\text{N}} = (1 + \mathfrak{R})\left(1 + \frac{v^2}{c^2}\right) q \cos^2\theta \,, \tag{99}$$

which, for low rotational velocities gives*

$$\mathfrak{F}_{\text{N}} = (1 + \mathfrak{R})\, q \cos^2\theta \,, \qquad q = \frac{\mu}{2\pi c}\,\Omega^2. \tag{100}$$

Most of rotations we observe are slow. The maximum of the known velocities is that for an electron in the 1st Bohr orbit ($v_{\text{b}} = 2.18\times10^8$ cm/sec). Therefore the ratio $\frac{v^2}{c^2}$, taking reaches a maximum numerical value of only 5.3×10^{-5}.

The presence of wave pressure in time density fields provides a way of measuring the numerical value of the energy-momentum constant μ, specific for such fields. For instance, a gyroscope, rotating around the axis $x^3 = z$, will be a source of circular waves of the field of time density propagating in the xy-plane. In this case the chr.inv.-field strength tensor U_{ik} has the non-zero components U_{11}, U_{12}, U_{21}. It is easy to calculate that the normal wave pressure of a circular wave will be different from the pressure of a plane wave (99) in only higher order terms. The same situation applies for spherical waves†. Therefore the normal pressure exerted by the waves on a wall orthogonal to the direction $x^1 = x$, shall be

$$\mathfrak{F}_{\text{N}} = \frac{\mu}{2\pi c}\,(1 + \mathfrak{R})\,\Omega^2 \tag{101}$$

to within the higher order terms withheld. Rotations at 6×10^3 rpm ($\Omega = 100$ rps) are achievable in modern gyroscopes, rotations in atoms are much greater, taking their maximum angular velocity to 4.1×10^{16} rps in the 1st Bohr orbit. A torsion balance registers forces, values of which are about 10^{-5} dynes. Then in accordance with the formula (101), if the wave pressure in an experiment is $\mathfrak{F}_{\text{N}} \approx 10^{-5}$ din/cm², derived from atomic transformations, the constant's numerical value will be in the order of $\mu \approx 10^{-28}$ gramme/sec.

Of course this is a crude supposition, based on the precision limits of measurement. Anyhow, the exact numerical value of the energy-momentum constant μ will be ascertained from special measurements with a torsion balance.

*Formula (100) is the same as $\mathfrak{F}_{\text{N}} = (1+\mathfrak{R})\, q \cos^2\theta$ — the normal pressure exerted by a plane electromagnetic wave in Minkowski space, (see §47 in *The Classical Theory of Fields* [1]). So the wave pressure of a time density field depends on the reflection coefficient $0 \leqslant \mathfrak{R} \leqslant 1$ in the same way as the pressure of electromagnetic waves.

†In a real experiment such a gyroscope, being an arbitrarily thin disc, will be a source of spherical waves of a time density field which propagates in all spatial directions. The waves will merely have a maximum amplitude in the gyroscope's rotation plane xy.

10 Physical conditions in atoms

So we have obtained formulae for chr.inv.-projections of the energy-momentum tensor of time density fields, which are physically observable characteristics of such fields — the energy density q (69), the momentum density J^i (70), and the strength tensor U_{ik} (71).

The formulae must be valid everywhere, the inside of atoms included. At the same time, physical conditions in atoms are subject to Bohr's quantum postulates. For an external observer, an atom can be represented as a tiny gyroscope, the rotations of which are ruled by the quantum laws. The quantised rotations of electrons are sources of a time density field, which shall be perceptible, because of the super-rapid angular velocities up to the maximum value in the 1st Bohr orbit $\Omega_{\text{b}} = 4.1\times10^{16}$ rps. This is a way of formulating the physical conditions under which a time density field exists in atoms.

Taking the above into account, we formulate the physical conditions with postulates, which result from the application of Bohr's postulates to a time density field in atoms.

POSTULATE I *A time density field in an atom remains unchanged in the absence of external influences. The atom radiates or absorbs waves of the time density field only in transitions of the electrons between their stationary orbits.*

Naturally, when an atom is in a stable state, all its electrons are located in their orbits. Such a stable atom, having a set of quantum orbital angular velocities, must possess numerous quantum states of the time density field. The quantum states are set up with the second postulate‡.

POSTULATE II *A time density field is quantised in atoms. Its energy density and the momentum density take quantum numerical values which, in accordance with the quantization of electron orbits, in n-th stationary orbit are*

$$q_{\text{n}} = \frac{\mu}{2\pi c}\left(1 + \frac{v_{\text{n}}^2}{4c^2}\right)\frac{v_{\text{n}}^2}{R_{\text{n}}^2}\,, \tag{102}$$

‡To introduce the second postulate we assume a reference frame in an atom, where an electron rotates around the nucleus at the angular velocity Ω in the xy-plane. Then $A_{12} = -A_{21} = -\Omega$, $A_{13} = 0$, $A_{23} = 0$. So out of all components of Ω^{*i} only Ω^{*3} is non-zero: $\Omega^{*3} = \frac{1}{2}\varepsilon^{3mn}A_{mn} = \frac{1}{2}\left(\varepsilon^{312}A_{12} + \varepsilon^{321}A_{21}\right) = \varepsilon^{312}A_{12} = \frac{e^{312}}{\sqrt{h}}A_{12} = -\frac{\Omega}{\sqrt{h}}$ and $\Omega_{*3} = \frac{1}{2}\varepsilon_{3mn}A^{mn} = \varepsilon_{312}A^{12} = e_{312}\sqrt{h}\,A_{12} = -\sqrt{h}\,\Omega$. In calculating $h = \det\|h_{ik}\|$, it should be noted that the components of the space rotation linear velocity $v_i = A_{ik}x^k$ in this reference frame are $v_1 = -\Omega y$, $v_2 = \Omega x$, $v_3 = 0$. We obtain $h_{11} = 1 + \frac{1}{c^2}\Omega^2 y^2$, $h_{22} = 1 + \frac{1}{c^2}\Omega^2 x^2$, $h_{12} = -\frac{1}{c^2}\Omega^2 xy$, $h_{33} = 1$. Then $h = \det\|h_{ik}\| = h_{11}h_{22} - (h_{12})^2 = 1 + \frac{1}{c^2}\Omega^2(x^2 + y^2)$. In the 1st Bohr orbit we have $\frac{1}{c^2}\Omega^2(x^2 + y^2) = \frac{1}{c^2}\Omega^2 R^2 = 5.3\times10^{-5}$, so we can set $h \approx 1$ to within the higher order terms withheld. Harking back to the formulae for Ω^{*3} and Ω_{*3}, we see that the space rotates in atoms at a constant angular velocity $\Omega^{*3} = -\Omega$, $\Omega_{*3} = -\Omega$, then in the assumed reference frame we have $A_{ik}A^{ik} = 2A_{12}A^{12} = 2\Omega_{*3}\Omega^{*3} = 2\Omega^2$, and also $F_1 = -\frac{\partial v_1}{\partial t} = \Omega^2 x$, $F_2 = -\frac{\partial v_2}{\partial t} = \Omega^2 y$, $F_3 = 0$, which is taken into account in Postulate II.

$$J_{\rm n} = \sqrt{(J_i J^i)_{\rm n}} = \frac{\mu}{2\pi c}\,\Omega_{\rm n}^3 R_{\rm n} = \frac{\mu}{2\pi c}\frac{v_{\rm n}^3}{R_{\rm n}^2}\,. \tag{103}$$

Calculating the field density in neighbouring levels n and n+1, we take into account that the n-th orbital radius relates to the 1st Bohr radius as $R_{\rm n} = {\rm n}^2 R_{\rm b}$. As a result we obtain

$$\overline{q} = q_{\rm n} - q_{\rm n+1} = = \frac{\mu}{2\pi c}\,\Omega_{\rm b}^2 \left\{\left[\frac{1}{{\rm n}^6} - \frac{1}{({\rm n}+1)^6}\right] + \frac{v_{\rm b}^2}{4c^2}\left[\frac{1}{{\rm n}^8} - \frac{1}{({\rm n}+1)^8}\right]\right\}, \tag{104}$$

so the difference between the field density in the neighbour levels is inversely proportional to ${\rm n}^7$, and ${\rm n} \gg 1$ gives

$$\overline{q} = q_{\rm n} - q_{\rm n+1} \approx \frac{1}{{\rm n}^7}\frac{3\mu}{\pi c}\,\Omega_{\rm b}^2\,, \tag{105}$$

and $\overline{q} \to 0$ for quantum numbers ${\rm n} \to \infty$.

Theoretically, the non-zero field density, $q \neq 0$, must result in a flux of the field momentum (this flux is set up by the field strength tensor $U_{ik} = \frac{1}{3}\,qc^2 h_{ik} - \beta_{ik}$). So an electron, moving in its orbit, should be radiating a momentum flux of the time density field (waves of the field). Because of the momentum loss in the radiation, the electron's own angular velocity would decrease, contradicting the experimental facts on the stability of atoms in the absence of external influences. To obviate this contradiction the third postulate is,

POSTULATE III *An atom radiates a quantum portion of momentum flux of a time density field, when an electron transits from the n-th quantum level to the (n+1)-th level in the atom. When an electron transits from the (n+1)-th level to the n-th level, the atom absorbs the same portion of the momentum flux, which is*

$$\overline{U}_{11} = U_{11}^{\rm n} - U_{11}^{\rm n+1} = = \frac{\mu c}{2\pi}\,\Omega_{\rm b}^2 \left\{\left[\frac{1}{{\rm n}^6} - \frac{1}{({\rm n}+1)^6}\right] - \frac{v_{\rm b}^2}{4c^2}\left[\frac{1}{{\rm n}^8} - \frac{1}{({\rm n}+1)^8}\right]\right\}. \tag{106}$$

We assume in this formula that the atom radiates/absorbs a plane wave of a time density field, which travels along the $x^1 = x$ axis. Taking this formula with ${\rm n} \gg 1$, we have

$$\overline{U}_{11} = U_{11}^{\rm n} - U_{11}^{\rm n+1} \approx \frac{1}{{\rm n}^7}\frac{3\mu c}{\pi}\,\Omega_{\rm b}^2\,, \tag{107}$$

which, for quantum numbers ${\rm n} \to \infty$, gives $\overline{U}_{11} \to 0$. So for quantum numbers ${\rm n} \gg 1$ we have the ratio

$$\overline{U}_{11} = \overline{q}c^2. \tag{108}$$

In accordance with the correspondence principle, any result of quantum theory at high quantum numbers must coincide with the relevant classical results; any difference being imperceptible. We therefore take into consideration the formulae for q (69) and U_{ik} (71) in atoms, obtained by the methods of the classical theory of fields, under the condition $h \approx 1$. As a result we get the formulae $q = \frac{\mu}{2\pi c}\Omega^2\left(1 + \frac{v^2}{4c^2}\right) \approx \frac{\mu}{2\pi c}\Omega^2$ and $U_{ik} = \frac{\mu c}{2\pi}\Omega^2\left(h_{11} - \frac{v^2}{2c^2} + \frac{v^2}{4c^2}h_{11}\right) \approx \frac{\mu c}{2\pi}\Omega^2$, leading to the same relationship $U_{11} = qc^2$ that quantum theory has given (108). So the correspondence principle is valid for time density fields in atoms.

Postulate III has two consequences:

CONSEQUENCE I *An atom undergoing excitation radiates the momentum flux of a time density field, producing a positive wave pressure in the field.*

Calculating this positive pressure, orthogonal to the surface of a wall (here θ is the angle of reflection, $\mathfrak{R}$ is the reflection coefficient) for quantum numbers ${\rm n} \gg 1$, we obtain

$$\overline{\mathfrak{F}}_{\rm N} = (1 + \mathfrak{R})\,\overline{q}\cos^2\theta\,. \tag{109}$$

CONSEQUENCE II *An atom undergoing relaxation absorbs the momentum flux of a time density field. In this case the wave pressure in a time density field near the atom becomes negative.*

As a matter of fact, this negative pressure around a relaxing atom should be

$$\overline{\mathfrak{F}}_{\rm N} = -(1 + \mathfrak{R})\,\overline{q}\cos^2\theta\,. \tag{110}$$

That is, in accordance with this theory, excitation of atoms causes radiation of waves of the time density field. An effect derived from the radiation should be the positive pressure of the waves. On the other hand, relaxing atoms, absorbing waves of the time density field, should be sources of negative wave pressure.

It is interesting that this effect is opposite to that which atoms produce in an electromagnetic field — it is well-known that relaxing atoms radiate electromagnetic waves, so they produce a positive wave pressure in an electromagnetic field.

The predicted repulsion/attraction produced by atomic processes, being outside the actions of electromagnetic or gravitational fields, are peculiarities of only the theory of the time density field herein. So the given conclusions open up wide possibilities for checking the whole theory in practice.

In particular, for instance, if a torsion balance registered the repulsing/attracting wave pressure $\overline{\mathfrak{F}}_{\rm N}$ derived from subatomic excitation/relaxation processes, we will have obtained the numerical value of the energy-momentum constant μ for time density fields. After substituting $\overline{q}$ (105) into the wave pressure $\overline{\mathfrak{F}}_{\rm N}$, assuming $\cos\theta = 1$, we arrive at a formula for experimental calculations

$$\mu = \frac{\pi c {\rm n}^7}{3\Omega_{\rm b}^2}\frac{\overline{\mathfrak{F}}_{\rm N}}{(1 + \mathfrak{R})}\,. \tag{111}$$

A torsion balance registered such forces at $\sim 10^{-5}$ dynes in prior experiments. The torsion balance had a 2-in long nylon thread, 15 μm in diameter, and a 3-in long wooden balance suspended in the ratio 8:1 of the length. The balance had a reflecting shield at the end of the long arm and a lead load on the short arm. The torsion balance was located inside a box isolated from air convection and light radiation. Chemical reactions of the opposite directions, processes of crystallization and dissolution were sources of a time density field acting on the torsion balance. Prof. Kyril Stanyukovich and Dr. Larissa Borissova assisted me in the experiments that were repeated a number of time during a period of 2 years in Moscow (Russia). The balance underwent deviations of up to 90° in directions predicted by this theory.

Even heating up bodies and cooling down bodies gave the same thermal influence, moving the balance in opposite directions, according to the theory, so the discovered phenomenon is outside thermal influences on torsion balance.

The techniques and measurements are very simply, and could therefore be reproduced in any physical laboratory. Anyway the experiments should be continued, with the aim of determining the exact numerical value of the energy-momentum constant μ for time density fields through formula (111).

11 Conclusions

Let us collect the main results of this analysis.

By projecting an interval of four-dimensional coordinates dx^α onto the time line of an observer, who accompanies his references ($b^i=0$), we obtain an interval of physical time $d\tau=\frac{1}{c}\,b_\alpha dx^\alpha$ he observes. Observations at the same spatial point give $d\tau=\sqrt{g_{00}}\,dt$, so the operator of projection on time lines b^α defines observable inhomogeneity of time references in the accompanying reference frame.

So, observable inhomogeneities of time references can be represented as a field of "density" of observable time τ. The projecting operator b^α is the field "potential", chr.inv.-projections of which are $\varphi=1$ and $q^i=0$.

The field tensor $F_{\alpha\beta}=\nabla_\alpha b_\beta-\nabla_\beta b_\alpha$ for time density fields was introduced as well as Maxwell's electromagnetic field tensor. Its chr.inv.-projections $E^i=-\frac{1}{c^2}F^i$ and $H_{ik}=-\frac{2}{c}A_{ik}$ are derived from the gravitational inertial force and rotation of the space. We referred to the E^i and H_{ik} as the "electric" and "magnetic" observable components of the time density field, respectively. We also introduced the field pseudotensor $F^{*\alpha\beta}$, dual of the $F_{\alpha\beta}$, and also the field invariants.

Equations of motion of a free mass-bearing particle, being expressed through the E^i and H_{ik}, group them into an acting force of a form similar to the Lorentz force. In particular if the particle moves only along time lines, it moves solely because of the "magnetic" component $H_{ik}\neq 0$ of a time density field. In other words, the space rotation A_{ik} effectively "screws" particles into the time lines. Because observable particles with the whole spatial section move from past into future, a "starting" non-holonomity, $A_{ik}\neq 0$, will exist in our real space that is a "primordial non-orthogonality" of the real spatial section to the time lines. Other physical conditions (gravitation, rotation, etc.) are only augmentations that intensify or reduce this starting-rotation of the space.

A system of equations of a time density field consists of Lorentz's condition $\nabla_\sigma b^\sigma=0$, two groups of Maxwell-like equations, $\nabla_\sigma F^{\alpha\sigma}=\frac{4\pi}{c}\,j^\alpha$ and $\nabla_\sigma F^{*\alpha\sigma}=0$, and the continuity equation $\nabla_\sigma j^\sigma=0$, which define the main properties of the field and its-inducing sources. All the equations have been deduced here in chr.inv.-form.

The energy-momentum tensor $T^{\alpha\beta}$ we have deduced for time density fields has the following observable projections: chr.inv.-scalar q of the field density; chr.inv.-vector J^i of the field momentum density, and chr.inv.-tensor U^{ik} of the field strengths. Their specific formulas define physical properties of such fields:

1. A time density field is non-stationary distributed medium $q\neq$ const, it becomes stationary, $q=$ const, under stationary rotation, $A_{ik}=$ const, of the space and stationary gravitational inertial force $F_i=$ const;
2. The field bears momentum ($J^i\neq 0$ in the general case), so it can transfer impulse;
3. In a rotating space, $A_{ik}\neq 0$, the field is an emitting medium;
4. The field is viscous. The viscosity α_{ik} is anisotropic. The anisotropy increases with the space rotation speed;
5. The equation of state of the field is $p=\frac{1}{3}qc^2$, so the field is like an ultrarelativistic gas: at positive density the pressure is positive — the medium compresses.

For a plane wave of the field considered, we have concluded that waves of the time density fields are transverse. The wave pressure in the fields is derived from atomic and sub-atomic transformations mainly, because of huge rotational velocities. Exciting atoms produces a positive wave pressure in the time density field, while the wave pressure resulting from relaxing atoms is negative. This effect is opposite to that of the electromagnetic field — relaxing atoms radiate γ-quanta, producing a positive pressure of light waves.

Experimental tests have a basis in the predicted repulsion/attraction, produced by sub-atomic processes, being outside of known effects of electromagnetic or gravitational fields, which are peculiarities only of this theory. A torsion balance registered such forces at $\sim 10^{-5}$ dynes in prior experiments. The registered repulsion/attraction is outside thermal effects on the torsion balance.

The results we have obtained in this study imply that even if inhomogeneity of time references is a tiny correction to ideal time, a field of the inhomogeneities that is a time density field, manifest as gravitational and inertial forces, has a more fundamental effect on observable phenomena, than those previouly supposed.

Acknowledgements

I began this research in 1984 when I, a young scientist in those years, started my scientific studies under the direction of Prof. Kyril Stanyukovich, who in the 1940's was already a well-known expert in the General Theory of Relativity. At that time we found an article of 1971, wherein Prof. Nikolai Kozyrev (1908–1983) reported on his experiments with a torsion balance [7].

His high precision torsion balance, having unequal arms, registered weak forces of attraction/repulsion at 10^{-5}–10^{-6} dynes; the forces derived from creative/destructive processes in his laboratory. Proceeding from his considerations, re-distribution of energy should produce a non-uniformity of time that generates a force field of attraction or repulsion depending on the creative/destructive direction of the re-distributing energy process.

Kozyrev did not put this idea into any mathematical form. So his propositions, having a purely phenomenological a basis, remained without a theory. Kozyrev was the famous experimental physicist and astronomer of the 20th century who discovered volcanic activity in the Moon (1958), the atmosphere of Mercury (1963), and many other phenomena. His authority in physical experiment was beyond any doubt.

In those years Stanyukovich was head of the Department of Fundamental Theoretical Metrology, Surface and Vacuum Scientific Centre, State Committee for Standards (Moscow, Russia). Dr. Larissa Borissova worked in his Department. We were all close friends, despite our age difference. We had a good time working with Stanyukovich, who was friendly in his conversations on different scientific problems.

I was interested by Kozyrev's experiments with the torsion balance, therefore Stanyukovich proposed me that I investigate them, meaning that it would be helpful for me to build the required theory. During 1984–1985 I twice visited the laboratory of late Kozyrev in Pulkovo Astronomical Observatory, near Leningrad. Then, in Moscow, we made a copy of his torsion balance, and modified it to make it more sensitive.

Stanyukovich was right — the experiments were approbated with strictly positive result. We repeated the experiments for some of his colleagues, in particular for Dr. Vitaly Schelest.

More than 15 years were required for the development of this theory. I finished the whole theory only in 2004. Stanyukovich had died; only Larissa Borissova and I remain. Our years of friendly conversations with Stanyukovich, and his patient personal instructions reached us by all his experience in theoretical physics. Actually, we are beholden to him. Today I would like to do only one thing — satisfy the hopes of my teacher.

Finally I am grateful to my colleagues: Stephen J. Crothers for some editing this paper, and Larissa Borissova who checked all my calculations that here.

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Gravitational Waves and Gravitational Inertial Waves in the General Theory of Relativity: A Theory and Experiments

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This research shows that gravitational waves and gravitational inertial waves are linked to a special structure of the Riemann-Christoffel curvature tensor. Proceeding from this a classification of the waves is given, according to Petrov's classification of Einstein spaces and gravitational fields located therein. The world-lines deviation equation for two free particles (the Synge equation) is deduced and that for two force-interacting particles (the Synge-Weber equation) in the terms of chronometric invariants — physical observable quantities in the General Theory of Relativity. The main result drawn from the deduced equations is that in the field of a falling gravitational wave there are not only spatial deviations between the particles but also deviations in the time flow. Therefore an effect from a falling gravitational wave can manifest only if the particles located on the neighbouring world-lines (both geodesics and non-geodesics) are in motion at the initial moment of time: gravitational waves can act only on moving neighbouring particles. This effect is purely parametric, not of a resonance kind. Neither free-mass detectors nor solid-body detectors (the Weber pigs) used in current experiments can register gravitational waves, because the experimental statement (freezing the pigs etc.) forces the particles of which they consist to be at rest. In aiming to detect gravitational waves other devices should be employed, where neighbouring particles are in relative motion at high speeds. Such a device could, for instance, consist of two parallel laser beams.

Contents

1 Introduction and advanced results

The fact that gravitational waves have not yet been discovered has attracted the attention of experimental physicists over the last decade. Initial interest in gravitational waves arose in 1968–1971 when Joseph Weber, professor at Maryland University (USA), carried out his first experiments with gravitational antennae. He registered weak signals, in common with all his independent antennae, which were separated by up to 1000 km [1]. He supposed that some processes at the centre of the Galaxy were the origin of the registered signals. However such an interpretation had a significant drawback: the frequency of the observed signals (more than 5 per month) meant that the energy spent by the signal's source, located at the centre of the Galaxy, should be more than $M_{\odot}c^2{\times}10^3$ per annum ($M_{\odot}$ is the mass of the Sun, c is the velocity of light). This energy expenditure is a fantastic value, if we accept today's bounds on the age of the Galaxy [2, 3, 4].

In 1972 the experiments were approbated by the a common group of researchers working at Moscow University and the Institute of Space Research (Moscow, Russia). Their antennae were similar to Weber's antennae, but they were separated by 20 km. The registering system in their antennae was better than that for the Weber detectors, making the whole system more sensitive. But. . . 20 days of observations gave no signals that would be more than noise [5].

The experiments were continued in 1973–1974 at laboratories in Rochester University, Bell Company, and IBM in USA [6, 7], Frascati, Münich, Meudon (Italy, Germany, France) [8], Glasgow University (Scotland) [9] and other laboratories around the world. The experimental systems used in these attempts were more sensitive than those of the Weber detectors, but none registered the Weber effect.

Because theoretical considerations showed that huge gravitational waves should be accompanied by other radiations, the researchers conducted a search for radio outbreaks [10] and neutron outbreaks [11]. The result was negative. At the same time it was found that Weber's registered effects were related to solar and geomagnetic activities, and also to outbreaks of space beams [12, 13].

The search for gravitational waves has continued. Higher precision and more sensitive modifications of the Weber antennae (solid detectors of the resonance kind) are used in this search. But even the second generation of Weber detectors have not led scientists to the expected results. Besides gravitational antennae of the Weber kind, there are antennae based on free masses. Such detectors consist of two freely suspended masses located far from one another, within the visibility of a laser range-finder. Supposed deviations of the masses, derived from a gravitational wave, should be registered by the laser beam.

So gravitational waves have not been discovered in experiments. Nonetheless it is accepted by most physicists that the discovery of gravitational waves should be one of the main verifications of the General Theory of Relativity. The main arguments in support of this thesis are:

1. Gravitational fields bear an energy described by the energy-momentum pseudotensor [14, 15];
2. A linearized form of the equations of Einstein's equations permits a solution describing weak plane gravitational waves, which are transverse;
3. An energy flux, radiated by gravitational waves, can be calculated through the energy-momentum pseudotensor of the field [14, 15];
4. Such waves, because of their physical nature, are derived from instability of components of the fundamental metric tensor (this tensor plays the part of the four-dimensional gravitational potential).

These theoretical considerations were placed into the foreground of the theory for detecting gravitational waves, the main part in the theory being played by the theoretical works of Joseph Weber, the pioneer and famous expert in the detection of gravitational waves [16]. His main theoretical claim was that he deduced equations of deviation of world-lines — equations that describe relative oscillations of *two non-free particles* in a gravitational field, particles which are connected by a force of non-gravitational nature. Equations of deviation of geodesic lines, describing relative oscillations of *two free particles*, was obtained earlier by Synge [17]. In general, relative oscillations of test-particles, both free particles and linked (interacting) particles, are derived from the space curvature*, given by the Riemann-Christoffel four-dimensional tensor. Equality to zero of all its components in an area is the necessary and sufficient condition for the four-dimensional space (space-time) to be flat in the area under consideration, so no gravitational fields exist in the area.

*As it is well-known, the space curvature is linked to the gravitational field by the Einstein equations.

Thus the Synge-Weber equation provides a means for the calculation of the relative oscillations of test-particles, derived from the presence of the space curvature (gravitational fields). Weber proposed a gravitational wave detector consisting of two particles connected by a spring that imitates a non-gravitational interaction between them. In his analysis he made the substantial supposition that the under action of gravitational waves the model will behave like a harmonic oscillator where the forcing power is in the Riemann-Christoffel curvature tensor. Weber made calculations and theoretical propositions for the behaviour of this model. This model is known as the *quadrupole mass-detector* [17].

The Weber calculations served as theoretical grounds for creating a whole industry, the main task of which has been the building of resonance type detectors, known as the Weber detectors (the Weber pigs). It is supposed that the body of a Weber detector, having cylindrical form, should be deformed under the action of a gravitational wave. This deformation should lead to a piezoelectric effect. Thus, oscillations of atoms in the cylindrical pig, resulting from a gravitational wave, could be registered. To amplify the effect in measurements, the level of noise was lowered by cooling the cylinder pigs down to temperature close to 0 K.

But the fact that gravitational waves have not yet been discovered does not imply that the waves do not exist in Nature. The corner-stone of this problem is that the Weber theory of detection is linked to a search for waves of only a specific kind — weak transverse waves of the space deformation (*weak deformation transverse waves*). However, besides the Weber theory, there is the *theory of strong gravitational waves*, which is independent of the Weber theory. Studies of the theory of strong gravitational waves reached its peak in the 1950's.

Generally speaking, all theoretical studies of gravitational waves can be split into three main groups:

1. The first group consists of studies whose task is to give an invariant definition for gravitational waves. These are studies made by Pirani [18, 19], Lichnerowicz [20, 21], Bel [22, 23, 24], Debever [25, 26, 27], Hély [28], Trautman [29], Bondi [19, 30], and others.
2. The second group joins studies around a search for such solutions to the Einstein equations for gravitational fields, which, proceeding from physical considerations, could describe gravitational radiations. These are studies made by Bondi [31], Einstein and Rosen [32, 33], Peres [34], Takeno [35, 36], Petrov [37], Kompaneetz [38], Robinson and Trautman [39], and others.
3. The task of works related to the third group is to study gravitational inertial waves, covariant with respect of transformations of spatial coordinates and also invariant with respect of transformations of time [40, 41].

The studies are based on the theory of physically observable quantities — Zelmanov's theory of chronometric invariants [42, 43].

Most criteria for gravitational waves are linked to the structure of the Riemann-Christoffel curvature tensor, hence one assumes space curvature the source of such waves.

Besides these three main considerations, the theory of gravitational waves is directly linked to the algebraical classification of spaces given by Petrov [37] (*Petrov classification*), according to which three kinds for spaces (gravitational fields) exist. They are dependent on the structure of the Riemann-Christoffel curvature tensor:

1. Fields of gravitation of the 1st kind are derived from island distributions of masses. An instance of such a field is the that of a spherical distribution of matter (a spherical mass island) described by the Schwarzschild metric [44]. Spaces containing such fields approach a flat space at an infinite distance from the gravitating island.

2–3. Spaces containing gravitational fields of the 2nd and 3rd kinds cannot asymptotically approach a flat space even, if they are empty. Such spaces can be curved themselves, independently of the presence of gravitating matter. Such fields satisfy most of the invariant definitions given to gravitational waves [40, 45, 46, 47].

It should be noted that the well-known solution that gives weak plane gravitational waves [14, 15] is related to fields of the sub-kind N of the 2nd kind by Petrov's classification (see p. 38). Hence the theory of weak plane gravitational waves is a particular case of the theory of strong gravitational waves. But, bseides this well-studied particular case, the theory of strong gravitational waves contains many other approaches to the problem and give other methods for the detection of gravitational waves, different to the Weber detectors in principle (see [48], for instance).

We need to look at the gravitational wave problem from another viewpoint, by studying other cases of the theory of strong gravitational waves not considered before. Exploring such new approaches to the theory of gravitational waves is the main task of this research.

At the present time there are many solutions of the gravitational wave problem, but none of them are satisfactory. The principal objective of this research is to extract that which is common to every one of the theoretical approaches.

We will see further that this analysis shows, according to most definitions given for gravitational waves, that a gravitational field is assumed a wave field if the space where it is located has the specific curvature described by numerous particular cases of the Riemann-Christoffel curvature tensor.

Note that we mean the Riemannian (four-dimensional) curvature, whose formula contains accelerations, rotations, and deformations of the observer's reference space. Analysis of most wave solutions to the gravitational field equations (Einstein's equations) shows that such gravitational waves have a *deformation nature* — they are waves of the space deformations. The true nature of gravitational waves can be found by employing the mathematical methods of chronometric invariants (the theory of physically observable quantities in the General Theory of Relativity), which show that the space deformation (non-stationarity of the spatial observable metric) consists of two factors:

1. Changes of the observer's scale of distance with time (deformations of the 1st kind);
2. Possible vortical properties of the acting gravitational inertial force field (deformations of the 2nd kind).

Waves of the space deformations (of the 1st or 2nd kind) underlie the detection attempts of the experimental physicists.

Because such gravitational waves are expected to be weak, one usually uses the metric for weak plane gravitational waves of the 1st kind (which are derived from changes of the distance scale with time).

The basis for all the experiments is the Synge-Weber equation (the world-lines deviation equation), which sets up a relation between relative oscillations of test-particles and the Riemann-Christoffel curvature tensor. Unfortunately Joseph Weber himself gave only a rough analysis of his equation, aiming to describe the behaviour of a quadrupole mass-detector in the field of weak plane gravitational waves. In his analysis he assumed (without substantial reasons) that space deformation waves of the 1st kind must produce a resonance effect in a quadrupole mass-detector.

However, it would be more logical way, making no assumptions or propositions, to solve the Synge-Weber equation aiming exactly. Weber did not do this, limiting himself instead to only rough bounds on possible solutions.

In this research we obtain exact solutions to the Synge-Weber equation in the fields of weak plane gravitational waves. As a result we conclude that the expected relative oscillations of test-particles, which originate in the space deformation waves of the 1st kind, *are not of the resonance kind* as Weber alleged from his analysis, but are instead *parametric oscillations*.

This deviation between our conclusion and Weber's false conclusion is very important, because oscillations of a parametric kind appear only if test-particles are moving*, whilst in Weber's statement of the experiment the particles are at rest in the observer's laboratory reference frame. All activities in search of gravitational waves using the Weber pigs are concentrated around attempts to isolate the bulk pigs from external affects — experimental physicists place them in mines in the depths of mountains and cool them to 2 K,

*In other words, if their velocities are different from zero. Parametric oscillations merely add their effect to the relative motion of the moving particles. Parametric oscillations cannot be excited in a system of particles which are at rest with respect to each other and the observer.

so particles of matter in the pigs can be assumed at rest with respect to one another and to the observer. At present dozens of Weber pigs are used in such experiments all around the world. Experimental physicists spend billions and billions of dollars yearly on their experiments with the Weber pigs.

Parametric oscillations do not appear in resting particles, so the space deformation waves of the 1st kind can not excite parametric oscillations in the Weber pigs. Therefore the *gravitational waves expected by scientists cannot be registered by solid-body detectors of the resonance kind* (*the Weber pigs*).

Even so, everything said so far does not mean rejection of the experimental search for gravitational waves. We merely need to look at the problem from another viewpoint. We need to remember the fact that our world is not a three-dimensional space, but a four-dimensional space-time. For this reason we need to turn our attention to the fact that relative deviations of particles in the field of gravitational waves have both spatial components and a time component. Therefore it would be reasonable to propose an experiment by which, having a detector under the influence of gravitational waves, we could register both relative displacements of particles in the detector and also corrections to time flow in the detector due to the waves (the second task is much easier from the technical viewpoint).

Here are two aspects for consideration. First, in solving the Synge-Weber equations we must take its time component into account; we must not neglect the time component. Second, we should turn our attention to possible experimental effects derived from gravitational waves of the 2nd (deformation) kind, which appear if the acting gravitational inertial force field is vortical, as it will be shown further that in this case there is a field of the space rotation (stationary or non-stationary)*. Such experiments, aiming to register gravitational waves of the 2nd kind are progressive because they are much simpler and cheaper than the search for waves of the 1st kind.

2 Theoretical bases for the possibility of registering gravitational waves

Gravitational waves were already predicted by Einstein [37], but what space objects could be sources of the waves is not a trivial problem. Some link the possibility of gravitational radiations to clusters of black holes. Others await powerful gravitational radiations from super-dense compact stars of radii close to their gravitational radii† $r \sim r_g$. Although the "black hole solution", being under substantial criticism from the purely mathematical viewpoint [53, 54, 55], makes objects like black holes very doubtful, the existence of super-dense neutron stars is outside of doubt between astronomers. Gravitational waves at frequencies of 10^2–10^4 Hz should also be radiated in super-nova explosions by explosion of their super-dense remains [56].

The search for gravitational waves, beginning with Weber's observations of 1968–1971, is realized by using gravitational antennae, the most promising of which are:

1. Solid-body detectors (the Weber cylinder pigs);
2. Antennae built on free masses.

A solid-body detector of the Weber kind is a massive cylindrical pig of 1–3 metres in length, made with high precision. This experiment supposes that gravitational waves are waves of the space deformation. For this reason the waves cause a piezoelectic effect in the pig, one consequence of which is mechanical oscillations at low frequencies that can be registered in the experiment. It is supposed that such oscillations have a resonance nature. An immediate problem is that such resonance in massive pigs can be caused by very different external processes, not only waves of the space deformation. To remove other effects, experimental physicists locate the pigs in deep tunnels in mountains and cool the pigs down to temperature close to 0 K.

An antenna of the second kind consists of two masses, separated by $\Delta l \sim 10^3$–10^4 metres, and a laser range-finder which should register small changes of Δl. Both masses are freely suspended. This experiment supposes that waves of the space deformation should change the distance between the free masses, and should be registered by the laser range-finder. It is possible to use two satellites located in the same orbit near the Earth, having a range-finder in each of the satellites. Such satellites, being in free fall along the orbit, should be an ideal system for measurements, if it were not for effects due to the terrestrial globe. In practice it would be very difficult to divorce the effect derived from waves of the space deformation (supposed gravitational waves) and many other factors derived from the inhomogeneity of the Earth's gravitational field (purely geophysical factors).

The mathematical model for such an antenna consists of two free test-particles moving on neighbouring geodesic lines located infinitely close to one another. The mathematical model for a solid-body detector (a Weber pig) consists of two test-masses connected by a spring that gives a model for elastic interactions inside a real cylindrical pig, in which changes reveal the presence of a wave of the space deformation.

From the theoretical perspective, we see that the possibility of registering waves of the space deformation (supposed gravitational waves) is based on the supposition that particles which encounter such a wave should be set into relative oscillations, the origin of which is the space curvature. The

*There are well-known Hafele-Keating experiments concerned with displacing standard clocks around the terrestrial globe, where rotation of the Earth space sensibly changes the measured time flow [49, 50, 51, 52].

†According to today's mainstream concepts, the gravitational radius r_g of an object is that minimal distance from its centre to its surface, starting from which this object is in a special state — *collapse*. One means that any object going into collapse becomes a "black hole". From the purely mathematical viewpoint, under collapse, the potential w of the gravitational field of the object merely reaches its upper ultimate numerical value $\mathrm{w} = c^2$.

strong solution for this problem had been given by Synge for free particles [17]. He considered a two-parameter family of geodesic lines $x^\alpha = x^\alpha(s,v)$, where s is a parameter along the geodesics, v is a parameter along the direction orthogonal to the geodesics (it is taken in the plane normal to the geodesics). Along each geodesic line $v=$ const.

He introduced two vectors

$$U^\alpha = \frac{\partial x^\alpha}{\partial s}\,, \qquad V^\alpha = \frac{\partial x^\alpha}{\partial v}\,, \tag{2.1}$$

where $\alpha = 0,1,2,3$ denotes four-dimensional (space-time) indexes. The vectors satisfy the condition

$$\frac{\mathrm{D}U^\alpha}{\partial v} = \frac{\mathrm{D}V^\alpha}{\partial s}\,, \tag{2.2}$$

(where D is the absolute derivative operator) that can be easy verified by checking the calculation. The parameter v is different for neighbouring geodesics; the difference is dv. Therefore, studying relative displacements of two geodesics $\Gamma(v)$ and $\Gamma(v+dv)$, we shall study the vector of their infinitesimal relative displacement

$$\eta^\alpha = \frac{\partial x^\alpha}{\partial v}\,dv = V^\alpha dv\,. \tag{2.3}$$

The deviation of the geodesic line $\Gamma(v+dv)$ from the geodesic line $\Gamma(v)$ can be found by solving the equation [17]

$$\begin{aligned} \frac{\mathrm{D}^2 V^\alpha}{ds^2} &= \frac{\mathrm{D}}{ds}\frac{\mathrm{D}V^\alpha}{ds} = \frac{\mathrm{D}}{ds}\frac{\mathrm{D}U^\alpha}{dv} = \\ &= \frac{\mathrm{D}}{dv}\frac{\mathrm{D}U^\alpha}{ds} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma, \end{aligned} \tag{2.4}$$

where $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ is the Riemann-Christoffel curvature tensor. This equality has been obtained using the relation [17]

$$\frac{\mathrm{D}^2 V^\alpha}{ds dv} - \frac{\mathrm{D}^2 V^\alpha}{dv ds} = R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma. \tag{2.5}$$

For two neighbour geodesic lines, the following relation is obviously true

$$\frac{\mathrm{D}U^\alpha}{ds} = \frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = 0\,, \tag{2.6}$$

where $\Gamma^\alpha_{\beta\gamma}$ are Christoffel's symbols of the 2nd kind. Then (2.4) takes the form

$$\frac{\mathrm{D}^2 V^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta V^\gamma = 0\,, \tag{2.7}$$

or equivalently,

$$\frac{\mathrm{D}^2 \eta^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta \eta^\gamma = 0\,. \tag{2.8}$$

It can be shown [17] that,

$$\frac{\partial}{\partial s}(U_\alpha V^\alpha) = U_\alpha\frac{\mathrm{D}V^\alpha}{ds} = U_\alpha\frac{\mathrm{D}U^\alpha}{dv} = \frac{1}{2}\frac{\partial}{\partial v}(U_\alpha U^\alpha)\,. \tag{2.9}$$

The quantity $U_\alpha U^\alpha = g_{\alpha\beta}U^\alpha U^\beta$ takes the numerical value +1 for non-isotropic geodesics (substantial particles) or 0 for isotropic geodesics (massless light-like particles). Therefore

$$U_\alpha V^\alpha = \text{const}\,. \tag{2.10}$$

In the particular case where the vectors U_α and V^α are orthogonal to each to other at a point, where $U_\alpha V^\alpha$ is true, the orthogonality remains true everywhere along the $\Gamma(v)$.

Thus relative accelerations of free test-particles are caused by the presence the space curvature ($R^{\alpha\cdots}_{\cdot\beta\gamma\delta} \neq 0$), and linear velocities of the particles are determined by the geodesic equations (2.6).

Relative accelerations of test-particles, connected by a force Φ^α of non-gravitational nature, are determined by the Synge-Weber equation [16]. The Synge-Weber equation is the generalization of equation (2.8) for that case where the particles, each having the rest-mass m_0, are moved along non-geodesic world-lines, determined by the equation

$$\frac{\mathrm{D}U^\alpha}{ds} = \frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = \frac{\Phi^\alpha}{m_0 c^2}\,. \tag{2.11}$$

In this case the world-lines deviation equation takes the form

$$\frac{\mathrm{D}^2 \eta^\alpha}{ds^2} + R^{\alpha\cdots}_{\cdot\beta\gamma\delta}U^\beta U^\delta \eta^\gamma = \frac{1}{m_0 c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}\,dv\,, \tag{2.12}$$

which describes relative accelerations of the interacting masses. In this case

$$\frac{\partial}{\partial s}(U_\alpha \eta^\alpha) = \frac{1}{m_0 c^2}\,\Phi_\alpha \eta^\alpha, \tag{2.13}$$

so the angle between the vectors U^α and η^α does not remain constant for the interacting particles.

Equations (2.8) and (2.12) describe relative accelerations of free particles and interacting particles, respectively. Then, to obtain formulae for the velocity U^α it is necessarily to solve the geodesic equations for free particles (2.6) and the world-line equations for interacting particles (2.11). We consider the equations (2.8) and (2.12) as a mathematical base, with which we aim to calculate gravitational wave detectors: (1) antennae built on free particles, and (2) solid-body detectors of the resonance kind (the Weber detectors).

3 Invariant criteria for gravitational waves and their link to Petrov's classification

From the discussion in the previous paragraphs, one concludes that a physical factor enforcing relative displacements of test-particles (both free particles and interacting particles) is the space curvature — a gravitational field wherein the particles are located.

Here the next question arises. How well justified is the statement of the gravitational wave problem?

Generally speaking, in the General Theory of Relativity, there is a problem in describing gravitational waves in a mathematically correct way. This is a purely mathematical problem, not solved until now, because of numerous difficulties. In particular, the General Theory of Relativity does not contain a satisfactory general covariant definition for the energy of gravitational fields. This difficulty gives no possibility of describing gravitational waves as traveling energy of gravitational fields.

The next difficulty is that when one attempts to solve the gravitational wave problem using the classical theory of differential equations, he sees that the gravitational field equations (the Einstein equations) are a system of 10 non-linear equations of the 2nd order written with partial derivatives. No universal boundary conditions exist for such equations.

The gravitational field equations (the Einstein equations) are

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = -\kappa T_{\alpha\beta} + \lambda\, g_{\alpha\beta}\,, \tag{3.1}$$

where $R_{\alpha\beta} = R^{\sigma\cdots}_{\cdot\alpha\sigma\beta}$ is Ricci's tensor, $R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, $\kappa = \frac{8\pi G}{c^2}$ is Einstein's constant for gravitational fields, G is Gauss' constant of gravitation, λ is the cosmological constant (λ-term).

When studying gravitational waves, one assumes $\lambda = 0$. Sometimes one uses a particular case of the Einstein equations (3.1)

$$R_{\alpha\beta} = \kappa\, g_{\alpha\beta}\,, \tag{3.2}$$

in which case the space, where the gravitational field is located, is called an *Einstein space*. If $\kappa = 0$, we have an *empty space* (without gravitating matter). But even in empty spaces ($\kappa = 0$) gravitational fields can exist, if the spaces are of the 2nd and 3rd kinds by Petrov's classification.

In accordance with the classical theory of differential equations, those gravitational fields that describe gravitational waves are determined by solutions of the Einstein equations with initial conditions located in a characteristic surface. A wave is a Hadamard break in the initial characteristic surface; such a surface is known as the *wave front*. The wave front is determined as the characteristic isotropic surface $S\,\{\Phi(x^\alpha) = 0\}$ for the Einstein equations. Here the scalar function Φ satisfies the eikonal equation [20, 21]

$$g^{\alpha\beta}\, \nabla_\alpha \nabla_\beta = 0\,, \tag{3.3}$$

where ∇_α denotes covariant differentiation with respect to Riemannian coherence with the metric $g_{\alpha\beta}$. The trajectories along which gravitational waves travel (gravitational rays) are bicharacteristics of the field equations, having the form

$$\frac{dx^\alpha}{d\tau} = g^{\alpha\sigma}\, \nabla_\sigma \Phi\,, \tag{3.4}$$

where τ is a parameter along lines of the geodesic family.

But the general solution of the Einstein equations with initial conditions in the hypersurface is unknown. For this reason the next problem arises: it is necessary to formulate an effective criterion which could determine solutions to the Einstein equations with initial conditions in the characteristic hypersurface.

There is another difficulty: there is no general covariant d'Alembertian which, being in its clear form, could be included into the Einstein equations.

Therefore, solving the gravitational wave problem reduces to the problem of formulating an invariant criterion which could determine this family of the field equations as wave equations.

Following this approach, analogous the classical theory of differential equations, we encounter an essential problem. Are functions $g_{\alpha\beta}(x^\sigma)$ smooth when we set up the Cauchy problem for the Einstein equation? A gravitational wave is interpreted as Hadamard break for the curvature tensor field in the initial characteristic hypersurface. The curvature tensor field permits a Hadamard break only if the functions $g_{\alpha\beta}(x^\sigma)$ permit breaks in their first derivatives. In accordance with Hadamard himself [20], the second derivatives of $g_{\alpha\beta}$ can have a break in a surface $S\,\{\Phi(x^\alpha) = 0\}$

$$[\partial_{\rho\sigma}\, g_{\alpha\beta}] = a_{\alpha\beta}\, l_\rho\, l_\sigma\,, \qquad (l_\alpha \equiv \partial_\alpha \Phi) \tag{3.5}$$

only if a Hadamard break in the curvature tensor field $[R_{\alpha\beta\gamma\delta}]$ satisfies the equations [21]

$$l_\lambda [R_{\mu\alpha\beta\nu}] + l_\alpha [R_{\mu\beta\lambda\nu}] + l_\beta [R_{\mu\lambda\alpha\nu}] = 0\,. \tag{3.6}$$

Proceeding from such an interpretation of the characteristic hypersurface for the Einstein equations, and also supposing that a break $[R_{\alpha\beta\gamma\delta}]$ in the curvature tensor $R_{\alpha\beta\gamma\delta}$ located in the front of a gravitational wave is proportional to the tensor itself, Lichnerowitz [20, 21] formulated this criterion for gravitational waves:

Lichnerowitz' criterion The space curvature $R_{\alpha\beta\gamma\delta} \neq 0$ determines the state of "full gravitational radiations", only if there is a vector $l^\alpha = 0$ satisfying the equations

$$\begin{aligned} &l^\mu R_{\mu\alpha\beta\nu} = 0\,, \\ &l_\lambda R_{\mu\alpha\beta\nu} + l_\alpha R_{\mu\beta\lambda\nu} + l_\beta R_{\mu\lambda\alpha\nu} = 0\,, \end{aligned} \tag{3.7}$$

and thus the vector l^α is isotropic ($l_\alpha l^\alpha = 0$). If $R_{\alpha\beta} = 0$ (the space is free of masses, so it is empty), the equations (3.7) determine the state of "clear gravitational radiations".

There is also Zelmanov's invariant criterion for gravitational waves [40]*, it is linked to the Lichnerowitz criterion.

*This criterion is named for Abraham Zelmanov, although it had been published by Zakharov [40]. This happened because Zelmanov gave many of his unpublished results, his unpublished criterion included, to Zakharov, who completed his dissertation under Zelmanov's leadership at that time.

Zelmanov proceeded from the general covariant generalization given for the d'Alembert wave operator

$$\Box^{\sigma}_{\sigma} \equiv g^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}\,. \qquad (3.8)$$

Zelmanov's criterion The space determines the state of gravitational radiations, only if the curvature tensor: (a) is not a covariant constant quantity ($\nabla_{\sigma}R_{\mu\alpha\beta\gamma}=0$); (b) satisfies the general covariant condition

$$\Box^{\sigma}_{\sigma}R_{\mu\alpha\beta\nu}=0. \qquad (3.9)$$

Thus, as it was shown in [40], any empty space that satisfies the Zelmanov criterion also satisfies the Lichnerowitz criterion. On the other hand, any empty space that satisfies the Lichnerowitz criterion (excluding that trivial case where $\nabla_{\sigma}R_{\mu\alpha\beta\gamma}=0$) also satisfies the Zelmanov criterion.

There are also other criteria for gravitational waves, introduced by Bel, Pirani, Debever, Maldybaeva and others [58]. Each of the criteria has its own advantages and drawbacks, therefore none of the criteria can be considered as the final solution of this problem. Consequently, it would be a good idea to consider those characteristics of gravitational wave fields which are common to most of the criteria. Such an integrating factor is Petrov's classification — the algebraic classification of Einstein spaces given by Petrov [37], in the frame of which those gravitational fields that satisfy the condition (3.2) are classified by their relation to the algebraic structure of the Riemann-Christoffel curvature tensor.

As is well known, the components of the Riemann-Christoffel tensor satisfy the identities

$$R_{\alpha\beta\gamma\delta}=-R_{\beta\alpha\gamma\delta}=-R_{\alpha\beta\delta\gamma}=R_{\gamma\delta\alpha\beta}\,,\quad R_{\alpha[\beta\gamma\delta]}=0\,. \qquad (3.10)$$

Because of (3.10), the curvature tensor is related to tensors of a special family, known as *bitensors*. They satisfy two conditions:

1. Their covariant and contravariant valencies are even;
2. Both covariant and contravariant indices of the tensors are split into pairs and inside each pair the tensor $R_{\alpha\beta\gamma\delta}$ is antisymmetric.

A set of tensor fields located in an n-dimensional Riemannian space is known as a *bivector set*, and its representation at a point is known as a *local bivector set*. Every antisymmetric pair of indices $\alpha\beta$ is denoted by a common index a, and the number of the common indices is $N=\dfrac{n(n-1)}{2}$. It is evident that if $n=4$ we have $N=6$. Hence a bitensor $R_{\alpha\beta\gamma\delta}\rightarrow R_{ab}$, located in a four-dimensional space, maps itself into a six-dimensional bivector space. It can be metrised by introducing the specific metric tensor

$$g_{ab}\rightarrow g_{\alpha\beta\gamma\delta}\equiv g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}\,. \qquad (3.11)$$

The tensor g_{ab} ($a,b=1,2,\dots N$) is symmetric and non-degenerate. The metric g_{ab}, given for the sign-alternating $g_{\alpha\beta}$, can be sign-alternating, having a signature dependent on the signature of the $g_{\alpha\beta}$. So, for Minkowski's signature $(+---)$, the signature of the g_{ab} is $(+++---)$.

Mapping the curvature tensor $R_{\alpha\beta\gamma\delta}$ onto the metric bivector space R_N, we obtain the symmetric tensor R_{ab} ($a,b=1,2,\dots N$) which can be associated with a lambda-matrix

$$\left(R_{ab}-\Lambda\, g_{ab}\right). \qquad (3.12)$$

Solving the classic problem of linear algebra (reducing the lambda-matrix to its canonical form along a real distance), we can find a classificaton for V_n under a given n. Here the *specific kind* of an Einstein space we are considering is set up by a *characteristic* of the lambda-matrix. This kind remains unchanged in that area where this characteristic remains unchanged.

Bases of elementary divisors of the lambda-matrix for any V_n have an ordinary geometric meaning as *stationary curvatures*. Naturally, the Riemannian curvature V_n in a two-dimensional direction is determined by an ordinary (single-sheet) bivector $V^{\alpha\beta}=V^{\alpha}_{(1)}V^{\beta}_{(2)}$, of the form

$$K=\frac{R_{\alpha\beta\gamma\delta}V^{\alpha\beta}V^{\gamma\delta}}{g_{\alpha\beta\gamma\delta}V^{\alpha\beta}V^{\gamma\delta}}\,. \qquad (3.13)$$

If $V^{\alpha\beta}$ is not ordinary, the invariant K is known as the *bivector curvature in the given vector's direction*. Mapping K onto the bivector space, we obtain

$$K=\frac{R_{ab}V^{a}V^{b}}{g_{ab}V^{a}V^{b}}\,,\qquad a,b=1,2,\dots N. \qquad (3.14)$$

Ultimate numerical values of the K are known as *stationary curvatures* taken at a given point, and the vectors V^a corresponding to the ultimate values are known as *stationary not simple bivectors*. In this case

$$V^{\alpha\beta}=V^{\alpha}_{(1)}V^{\beta}_{(2)}\,, \qquad (3.15)$$

so the stationary curvature coincides with the Riemannian curvature V_n in the given two-dimension direction.

The problem of finding the ultimate values of K is the same as finding those vectors V^a where the K takes the ultimate values, that is, the same as finding *undoubtedly stationary directions*. The necessary and sufficient condition of stationary state of the V^a is

$$\frac{\partial}{\partial V^{a}}\,K=0\,. \qquad (3.16)$$

The problem of finding the stationary curvatures for Einstein spaces had been solved by Petrov [40]. If the space signature is sign-alternating, generally speaking, the stationary curvatures are complex as well as the stationary bivectors relating to them in the V_n.

For four-dimensional Einstein spaces with Minkowski signature, we have the following theorem [40]:

THEOREM *Given an ortho-frame* $g_{\alpha\beta}=\{+1,-1,-1,-1\}$, *there is a symmetric paired matrix* (R_{ab})

$$R_{ab}=\left(\begin{array}{c|c} M & N \\ \hline N & -M \end{array}\right), \qquad (3.17)$$

where M *and* N *are two symmetric square matrices of the 3rd order, whose components satisfy the relationships*

$$m_{11}+m_{22}+m_{33}=-\kappa\,, \qquad n_{11}+n_{22}+n_{33}=0\,. \qquad (3.18)$$

After transformations, the lambda-matrix $(R_{ab}-\Lambda g_{ab})$ where $g_{\alpha\beta}=\{+1,+1,+1,-1,-1,-1\}$ takes the form

$$\begin{gathered}(R_{ab}-\Lambda g_{ab})=\\ =\left(\begin{array}{c|c} M+iN+\Lambda\varepsilon & 0 \\ \hline 0 & M-iN+\Lambda\varepsilon \end{array}\right)\equiv\\ \equiv\begin{pmatrix} Q(\Lambda) & 0 \\ 0 & \bar{Q}(\Lambda)\end{pmatrix},\end{gathered} \qquad (3.19)$$

where $Q(\Lambda)$ and $\bar{Q}(\Lambda)$ are three-dimensional matrices, the elements of which are complex conjugates, ε is the three-dimensional unit matrix. The matrix $Q(\Lambda)$ can have only one of the following types of characteristics:

(1) [111]; (2) [21]; (3) [3]. It is evident that the initial lambda-matrix can have only one characteristic drawn from:

(1) $[111,\overline{111}]$; (2) $[21,\overline{21}]$; (3) $[3,3]$.

The bar in the second half of a characteristic implies that elementary divisors in both matrices are complex conjugates. There is no bar in the third kind because the elementary divisors there are always real.

Taking a lambda-matrix of each of the three possible kinds, Petrov deduced the canonical form of the matrix (R_{ab}) in a non-holonomic ortho-frame [40]

The 1st Kind

$$\begin{gathered}(R_{ab})=\begin{pmatrix} M & N \\ N & -M\end{pmatrix},\\ M=\begin{pmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & \alpha_3\end{pmatrix},\\ N=\begin{pmatrix}\beta_1 & 0 & 0\\ 0 & \beta_2 & 0\\ 0 & 0 & \beta_3\end{pmatrix},\end{gathered} \qquad (3.20)$$

where $\sum_{s=1}^{3}\alpha_s=-\kappa\,,\ \sum_{s=1}^{3}\beta_s=0$ (so in this case there are 4 independent parameters, determining the space structure by an invariant form),

The 2nd Kind

$$\begin{gathered}(R_{ab})=\begin{pmatrix} M & N \\ N & -M\end{pmatrix},\\ M=\begin{pmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2+1 & 0\\ 0 & 0 & \alpha_2-1\end{pmatrix},\\ N=\begin{pmatrix}\beta_1 & 0 & 0\\ 0 & \beta_2 & 1\\ 0 & 1 & \beta_2\end{pmatrix},\end{gathered} \qquad (3.21)$$

where $\alpha_1+2\alpha_2=-\kappa\,,\ \beta_1+2\beta_2=0$ (so in this case there are 2 independent parameters determining the space structure by an invariant form),

The 3rd Kind

$$\begin{gathered}(R_{ab})=\begin{pmatrix} M & N \\ N & -M\end{pmatrix},\\ M=\begin{pmatrix}-\frac{\kappa}{3} & 1 & 0\\ 1 & -\frac{\kappa}{3} & 0\\ 0 & 0 & -\frac{\kappa}{3}\end{pmatrix},\\ N=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix},\end{gathered} \qquad (3.22)$$

so no independent parameters determining the space structure by an invariant form exist in this case.

Thus Petrov had solved the problem of reducing a lambda-matrix to its canonical form along a real path in a space of the sign-alternating metric. Although this solution is obtained only at given point, the classification obtained is invariant because the results are applicable to any point in the space.

Real curvatures take the form

$$\Lambda_s=\alpha_s+i\beta_s\,, \qquad (3.23)$$

in gravitational fields (spaces) of the 3rd kind, where the quantities Λ_s are real: $\Lambda_1=\Lambda_2=\Lambda_3=-\frac{\kappa}{3}$.

Values of some stationary curvatures in gravitational fields (spaces) of the 1st and 2nd kinds can be coincident. If they coincide, we have sub-kinds of the fields (spaces). The 1st kind has 3 sub-kinds: I ($\Lambda_1\neq\Lambda_2\neq\Lambda_3$); D ($\Lambda_2=\Lambda_3$); O ($\Lambda_1=\Lambda_2=\Lambda_3$). If the space is empty ($\kappa=0$) the kind O means the flat space. The 2nd kind has 2 sub-kinds: II ($\Lambda_1\neq\Lambda_2$); N ($\Lambda_1=\Lambda_2$). Kinds I and II are called basic kinds.

In empty spaces (empty gravitational fields) the stationary curvatures become the unit value $\Lambda=0$, so the spaces (fields) are called *degenerate*.

Studying the algebraic structure of the curvature tensor for known solutions to the Einstein equations, it was shown that the most of the solutions are of the 1st kind by Petrov's classification. The curvature decreases with distance from a gravitating mass. In the extreme case where the distance becomes infinite the space approaches the Minkowski flat space. The well-known Schwarzschid solution, describing a spherically symmetric gravitational field derived from a spherically symmetric island of mass located in an empty space, is classified as the sub-kind D of the 1st kind [44].

Invariant criteria for gravitational waves are linked to the algebraic structure of the curvature tensor, which should be associated with a given criterion from the aforementioned types. The most well-known solutions, which are interpreted as gravitational waves, are attributed to the sub-kind N (of the 1st kind). Other solutions are attributed to the 2nd kind and the 3rd kind. It should be noted that spaces of the 2nd and 3rd kinds cannot be flat anywhere, because components of the curvature tensor matrix $\|R_{ab}\|$ contain $+1$ and -1. This makes asymptotical approach to a curvature of zero impossible, i.e. excludes asymptotical approach to Minkowski space. Therefore, because of the structure of such fields, gravitational fields in a space of the 2nd kind (the sub-kind N) or the 3rd kind, are gravitational waves of the curvature traveling everywhere in the space. Pirani [18] holds that gravitational waves are solutions to gravitational fields in spaces of the 2nd kind (the sub-kind N) or the 3rd kind by Petrov's classification. The following solutions are classified as sub-kind N: Peres' solution [34] where he describes flat gravitational waves

$$ds^2 = (dx^0)^2 - 2\alpha(dx^0 + dx^3)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2; \tag{3.24}$$

Takeno's solution [35]

$$ds^2 = (\gamma + \rho)(dx^0)^2 - 2\rho dx^0 dx^3 - \alpha(dx^1)^2 - 2\delta dx^1 dx^2 - \beta(dx^2)^2 + (\rho - \gamma)(dx^3)^2, \tag{3.25}$$

where $\alpha = \alpha(x^1 - x^0)$, and γ, ρ, β, δ are functions of $(x^3 = x^0)$; Petrov's solution [37], studied also by Bondi, Pirani and Robertson in another coordinate system [19]

$$ds^2 = (dx^0)^2 - (dx^1)^2 + \alpha(dx^2)^2 + 2\beta dx^2 dx^3 + \gamma(dx^3)^2, \tag{3.26}$$

where α, β, γ are functions of $(x^1 + x^0)$.

A detailed study of relations between the invariant criteria for gravitational waves and Petrov's classification had been undertaken by Zakharov [40]. He proved:

THEOREM *In order that a given space satisfies the state of "pure gravitational radiations" (in the Lichnerowicz sense), it is a necessary and sufficient condition that the space should be of the sub-kind N by Petrov's algebraical classification, characterized by equality to zero of the values of the curvature tensor matrix $\|R_{ab}\|$ in the bivector space.*

THEOREM *An Einstein space that satisfies Zelmanov's criterion can only be an empty space ($\varkappa = 0$) of the sub-kind N. And conversely, any empty space V_4 of the sub-kind N (excluding the sole symmetric space* of this kind), that is described by the metric*

$$ds^2 = 2dx^0 dx^1 - \mathrm{sh}^2 dx^0 (dx^2)^2 - \sin^2 dx^0 (dx^3)^2, \tag{3.27}$$

satisfies the Zelmanov criterion.

*A space is called *symmetric*, if its curvature tensor is a covariant constant, i.e. if it satisfies the condition $\nabla_\sigma R_{\alpha\beta\gamma\delta} = 0$.

With these theorems we obtain the general relation between the Zelmanov criterion for gravitational wave fields located in empty spaces and the Lichnerowicz criterion for "pure gravitational radiations":

> An empty V_4, satisfying the Zelmanov criterion for gravitational wave fields, also satisfies the Lichnerowicz criterion for "pure gravitational radiations". Conversely, any empty V_n, satisfying the Lichnerowicz criterion (excluding the sole trivial V_n described by the metric 3.27), satisfies the Zelmanov criterion. The relation between the criteria in the general case is still an open problem.

In [40] it was shown that all known solutions to the Einstein equations in vacuum, which satisfy the Zelmanov and Lichnerowicz criteria, can be obtained as particular cases of the more generalized metric whose space permits a covariant constant vector field l^α

$$\nabla_\sigma l^\alpha = 0\,. \tag{3.28}$$

It is evident that condition (3.10) leads automatically to the first condition (3.7), hence this empty V_4 is classified as sub-kind N by Petrov's classification and, also, there the vector l^α, playing a part of the gravitational field wave vector, is isotropic $l_\alpha l^\alpha = 0$ and unique. According to Eisenhart's theorem [60], the space V_4 containing the unique isotropic covariant constant vector l^α (the absolute parallel vector field l^α, in other words), has the metric

$$ds^2 = \varepsilon(dx^0)^2 + 2dx^0 dx^1 + 2\varphi dx^0 dx^2 + 2\psi dx^0 dx^3 + \alpha(dx^2)^2 + 2\gamma dx^2 dx^3 + \beta(dx^3)^2, \tag{3.29}$$

where ε, φ, ψ, α, β, γ are functions of x^0, x^2, x^3, and $l^\alpha = \delta_1^\alpha$. The metric (3.29), satisfying equations (3.2), is the exact solution to the Einstein equations for vacuum, and satisfies the Zelmanov and Lichnerowicz gravitational wave criteria. This solution generalizes well-known solutions deduced by Takeno, Peres, Bondi, Petrov and others, that satisfy the aforementioned criteria [40].

The metric (3.29), taken under some additional conditions [30], satisfies the Einstein equations in their general form (3.1) in the case where $\lambda = 0$ and the energy-momentum tensor $T_{\alpha\beta}$ describes an isotropic electromagnetic field where Maxwell's tensor $F_{\mu\nu}$ satisfies the conditions

$$F_{\mu\nu}F^{\mu\nu} = 0\,, \qquad F_{\mu\nu}F^{*\mu\nu} = 0\,, \tag{3.30}$$

$F^{*\mu\nu} = \frac{1}{2}\eta^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the pseudotensor dual of the Maxwell tensor, $\eta^{\mu\nu\rho\sigma}$ is the discriminant tensor. Direct substitution shows that this metric satisfies the following requirements: the Riner-Wheeler condition [61]

$$R = 0\,, \qquad R_{\alpha\rho}R^{\rho\beta} = \frac{1}{4}\,\delta_\alpha^\beta\,(R_{\rho\sigma}R^{\rho\sigma}) = 0\,, \tag{3.31}$$

and also the Nordtvedt-Pagels condition [62]

$$\eta_{\mu\varepsilon\gamma\sigma}\left(R^{\delta\gamma,\sigma}R^{\varepsilon\tau}-R^{\delta\varepsilon,\sigma}R^{\gamma\tau}\right), \tag{3.32}$$

where $R^{\delta\gamma,\sigma}=g^{\sigma\mu}\nabla_{\mu}R^{\delta\gamma}$, $\delta^{\alpha}_{\beta}=g^{\alpha}_{\beta}$.

From the physical viewpoint we have an interest in isotropic electromagnetic fields because an observer who accompanies it should be moving at the velocity of light [18, 21]. Hence, isotropic electromagnetic fields can be interpreted as fields of electromagnetic radiation without sources. On the other hand, according to Eisenhart theorem [60], a space V_4 with the metric (3.29) permits an absolute parallel vector field $l^{\alpha}=\delta^{\alpha}_{1}$. Taking this fact and also the Einstein equations into account, we conclude that the vector l^{α} considered in this case satisfies the Lichnerowicz criterion for "full gravitational radiations".

Thus the metric (3.29), satisfying the conditions

$$\begin{gathered}R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=-\kappa T_{\alpha\beta}\,,\\ T_{\alpha\beta}=\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\alpha\beta}-F_{\alpha\sigma}F_{\beta\cdot}^{\cdot\sigma}\,,\\ F_{\alpha\beta}F^{\alpha\beta}=0\,,\qquad F_{\alpha\beta}F^{*\alpha\beta}=0\end{gathered} \tag{3.33}$$

and under the additional condition [30]

$$R_{2323}=R_{0232}=R_{0323}=0\,, \tag{3.34}$$

is the exact solution to the Einstein equations which describes co-existence of both gravitational waves and electromagnetic waves. This solution does not satisfy the Zelmanov criterion in the general case, but the solution satisfies it in some particular cases where $T_{\alpha\beta}\neq 0$, and also under $R_{\alpha\beta}=0$.

Wave properties of recursion curvature spaces were studied in [63]. A recursion curvature space is a Riemannian space having a curvature which satisfies the relationship

$$\nabla_{\sigma}R_{\alpha\beta\gamma\delta}=l_{\sigma}R_{\alpha\beta\gamma\delta}\,. \tag{3.35}$$

Because of Bianchi's identity, such spaces satisfy

$$l_{\sigma}R_{\alpha\beta\gamma\delta}+l_{\alpha}R_{\beta\sigma\gamma\delta}+l_{\beta}R_{\sigma\alpha\gamma\delta}=0\,. \tag{3.36}$$

Total classification for recursion curvature spaces had been given by Walker [64]. His results [64] were applied to the basic space-time of the General Theory of Relativity, see [65] for the results. For the class of prime recursion spaces*, we are particularly interested in the two metrics

$$ds^2=\psi(x^0,x^2)(dx^0)^2+2dx^0dx^1-(dx^2)^2-(dx^3)^2, \tag{3.37}$$

$$\begin{aligned}ds^2=2dx^0dx^1+\psi(x^1,x^2)(dx^1)^2-\\ -(dx^2)^2-(dx^3)^2,\quad \psi>0\,.\end{aligned} \tag{3.38}$$

*A recursion curvature space is known as prime or simple, if it contains $n-2$ parallel vector fields, which could be isotropic or non-isotropic. Here n is the dimension of the space.

For the metric (3.37) there is only one component of the Ricci tensor that is not zero, $R_{00}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^{2\,2}}$, in the metric (3.38) only $R_{11}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^{2\,2}}$ is not zero. Einstein spaces with such metrics can only be empty ($\kappa=0$) and flat ($R_{\alpha\beta\gamma\delta}=0$). This can be proven by checking that both metrics satisfy conditions (3.31) and (3.32), which describe isotropic electromagnetic fields.

Both metrics are interesting from the physical viewpoint: in these cases the origin of the space curvature is an isotropic electromagnetic field. Moreover, if we remove this field from the space, the space becomes flat. Besides these there are few metrics which are exact solutions to the Einstein-Maxwell equations, related to the class of isotropic electromagnetic fields. Neither of the said metrics satisfy the Zelmanov and Lichnerowicz criteria.

Minkowski's signature permits only two metrics for non-simple recursion curvature spaces. They are the metric

$$\begin{gathered}ds^2=\psi(x^0,x^2,x^3)(dx^0)^2+2dx^0dx^1+\\ +K_{22}(dx^2)^2+2K_{23}dx^2dx^3+K_{33}(dx^3)^2,\\ K_{22}<0\,,\qquad K_{22}K_{33}-K_{23}^2<0\,,\end{gathered} \tag{3.39}$$

wherein $\psi=\chi_1(x_0)(a_{22}(x^2)^2+2a_{23}x^2x^3+a_{33}(x^3)^2)+$ $+\chi_2(x^0)x^2+\chi_3(x^0)x^3$, and the metric

$$\begin{gathered}ds^2=2dx^0dx^1+\psi(x^1,x^2,x^3)(dx^1)^2+\\ +K_{22}(dx^2)^2+2K_{23}dx^2dx^3+K_{33}(dx^3)^2,\end{gathered} \tag{3.40}$$

wherein $\psi=\chi_1(x_1)(a_{22}(x^2)^2+2a_{23}x^2x^3+a_{33}(x^3)^2)+$ $+\chi_2(x^1)x^2+\chi_3(x^1)x^3$. Here a_{ij}, K_{ij} $(i,j=2,3)$ are constants.

Both metrics satisfy the conditions $R_{\alpha\beta}=\kappa g_{\alpha\beta}$ only if $\kappa=0$, reducing to the single relationship

$$K_{33}a_{22}+K_{22}a_{33}-2K_{23}a_{23}=0\,. \tag{3.41}$$

In this case both metrics are of the sub-kind N by Petrov's classification. It is interesting to note that the metric (3.40) is stationary and, at the same time, describes "pure gravitational radiation" by Lichnerowicz. Such a solution was also obtained in [65].

In the general case ($R_{\alpha\beta}\neq\kappa g_{\alpha\beta}$) the metrics (3.39) and (3.40) satisfy conditions (3.32) and (3.33), so the metrics are solutions to the Einstein-Maxwell equations that describe co-existing gravitational waves and electromagnetic waves without sources. In this general case both metrics satisfy the Zelmanov and Lichnerowicz invariant criteria. The solution (3.40) is stationary.

All that has been detailed above applies to gravitational waves as waves of the space curvature, which exist in any reference frame.

Additionally it would be interesting to study another approach to the gravitational radiation problem, where the main issue is gravitational inertial waves, connected to the given reference frame of an observer. This new approach is linked directly to the mathematical apparatus of physically observable quantities (the theory of chronometric invariants), introduced by Zelmanov in 1944 [42, 43]. In order to understand the true results given by gravitational wave experiments it is necessary to master this mathematical apparatus, which is described concisely in the in the next section.

4 Basics of the theory of physical observable quantities

In brief, the essence of the mathematical apparatus of physically observable quantities (the theory of chronometric invariants), developed by Zelmanov in 1940's [42, 43] is that, if an observer accompanies his reference body, his observable quantities are projections of four-dimensional quantities on his time line and the spatial section — *chronometrically invariant quantities*, made by the projecting operators $b^\alpha = \frac{dx^\alpha}{ds}$ and $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$ which fully define his real reference space (here b^α is his velocity with respect to his real references). The chr.inv.-projections of a world-vector Q^α are $b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}}$ and $h^i_\alpha Q^\alpha = Q^i$, while chr.inv.-projections of a world-tensor of the 2nd rank $Q^{\alpha\beta}$ are $b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}$, $h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q_0^i}{\sqrt{g_{00}}}$, $h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}$. Physically observable properties of the space are derived from the fact that the chr. inv.-differential operators $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}v_i\frac{{}^*\partial}{\partial t}$ are non-commutative, so that $\frac{{}^*\partial^2}{\partial x^i\partial t} - \frac{{}^*\partial^2}{\partial t\,\partial x^i} = \frac{1}{c^2}F_i\frac{{}^*\partial}{\partial t}$ and $\frac{{}^*\partial^2}{\partial x^i\partial x^k} - \frac{{}^*\partial^2}{\partial x^k\partial x^i} = \frac{2}{c^2}A_{ik}\frac{{}^*\partial}{\partial t}$, and also from the fact that the chr.inv.-metric tensor h_{ik} may not be stationary. The observable characteristics are the chr.inv.-vector of gravitational inertial force F_i, the chr.inv.-tensor of angular velocities of the space rotation A_{ik}, and the chr.inv.-tensor of rates of the space deformations D_{ik}, namely

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \quad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2} \tag{4.1}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \tag{4.2}$$

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \quad D^k_k = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \tag{4.3}$$

where w is the gravitational potential, $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation, $h_{ik} = -g_{ik} + \frac{1}{c^2}v_i v_k$ is the metric chr.inv.-tensor, and $h = \det\|h_{ik}\|$, $h g_{00} = -g$, $g = \det\|g_{\alpha\beta}\|$. Observable inhomogeneity of the space is set up by the chr.inv.-Christoffel symbols $\Delta^i_{jk} = h^{im}\Delta_{jk,m}$, which are built just like Christoffel's regular symbols $\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$, but using h_{ik} instead of $g_{\alpha\beta}$.

In this way, any equations obtained using general covariant methods we can calculate their physically observable projections on the time line and the spatial section of any particular reference body and formulate the projections with its real physically observable properties. From this we arrive at equations containing only quantities measurable in practice. Expressing $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$ through the observable time interval

$$d\tau = \frac{1}{c}\,b_\alpha dx^\alpha = \left(1 - \frac{\mathrm{w}}{c^2}\right)dt - \frac{1}{c^2}\,v_i dx^i \tag{4.4}$$

and also the observable spatial interval $d\sigma^2 = h_{\alpha\beta}\,dx^\alpha dx^\beta = h_{ik}\,dx^i dx^k$ (note that $b^i = 0$ for an observer who accompanies his reference body). We arrive at the formula

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{4.5}$$

From an"external" viewpoint, an observer's three-dimensional space is the *spatial section* $x^0 = ct = \text{const}$. At any point of the space-time a local spatial section (a local space) can be placed orthogonal to the *time line*. If there exists a space-time enveloping curve for such local spaces, then it is a spatial section everywhere orthogonal to the time lines. Such a space is called *holonomic*. If no enveloping curve exists for such local spaces, so there only exist spatial sections locally orthogonal to the time lines, such a space is called *non-holonomic*. A spatial section, placed in a holonomic space, is everywhere orthogonal to the time lines, i. e. $g_{0i} = 0$ is true there. In the presence of $g_{0i} = 0$ we have $v_i = 0$, hence $A_{ik} = 0$. This implies that non-holonomity of the space and its three-dimensional rotation are the same. In a non-holonomic space $g_{0i} \neq 0$ and $A_{ik} \neq 0$. Hence $A_{ik} = 0$ is the necessary and sufficient condition of holonomity of the space. So A_{ik} is the *tensor of the space non-holonomity*.

Zelmanov had also found that the chr.inv.-quantities F_i and A_{ik} are linked to one another by two identities

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{4.6}$$

$$\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0\,, \tag{4.7}$$

which are known as *Zelmanov's identities*.

Components of the usual Christoffel symbols

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right). \tag{4.8}$$

are linked to the chr.inv.-Christoffel symbols

$$\Delta^i_{jk} = \frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^j} - \frac{{}^*\partial h_{jk}}{\partial x^m}\right), \tag{4.9}$$

and other chr.inv.-chractersitics of the accompanying reference space of the given observer by the relations

$$D_k^i + A_{k\cdot}^{\cdot i} = \frac{c}{\sqrt{g_{00}}}\left(\Gamma_{0k}^i - \frac{g_{0k}\Gamma_{00}^i}{g_{00}}\right), \qquad (4.10)$$

$$F^k = -\frac{c^2\,\Gamma_{00}^k}{g_{00}}\,, \qquad g^{i\alpha}g^{k\beta}\,\Gamma_{\alpha\beta}^m = h^{iq}h^{ks}\Delta_{qs}^m. \qquad (4.11)$$

Here is the four-dimensional generalization of the chr.inv.-quantities F_i, A_{ik}, and D_{ik} (by Zelmanov, the 1960's [57]): $F_\alpha = -2c^2 b^\beta a_{\beta\alpha}$, $A_{\alpha\beta} = ch_\alpha^\mu h_\beta^\nu a_{\mu\nu}$, $D_{\alpha\beta} = ch_\alpha^\mu h_\beta^\nu d_{\mu\nu}$, where $a_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha b_\beta - \nabla_\beta b_\alpha\right)$, $d_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha b_\beta + \nabla_\beta b_\alpha\right)$.

Zelmanov also deduced formulae for chr.inv.-projections of the Riemann-Christoffel tensor [42]. He followed the same procedure by which the Riemann-Christoffel tensor was built, proceeding from the non-commutativity of the second derivatives of an arbitrary vector taken in the given space. Taking the second chr.inv.-derivatives of an arbitrary vector

$${}^*\nabla_i{}^*\nabla_k\,Q_l - {}^*\nabla_k{}^*\nabla_i\,Q_l = \frac{2A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t} + H_{lki\cdot}^{\cdot\cdot\cdot j}Q_j\,, \quad (4.12)$$

he obtained the chr.inv.-tensor

$$H_{lki\cdot}^{\cdot\cdot\cdot j} = \frac{{}^*\partial\Delta_{il}^j}{\partial x^k} - \frac{{}^*\partial\Delta_{kl}^j}{\partial x^i} + \Delta_{il}^m\Delta_{km}^j - \Delta_{kl}^m\Delta_{im}^j\,, \quad (4.13)$$

which is like Schouten's tensor from the theory of non-holonomic manifolds [59]. The tensor $H_{lki}^{\cdot\cdot\cdot j}$ differs algebraically from the Riemann-Christoffel tensor because of the presence of rotation of the space A_{ik} in the formula (4). Nevertheless its generalization gives the chr.inv.-tensor

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right), \qquad (4.14)$$

which possesses all the algebraic properties of the Riemann-Christoffel tensor in this three-dimensional space. Therefore Zelmanov called C_{iklj} the *chr.inv.-curvature tensor*, which actually is the tensor of the observable curvature of the observer's spatial section. This tensor, describing the observable curvature of the three-dimensional space of an observer, possesses all the properties of the Riemann-Christoffel curvature tensor in the three-dimensional space and, at the same time, the property of chronometric invariance. Its contraction

$$C_{kj} = C_{kij\cdot}^{\cdot\cdot\cdot i} = h^{im}C_{kimj}\,, \qquad C = C_j^j = h^{lj}C_{lj} \quad (4.15)$$

gives the chr.inv.-scalar C whose sense is the *observable three-dimensional curvature* of this space.

Substituting the necessary components of the Riemann-Christoffel tensor into the formulae for its chr.inv.-projections $X^{ik} = -c^2\frac{R_{\cdot 0\cdot 0\cdot}^{\cdot i\cdot k}}{g_{00}}$, $Y^{ijk} = -c\frac{R_{0\cdot\cdot\cdot}^{\cdot ijk}}{\sqrt{g_{00}}}$, $Z^{ijkl} = c^2 R^{ijkl}$, and by lowering indices Zelmanov obtained the formulae

$$\begin{aligned} X_{ij} = \frac{{}^*\partial D_{ij}}{\partial t} &- \left(D_i^l + A_{i\cdot}^{\cdot l}\right)\left(D_{jl} + A_{jl}\right) + \\ &+ \frac{1}{2}\left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2}\,F_iF_j\,, \end{aligned} \qquad (4.16)$$

$$\begin{aligned} Y_{ijk} = {}^*\nabla_i\left(D_{jk} + A_{jk}\right) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \\ + \frac{2}{c^2}A_{ij}F_k\,, \end{aligned} \qquad (4.17)$$

$$\begin{aligned} Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - A_{il}A_{kj} + \\ + 2A_{ij}A_{kl} - c^2 C_{iklj}\,, \end{aligned} \qquad (4.18)$$

where we have $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$ just like the Riemann-Christoffel tensor. Contraction of the spatial observable projection Z_{iklj} step-by-step gives

$$Z_{il} = D_{ik}D_l^k - D_{il}D + A_{ik}A_{l\cdot}^{\cdot k} + 2A_{ik}A_{\cdot l}^{k\cdot} - c^2C_{il}\,, \quad (4.19)$$

$$Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2C\,. \quad (4.20)$$

Besides these considerations, taken in an observer's accompanying reference frame, Zelmanov considered a *locally geodesic reference frame* that can be introduced at any point of the pseudo-Riemannian space. Within infinitesimal vicinities of any point of such a reference frame the fundamental metric tensor is

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2\tilde{g}_{\alpha\beta}}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \ldots\,, \quad (4.21)$$

i. e. its components at a point, located in the vicinities, are different to those at the point of reflection to within only the higher order terms, values of which can be neglected. Therefore, at any point of a locally geodesic reference frame the fundamental metric tensor can betaken as constant, while the first derivatives of the metric (the Christoffel symbols) are zero.

As a matter of fact, within infinitesimal vicinities of any point located in a Riemannian space, a locally geodesic reference frame can be defined. At the same time, at any point of this locally geodesic reference frame, a tangential flat Euclidean space can be defined so that this reference frame, being locally geodesic for the Riemannian space, is the global geodesic for that tangential flat space.

The fundamental metric tensor of a flat Euclidean space is constant, so values of $\tilde{g}_{\mu\nu}$, taken in the vicinities of a point of the Riemannian space converge to values of the tensor $g_{\mu\nu}$ in the flat space tangential at this point. Actually, this means that we can build a system of basis vectors $\vec{e}_{(\alpha)}$, located in this flat space, tangential to curved coordinate lines of the Riemannian space.

In general, coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other (if the space is non-holonomic). So the lengths of the basis vectors may be sometimes very different from unity.

We denote a four-dimensional vector of infinitesimal displacement by $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$. Then $d\vec{r} = \vec{e}_{(\alpha)}dx^\alpha$, where components of the basis vectors $\vec{e}_{(\alpha)}$ tangential to the coordinate lines are $\vec{e}_{(0)} = \{e_{(0)}^0, 0, 0, 0\}$, $\vec{e}_{(1)} = \{0, e_{(1)}^1, 0, 0\}$, $\vec{e}_{(2)} = \{0, 0, e_{(2)}^2, 0\}$, $\vec{e}_{(3)} = \{0, 0, 0, e_{(3)}^2\}$. The scalar product of the vector $d\vec{r}$ with itself is $d\vec{r}d\vec{r}=ds^2$. On the other hand, the same quantity is $ds^2=g_{\alpha\beta}\,dx^\alpha dx^\beta$. As a result we have $g_{\alpha\beta}=\vec{e}_{(\alpha)}\vec{e}_{(\beta)}=e_{(\alpha)}e_{(\beta)}\cos\left(x^\alpha;x^\beta\right)$. So we obtain

$$g_{00}=e^2_{(0)}\,,\qquad g_{0i}=e_{(0)}e_{(i)}\cos\left(x^0;x^i\right), \tag{4.22}$$

$$g_{ik}=e_{(i)}e_{(k)}\cos\left(x^i;x^k\right). \tag{4.23}$$

The gravitational potential is $\mathrm{w}=c^2(1-\sqrt{g_{00}})$. So, the time basis vector $\vec{e}_{(0)}$ tangential to the time line $x^0=ct$, having the length $e_{(0)}=\sqrt{g_{00}}=1-\frac{\mathrm{w}}{c^2}$ is smaller than unity the greater is the gravitational potential w.

The space rotation linear velocity v_i and, according to it, the chr.inv.-metric tensor h_{ik} are

$$v_i=-c\,e_{(i)}\cos\left(x^0;x^i\right), \tag{4.24}$$

$$h_{ik}=e_{(i)}e_{(k)}\Big[\cos(x^0;x^i)\cos(x^0;x^k)-\cos(x^i;x^k)\Big]. \tag{4.25}$$

This representation enablkes us to see the geometric sense of physical quantities measurable in experiments, because we represent them through pure geometric characteristics of the observer's space — the angles between coordinate axes etc.

This completes the basics of Zelmanov's mathematical apparatus of chronometric invariants (physically observable quantities) that will be employed below with the aim of studying the gravitational wave problem.

5 Gravitational inertial waves and their link to the chronometrically invariant representation of Petrov's classification

Of all the experimental statements on the General Theory of Relativity, including the search for gravitational wave experiments, the most important case is that where the observer is at rest with respect to his laboratory reference frame and all physical standards located in it. Quantities measured by the observer in an *accompanying reference frame* are *chronometrically invariant quantities* (see the previous paragraph for the details). Keeping this fact in mind, Zelmanov formulated his *chronometrically invariant criterion for gravitational waves*. This criterion is invariant only for transformations of coordinates of that reference system which is at rest with respect to the laboratory references (the body of reference). Such an approach, in contrast to the invariant approach, permits us to interpret the results of measurement in terms of physically observable quantities, providing thereby a means of comparing results given by the theory of gravitational waves to results obtained from real physical experiments.

In order to solve the problem of interpretation of experimental data on gravitational waves it is appropriate to consider a more general case — fields of gravitational inertial waves. Such fields are more general because they are applicable to both gravitational fields and the inertial field of the observer's reference frame. The mathematical method that we propose to apply to this problem joins both fields into a common field. The method itself does not differ for each field: to set an invariant difference between gravitational fields and the observer's inertial field would be possible only by introducing an additional invariant criterion.

Gravitational waves are determined independently of both spatial coordinate frames and space-time reference frames. In contrast to gravitational waves, gravitational inertial waves are determined only in the reference frame of an observer, who observes them. They are determined with precision to within so-called "inner" transformations of coordinates

$$\left.\begin{aligned} &\text{(a)}\quad \tilde{x}^0=\tilde{x}^0(x^0,x^1,x^2,x^3)\\ &\text{(b)}\quad \tilde{x}^i=\tilde{x}^i(x^1,x^2,x^3)\,,\qquad \frac{\partial\tilde{x}^i}{\partial x^0}=0 \end{aligned}\right\} \tag{5.1}$$

which does not change the space-time reference frame itself.

Invariance with respect to (5.1) splits into invariance with respect to (5.1a), so-called *chronometric invariance*, and also invariance with respect to (5.1b), so-called *spatial invariance*. Therefore a definition given for gravitational inertial waves should be:

(1) chronometrically invariant;

(2) spatially covariant.

We then have a basis by which we introduce the chronometrically invariant spatially covariant d'Alembert operator [40]*

$$^*\Box=h^{ik}\,{}^*\nabla_i{}^*\nabla_k-\frac{1}{a^2}\frac{^*\partial^2}{\partial t^2}\,, \tag{5.2}$$

where $h^{ik}=-g^{ik}$ is the chr.inv.-metric tensor (the physically observable metric tensor) in its contravariant (upper-index) form, $^*\nabla_i$ is the symbol for the chr.inv.-derivative (the chr.inv.-analogue to the covariant derivative symbol ∇_σ), a is the linear velocity at which attraction of gravity spreads, $\frac{^*\partial}{\partial t}$ is the symbol for the chr.inv.-derivative with respect to time.

A chronometrically invariant criterion for gravitational inertial waves, formulated according to Zelmanov's idea, is:

Zelmanov's chr.inv.-criterion Chr.inv.-quantities f, characterising the observer's reference space, such as the gravitational inertial force vector F_i, the space non-holonomity (self-rotation) tensor A_{ik}, the space deformation rate tensor D_{ik}, the spatial curvature tensor C_{iklj}, and also scalar quantities, built on them, and also the Riemann-Christoffel curvature tensor's chr.inv.-components X^{ij}, Y^{ijk}, Z^{iklj} must satisfy equations of the form

$$^*\Box f=A, \tag{5.3}$$

*This approach to the gravitational inertial wave problem was developed by Zelmanov, although it had first been published by Zakharov because the latter prepared his dissertation under Zelmanov's leadership: see footnote on page 35.

where A is an arbitrary function of four-dimensional world-coordinates, which has no more than first order derivatives of the f.

The Zelmanov chr.inv.-criterion (5.3) was applied in analyzing well-known solutions to the Einstein equations in emptiness [40]. This criterion is true for the metrics (3.25) in that case where the gravitational inertial force vector F^i is the wave function. But, at the same time, most of the invariant criteria for gravitational waves are related to some conditions and limitations imposed on the curvature tensor. Therefore it would be most interesting to study relations between gravitational wave criteria and gravitational inertial wave criteria in that case where the Riemann-Christoffel curvature tensor's chr.inv.-components X^{ij}, Y^{ijk}, Z^{iklj} are the wave functions.

What is the relation between the Zelmanov invariant criterion (3.9) and his chr.inv.-criterion (5.3)? This problem was solved by Zakharov [40, 58]. His method was to express equation (3.9) in chr.inv.-form. In chr.inv..-form (in the terms of physically observable quantities) equation (3.9) takes the form

$$^*\Box X^{ij} = A^{ij}_{(1)}\,, \quad ^*\Box Y^{ijk} = A^{ijk}_{(2)}\,, \quad ^*\Box Z^{iklj} = A^{iklj}_{(3)}\,, \tag{5.4}$$

where $A^{ij}_{(1)}$, $A^{ijk}_{(2)}$, $A^{iklj}_{(3)}$ are chronometrically invariant and spatially invariant tensors, which have no more than first order derivatives of the wave functions X^{ij}, Y^{ijk}, Z^{iklj}. Thus those gravitational fields that satisfy the Zelmanov invariant criterion also satisfy the Zelmanov chr.inv.-criterion (5.3), where the Riemann-Christoffel curvature tensor's physically observable components X^{ij}, Y^{ijk}, Z^{iklj} play the part of wave functions.

The necessary condition for gravitational inertial waves is the fact that the chr.inv.-d'Alembert operator (5.2) is non-trivial, mathematically expressed as follows:

1. Chr.inv.-quantities f are non-stationary, i. e. $\frac{^*\partial f}{\partial t} \neq 0$;
2. The quantities f are inhomogeneous, i. e. $^*\nabla_i f_k \neq 0$.

The wave functions X_{ij} (4.16), Y_{ijk} (4.17) and Z_{iklj} (4.18) satisfy these requirements only if the mechanical chr.inv.-characteristics of the observer's reference space (the chr.inv.-quantities F_i, A_{ik}, D_{ik}) and the geometric chr.inv.-characteristic of the space (the chr.inv.-quantity C_{iklj}) also satisfy these requirements. Zelmanov himself in [42] formulated conditions of inhomogeneity inside a finite region located in the observer's space

$$\begin{aligned} &^*\nabla_i F_k \neq 0\,, \quad ^*\nabla_j A_{ik} \neq 0\,, \\ &^*\nabla_j D_{ik} \neq 0\,, \quad ^*\nabla_j C_{ik} \neq 0\,. \end{aligned} \tag{5.5}$$

It is evident that under these conditions the wave functions X^{ij}, Y^{ijk}, Z^{iklj} shall be inhomogeneous.

The origin of non-stationary states of the gravitational inertial force vector F_i (4.1) is the non-stationarity of the gravitational potential w or the linear velocity of the space rotation v_i, consisting the force. Identities (4.6) and (4.7), linking quantities F_i and A_{ik}, lead us to conclude that the source of non-stationary states of v_i is the vortical nature of the vector F_i, i. e. $^*\nabla_k F_i - {^*\nabla_i} F_k \neq 0$. The origin of non-stationary states of the space deformation rate D_{ik} (4.3) and the space observable curvature C_{iklj} (4.14) is non-stationarity of the physical observable metric tensor h_{ik}, see [42],

$$h_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}} = -g_{ik} + \frac{1}{c^2}\, v_i v_k\,. \tag{5.6}$$

Thus, the origin of non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} is the non-stationarity of components of the fundamental metric tensor $g_{\alpha\beta}$, namely:

(1) $g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2$;

(2) $g_{0i} = -\frac{1}{c}\, v_i \left(1 - \frac{\mathrm{w}}{c^2}\right)$;

(3) $g_{ik} = -h_{ik} + \frac{1}{c^2}\, v_i v_k$.

We consider each of the cases here,mindful of the need to find theoretical grounds for gravitational wave experiments:

1. Non-stationary states of g_{00} manifest as a result of time changes of the gravitational potential w. In experiments this non-stationarity is derived from very different geophysical sources, which, in a particular case, are due to changes in solar activity;
2. Non-stationary states of mixed components g_{0i} are derived from the non-stationarity of the space rotation linear velocity v_i and the gravitational potential w. The quantities g_{00} and g_{0i} are included in the formula for an interval of observable time $d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{\sqrt{g_{00}}}\, dx^i$ [42, 43]. Thus under non-stationary states of g_{00} and g_{0i} in the observer's laboratory (his reference frame) a standard clock located there should have some corrections (which change with time) with respect to a standard clock located in an region where the quantities g_{00} and g_{0i} are stationary.
3. Non-stationary states of g_{ik} are usually considered as deformations of the three-dimensional space. But the theory of physically observable quantities introduces substantial corrections to this thesis. The approach of Classical Mechanic looks at the spatial deformations as $\frac{1}{2}\frac{\partial g_{ik}}{\partial t}$, but the theory of physically observable quantities, taking properties of the observer into account, gives rise to a corrected formula for the spatial deformations which is $D_{ik} = \frac{1}{2\sqrt{g_{00}}}\frac{\partial}{\partial t}\left(-g_{ik} + \frac{1}{c^2}\, v_i v_k\right)$.*

*The presence of the minus sign here is a consequence of the fact that we use the signature $(+---)$, where plus is related to the time coordinate while minus is attributed to spatial coordinates. The minus sign has been chosen for the g_{ik} in the h_{ik} formula, because in this case the observable spatial interval $d\sigma = h_{ik}\, dx^i dx^k$ is positive, which is an important fact in the theory of physically observable quantities [42, 43].

The formulae coincide in that particular case where $g_{00} = 1$ (w = 0) and $g_{0i} = 0$ ($v_i = 0$). If $F_i = 0$, according (to 4.6) the space rotation is stationary. If $v_i = 0$, $A_{ik} = 0$. Thus the necessary and sufficient condition to make w and v_i simultaneously zero is $F_i = 0$ and $A_{ik} = 0$ [42, 43]. In this case the observer's reference frame falls freely and is free of rotations. Such reference frames are known as *synchronous* [15], because there all clocks can be synchronized. Moreover, in this case time can be integrated: in calculations of the time interval $d\tau = dt$ between any two events, the integral of $d\tau$ is independent of the way we take this integral between the events (the path of integration). If $F_i \neq 0$ but $A_{ik} = 0$, it is impossible to synchronize all the clocks simultaneously, but the synchronization itself can be realized because of the proportionality $d\tau = \sqrt{g_{00}}\,dt$ there. If $A_{ik} \neq 0$, the synchronization is impossible in principle, because the integral of $d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{\sqrt{g_{00}}}\,dx^i$ depends on the path of integration [42, 43].

Synchronous reference frames, because of their simplicity and associated simple calculations, are of broad utility in the General Theory of Relativity. In particular, they are used in relativistic cosmology and the gravitational wave problem. For instance, the well-known metric of weak plane gravitational waves takes the form [14, 15]

$$ds^2 = c^2dt^2 - (dx^1)^2 - (1-a)(dx^2)^2 + \\ + 2b\,dx^2dx^3 - (1+a)(dx^3)^2, \tag{5.7}$$

where $a = a(ct \pm x^1)$, $b = b(ct \pm x^1)$. So in this metric there is no gravitational potential (w = 0) as soon as there is no space rotation ($v_i = 0$). The condition w = 0 prohibits the ultimate transit to Newton's theory of gravity. For this reason we arrive at an important conclusion:

> Weak plane gravitational waves are derived from sources other than gravitational fields of masses*.

An analogous situation arises in relativistic cosmology, where, until now, the main part is played by the theory of a homogeneous isotropic universe. Foundations of this theory are built on the metric of a homogeneous isotropic space [42]

$$ds^2 = c^2dt^2 - \\ - R^2 \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{\left[1 + \frac{k}{4}\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right]\right]^2}, \tag{5.8}$$

$$R = R(t)\,, \qquad k = 0, \pm 1\,.$$

When one substitutes this metric into the Einstein equations taken with a specific value of the cosmological constant ($\lambda = 0$, $\lambda < 0$, $\lambda > 0$), he obtains a spectra of solutions, which are known as *Friedmann's cosmological models* [42].

Taking our previous conclusion on the origin of weak plane gravitational waves into account, we come to a new and important conclusion:

> No gravitational fields derived from masses exist in any Friedmann universe. Moreover, any Friedmann universe is free of space rotations.

Currently there is no indubitable observational data supporting the absolute rotation of the Universe. This problem has been under considerable discussion between astronomers and physicists over last decade, and remains open. Rotations of bulk space bodies like planets, stars, and galaxies are beyond any doubt, but these rotations do not imply the absolute rotation of the whole Universe, including the absolute rotation of its gravitational field if one will describe it by the Friedmann models.

Looking back at the question of whether or not gravitational inertial waves exist, or whether or not non-stationary states of the wave functions X^{ij}, Y^{ijk}, Z^{iklj} exist, we conclude that non-stationary states of the quantities are derived from:

1. A vortical nature of the field of the acting gravitational inertial force F_i;
2. Non-stationary states of the spatial components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$.

In the first case, the effect of gravitational inertial waves manifests as non-stationary corrections to the observer's time flow.

In the second case, the observer's time flow remains unchanged, but gravitational waves are waves of only the space deformation. Such pure deformation waves will deform a detector itself, so one simply waits for a gravitational wave to cause a resonance effect in a solid-body detector of the Weber kind [16]. Whether this conclusion is true or false will be considered in §7 and §8. Here we consider only the general theory of gravitational inertial waves and its relation to the invariant theory of gravitational waves.

As we showed above, those gravitational fields that satisfy the Zelmanov invariant criterion (3.9) also satisfy the Zelmanov chr.inv.-criterion (5.3), where the wave functions f are the Riemann-Christoffel tensor's observable components X^{ij}, Y^{ijk}, Z^{iklj}. As it was shown in the previous paragraph, "empty gravitational fields" (we mean gravitational fields permeating empty spaces, where no mass islands of matter exist) that satisfy the Zelmanov invariant criterion (3.9) are related to the 2nd kind (the sub-kind N) by Petrov's classification. Therefore it is appropriate to specify the algebraical kinds of the Riemann-Christoffel tensor in terms of physically observable quantities (chronometric invariants).

The whole problem of representing Petrov's classification in chronometrically invariant form has been solved in [66]. This solution, obtained Petrov in general covariant form [37],

*See §7 and §8 below for detailed calculations for the effect due to weak plane gravitational waves in solid-body detectors of the Weber kind (the Weber pigs) and also in antennae built on free masses.

was obtained for an ortho-frame, taken at an arbitrary fixed point of the space.

Chr.inv.-components of the Riemann-Christoffel curvature tensor have the properties

$$X_{ij} = X_{ji}\,, \quad X_k^k = -\kappa\,, \qquad Y_{[ijk]} = 0\,, \quad Y_{ijk} = -Y_{ikj}\,. \tag{5.9}$$

Equations (4.16), (4.17), (4.18) in an ortho-frame are

$$X_{ij} = -c^2 R_{0i0j}\,, \qquad Y_{ijk} = -cR_{0ijk}\,, \qquad Z_{iklj} = c^2 R_{iklj}\,. \tag{5.10}$$

When we write equations $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ in the orth-frame, we take the relationships (5.10) into account. Then, introducing three-dimensional matrices x and y such that

$$x \equiv \|x_{ik}\| = -\frac{1}{c^2}\,\|X_{ik}\|\,, \qquad y \equiv \|y_{ik}\| = -\frac{1}{2c}\,\|\varepsilon_{imn} Y_{k\cdot\cdot}^{\cdot mn}\|\,, \tag{5.11}$$

where ε_{imn} is the three-dimensional discriminant tensor, we represent the six-dimensional matrix R_{ab} as follows

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|, \qquad a, b = 1, 2, \dots 6\,, \tag{5.12}$$

satisfying the relations

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \qquad y_{11} + y_{22} + y_{33} = 0\,. \tag{5.13}$$

Now, let us compose a lambda-matrix

$$\|R_{ab} - \Lambda g_{ab}\| = \left\| \begin{matrix} x + \Lambda\varepsilon & y \\ y & -x - \Lambda\varepsilon \end{matrix} \right\|, \tag{5.14}$$

where ε is the three-dimensional unit matrix. Then, after transformations, we reduce this lambda-matrix to the form

$$\left\| \begin{matrix} x + iy + \Lambda\varepsilon & 0 \\ 0 & -x - iy - \Lambda\varepsilon \end{matrix} \right\| = \left\| \begin{matrix} \bar{Q}(\Lambda) & 0 \\ 0 & \bar{Q}(\Lambda) \end{matrix} \right\|. \tag{5.15}$$

The initial lambda-matrix can have one of the following characteristics:

$$(1)\;\; [111, \overline{111}]; \quad (2)\;\; [21, \overline{21}]; \quad (3)\;\; [3, 3]. \tag{5.16}$$

Then, using Petrov's had obtained the canonical form of the matrix $\|R_{ab}\|$ in the non-holonomic ortho-frame for each of the three kinds of the curvature tensor [37], we express the matrix $\|R_{ab}\|$ through components of the chr.inv.-tensors X_{ij} and Y_{ijk} [66]. We obtain

<u>The 1st Kind</u>

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|,$$

$$x = \left\| \begin{matrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{matrix} \right\|, \qquad y = \left\| \begin{matrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{matrix} \right\|, \tag{5.17}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \quad y_{11} + y_{22} + y_{33} = 0\,. \tag{5.18}$$

Using (5.11) we also express values of the stationary curvatures Λ_i $(i = 1, 2, 3)$ through the Riemann-Christoffel tensor's physically observable components

$$\Lambda_1 = -\frac{1}{c^2}\,X_{11} + \frac{i}{c}\,Y_{123}\,, \qquad \Lambda_2 = -\frac{1}{c^2}\,X_{22} + \frac{i}{c}\,Y_{231}\,, \qquad \Lambda_3 = -\frac{1}{c^2}\,X_{33} + \frac{i}{c}\,Y_{312}\,. \tag{5.19}$$

Thus, the components X_{ik} are included in the real parts of the stationary curvatures Λ_i, and components Y_{ijk} are included in the imaginary parts. In spaces of the sub-kind D $(\Lambda_2 = \Lambda_3)$ we have: $X_{22} = X_{33}$, $Y_{231} = Y_{312}$. In spaces of the sub-kind O $(\Lambda_1 = \Lambda_2 = \Lambda_3)$ we have: $X_{11} = X_{22} = X_{33} = -\frac{\kappa}{3}$, $Y_{123} = Y_{231} = Y_{312}$. Hence Einstein spaces of the sub-kind O have only real curvatures, while being empty they are flat.

For the 2nd kind we have

<u>The 2nd Kind</u>

$$\|R_{ab}\| = \left\| \begin{matrix} x & y \\ y & -x \end{matrix} \right\|,$$

$$x = \left\| \begin{matrix} x_{11} & 0 & 0 \\ 0 & x_{22}+1 & 0 \\ 0 & 0 & x_{33}-1 \end{matrix} \right\|, \qquad y = \left\| \begin{matrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 1 \\ 0 & 1 & y_{22} \end{matrix} \right\|, \tag{5.20}$$

where

$$x_{11} + x_{22} + x_{33} = -\kappa\,, \qquad x_{22} - x_{33} = 2\,, \quad y_{11} + 2y_{22} = 0\,. \tag{5.21}$$

The stationary curvatures are

$$\Lambda_1 = -\frac{1}{c^2}\,X_{11} + \frac{i}{c}\,Y_{123}\,, \qquad \Lambda_2 = -\frac{1}{c^2}\,X_{22} - 1 + \frac{i}{c}\,Y_{231}\,, \qquad \Lambda_3 = -\frac{1}{c^2}\,X_{33} + 1 + \frac{i}{c}\,Y_{312}\,. \tag{5.22}$$

From this we conclude that values of the stationary curvatures Λ_2 and Λ_3 can never become zero, so Einstein spaces (gravitational fields) of the 2nd kind are curved in any case — they cannot approach Minkowski flat space.

In spaces of the sub-kind N ($\Lambda_1 = \Lambda_2$) in an ortho-frame the relations are true

$$X_{11} = X_{22} - c^2 = X_{33} + c^2, \qquad Y_{123} = Y_{231} = Y_{312} = 0\,, \tag{5.23}$$

so the stationary curvatures are real. In an empty space the matrices x and y become degenerate (its determinant becomes zero). For this reason spaces of the sub-kind N are *degenerate*, and, respectively, gravitational fields in spaces of the sub-kind N are known as *gravitational fields of the 2nd degenerate kind by Petrov's classification*. In emptiness ($\kappa = 0$) some elements of the matrices x and y take the numerical values $+1$ and -1 thereby making an ultimate transition to the Minkowski flat space impossible.

For the 3rd kind we have

The 3rd Kind

$$\|R_{ab}\| = \begin{Vmatrix} x & y \\ y & -x \end{Vmatrix}, \quad x = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad y = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{Vmatrix}. \tag{5.24}$$

Here the stationary curvatures are zero and both of the matrices x and y are degenerate. Einstein spaces of the 3rd kind can only be empty ($\kappa = 0$), but, at the same time, they can never be flat.

From the equations deduced for the canonical form of the matrix $\|R_{ab}\|$, we conclude: $Y_{ijk} = 0$ can be true only in gravitational fields of the 1st kind, which are derived from island masses of matter in emptiness or vacuum. Therefore we conclude that those gravitational fields where $Y_{ijk} = 0$ is true in the observer's accompanying reference frame can only be of the 1st kind, having stationary curvatures which are real.

Furthermore, in accordance with most of the criteria, the presence of gravitational waves is linked to spaces of the 2nd (N) kind and the 3rd kind, where the matrix y_{ik} has components equal to $+1$ or -1. Moreover, in fields of the 2nd (N) and 3rd kinds the values $+1$ or -1 are attributed also to components of the matrix x. This implies that:

> Those spaces which contain gravitational fields, satisfying the invariant criteria for gravitational waves, are curved independently of whether or not they are empty ($T_{\alpha\beta} = 0$) or filled with matter (in such spaces $T_{\alpha\beta} = g_{\alpha\beta}$). In any case, gravitational radiations are derived from interaction between two observable components X_{ij}, Y_{ijk} of the Rimeann-Christoffel curvature tensor.

The classification of gravitational fields built here applies only to Einstein spaces, because solving this problem for spaces of general kind, where $T_{\alpha\beta} \neq \kappa g_{\alpha\beta}$, would be very difficult, for mathematical reasons. Considering the details of these difficulties, we see that, having an arbitrary distribution of matter in a space, the matrix $\|R_{ab}\|$, taken in a non-holonomic ortho-frame, is not symmetrically doubled; on the contrary, the matrix takes the form

$$\|R_{ab}\| = \begin{Vmatrix} x & y \\ y' & z \end{Vmatrix}, \tag{5.25}$$

where the three-dimensional matrices x, y, z are built on the following elements, respectively*

$$x_{ik} = -\frac{1}{c^2}\,X_{ik}\,, \qquad z_{ik} = \frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z^{mnpq}, \qquad y_{ik} = \frac{1}{2c}\,\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}\,, \tag{5.26}$$

and y' implies transposition. It is evident that reduction of this matrix to its canonical form is a very difficult problem.

Nevertheless Petrov's classification permits us to conclude:

> The physically observable components X^{ij} and Y^{ijk} of the Riemann-Christoffel curvature tensor are different in their physical origin†. Metrics can exist where $Y^{ijk} = 0$ but $X^{ij} \neq 0$ and $Z^{iklj} \neq 0$. Such spaces are of the 1st kind by Petrov's classification; they have real stationary curvatures. Such spaces do not satisfy the invariant criteria for gravitational waves. Thus no wave fields of gravity exist in spaces where $Y^{ijk} = 0$ but $X^{ij} \neq 0$ and $Z^{iklj} \neq 0$.

And further:

> In solutions of the Einstein equations there are no metrics where $Y^{ijk} \neq 0$ but $X^{ij} = 0$ and $Z^{iklj} = 0$. Thus in wave fields of gravity $Y^{ijk} \neq 0$ and $X^{ij} \neq 0$ (and as well $Z^{iklj} \neq 0$: see the footnote) everywhere and always.

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*In ortho-frames there is no difference between upper and lower indices (see [37]). For this reason we can write $z_{ik} = \frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z_{mnpq}$ and $y_{ik} = \frac{1}{2c}\,\varepsilon_{imn}Y_{kmn}$ instead of $z_{ik} = \frac{1}{c^2}\,\varepsilon_{imn}\varepsilon_{kpq}Z^{mnpq}$ and $y_{ik} = \frac{1}{2c}\,\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}$ in formula (3.26). This note relates to all formulae written in an ortho-frame. We met a similar case in formula (5.11), where we can also write $y \equiv \|y_{ik}\| = -\frac{1}{2c}\,\|\varepsilon_{imn}Y_{kmn}\|$ instead of $y \equiv \|y_{ik}\| = -\frac{1}{2c}\,\left\|\varepsilon_{imn}Y_{k\cdot\cdot}^{\cdot mn}\right\|$.

†We do not mention the third observable component Z^{iklj}, because in an ortho-frame the matrices x and z are connected by the equation $x = -z$.

We will show that in Einstein spaces filled with gravitational fields where the Riemann-Christoffel tensor's observable components X^{ij}, Y^{ijk}, Z^{iklj} play a part of the wave functions, the quantity X^{ij} is analogous to the electric component of an electromagnetic field, while Y^{ijk} is analogous to its magnetic component. All this will be discussed in §7.

6 Wave properties of Einstein's equations

In §2 we have showed that the gravitational field equations (the Einstein equations) do not contain a general covariant d'Alembert operator derived from the fundamental metric tensor $g_{\alpha\beta}$ (where $g_{\alpha\beta}$ is considered as a "four-dimensional gravitational potential"). Nevertheless this problem has been solved in linear approximation in the case where gravitational fields are occupy an empty space ($R_{\alpha\beta}=0$, "empty gravitational fields") [14, 15]. In this case a gravitational field is considered as a tiny addition to a flat space background described by the Minkowski metric. Thus

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \gamma_{\alpha\beta}\,, \tag{6.1}$$

where $\delta_{\alpha\beta}$ are components of the fundamental metric tensor in a Galilean reference frame $\delta_{\alpha\beta}=\{+1,-1,-1,-1\}$, and $\gamma_{\alpha\beta}$ describes weak corrections for the gravitational fields. The contravariant fundamental metric tensor $g^{\alpha\beta}$ to within the first order approximation of the $\gamma_{\alpha\beta}$ is

$$g^{\alpha\beta} = \delta^{\alpha\beta} - \gamma^{\alpha\beta}, \tag{6.2}$$

so the determinant of the tensor $g_{\alpha\beta}$ is

$$g = -\,(1+\gamma)\,, \qquad \gamma = \det\|\gamma_{\alpha\beta}\|\,. \tag{6.3}$$

The requirement that components of the "additional" metric $\gamma_{\alpha\beta}$ must be infinitesimal fixes a prime reference frame. If this requirement is true in a reference frame, it will also be true after transformations

$$\tilde{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}, \tag{6.4}$$

where ξ^{α} are infinitesimal quantities $\xi^{\alpha}\ll 1$. Then we have

$$\tilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{\partial \xi_{\alpha}}{\partial x^{\beta}} - \frac{\partial \xi_{\beta}}{\partial x^{\alpha}}\,. \tag{6.5}$$

Because of (6.1), we impose an additional requirement on the tensor $\gamma_{\alpha\beta}$; this requirement is [15]

$$\frac{\partial \psi^{\alpha}}{\partial x^{\beta}} = 0\,, \qquad \psi^{\alpha}_{\beta} = \gamma^{\alpha}_{\beta} - \frac{1}{2}\,\delta^{\alpha}_{\beta}\gamma\,. \tag{6.6}$$

Taking (6.6) into account, the Ricci tensor takes the form

$$R_{\alpha\beta} = \frac{1}{2}\,\Box\gamma_{\alpha\beta}\,, \tag{6.7}$$

where

$$\Box \equiv g^{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,, \qquad \Delta = \frac{\partial^2}{\partial x^{1\,2}} + \frac{\partial^2}{\partial x^{2\,2}} + \frac{\partial^2}{\partial x^{3\,2}}\,. \tag{6.8}$$

Here $\Box$ is the d'Alembert operator, Δ is the Laplace operator. The calibrating requirements (6.6) are true in any metric $\gamma_{\alpha\beta}$ only if the quantities ξ^{α} are solutions of the equation

$$\Box\xi^{\alpha} = 0\,. \tag{6.9}$$

In [15] the requirement

$$\Box\gamma_{\alpha\beta} = 0 \tag{6.10}$$

was imposed on the quantities $\gamma_{\alpha\beta}$, which is interpreted as the *equation of weak gravitational waves in emptiness* — this formula (6.10) is a standard wave equation that describes a wave of the tensor field $\gamma_{\alpha\beta}$, traveling at the velocity c in emptiness.

One usually considers the equation (6.10) as the basis for the claim that the General Theory of Relativity predicts gravitational waves, which travel at the speed of light.

If we have a weak plane gravitational wave, so the field has changes along a single spatial direction (the x^1 axis, for instance), the formula (6.10) takes the form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^{1\,2}}\right)\gamma_{\alpha\beta} = 0\,, \tag{6.11}$$

and solutions of it can be any function of $ct \pm x^1$. After numerous transformations of the function $\gamma_{\alpha\beta}$ [14, 15] it obtains that in the field of a weak plane gravitational wave only the following components are non-zero: $\gamma_{22}=-\gamma_{33}\equiv a$, $\gamma_{23}\equiv b$. Thus, those weak plane gravitational waves that satisfy the Einstein equations in emptiness are transverse.

Thus if some additional requirements are imposed upon the Einstein equations in emptiness, the equations describe weak plane waves of the space deformation, the space metric of which is [15]

$$\begin{aligned} ds^2 = c^2dt^2 - (dx^1)^2 - (1+a)(dx^2)^2 + \\ + 2b\,dx^2dx^3 - (1-a)(dx^3)^2, \end{aligned} \tag{6.12}$$

where a and b are functions of $ct \pm x^1$. The field of gravitation, described by the metric (6.12), is of the sub-kind N by Petrov's classification, so it satisfies most of invariant criteria for gravitational waves.

The metric (6.12) has been written in a synchronous reference frame, so its space deforms, falls freely, and, at the same time, has no rotations. Hence, *under the given assumptions, weak plane gravitational waves are waves of "pure" deformation of the space*. This conclusion is the main reason why experimental physicists, and Weber in particular [16], expect that gravitational waves will cause a "pure" deformation effect in detectors.

Calculations for the interaction between a Weber solid-body detector and a weak plane gravitational wave field will be given in §7. Here we continue our argument for the wave nature of the Einstein equations in *strong gravitational fields* in the case where matter is arbitrarily distributed in the space. This research will be given in the terms of physically observable quantities for the reason that we will consider situations derived from different factors, generating gravitational wave fields, not only the space deformation.

The Einstein equations in the case where matter is arbitrarily distributed are [42]

$$\frac{{}^*\partial D}{\partial t}+D_{jl}D^{jl}+A_{jl}A^{lj}+\left({}^*\nabla_j-\frac{1}{c^2}F_j\right)F^j= =-\frac{\varkappa}{2}\left(\rho c^2+U\right)+\lambda c^2, \tag{6.13}$$

$${}^*\nabla_j\left(h^{ij}D-D^{ij}-A^{ij}\right)+\frac{2}{c^2}F_jA^{ij}=\varkappa J^i, \tag{6.14}$$

$$\frac{{}^*\partial D_{ik}}{\partial t}-\left(D_{ij}+A_{ij}\right)\left(D^j_k+A^{\cdot j}_{k\cdot}\right)+DD_{ik}+ +3A_{ij}A^{\cdot j}_{k\cdot}+\frac{1}{2}\left({}^*\nabla_iF_k+{}^*\nabla_kF_i\right)-\frac{1}{c^2}F_iF_k- -c^2C_{ik}=\frac{\varkappa}{2}\left(\rho c^2h_{ik}+2U_{ik}-Uh_{ik}\right)+\lambda c^2h_{ik}. \tag{6.15}$$

Here ${}^*\nabla_j$ denotes the chr.inv.-derivative, while the quantities $\rho=\frac{T_{00}}{g_{00}}$, $J^i=\frac{cT^i_0}{\sqrt{g_{00}}}$, $U^{ik}=c^2T^{ik}$ (from which we have $U=h^{ik}U_{ik}$) are the chr.inv.-components of the energy-momentum tensor $T_{\alpha\beta}$ of matter: the physically observable density ρ, the physically observable impulse density vector J^i, and the physically observable stress-tensor U^{ik}.

Zelmanov had deduced [42] that the chr.inv.-spatial curvature tensor C_{iklj} is linked to a chr.inv.-tensor H_{iklj}, which is like Schouten's tensor [67], by the equation

$$H_{lkij}=C_{lkij}+\frac{1}{c^2}\big(2A_{kj}D_{jl}+A_{ij}A_{kl}+ +A_{jk}D_{il}+A_{kl}D_{ij}+A_{li}D_{jk}\big) \tag{6.16}$$

and contracted tensors $H_{lk}=H_{lik\cdot}^{\cdots i}$ and $C_{lk}=C_{lik\cdot}^{\cdots i}$ are related as follows

$$H_{lk}=C_{lk}+\frac{1}{c^2}\left(A_{kj}D^j_l+A_{lj}D^j_k+A_{kl}D\right). \tag{6.17}$$

Taking the definition $D_{ik}=\frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}$ into account, and C_{lk} from (6.17), we reduce (6.15) to the form

$$\frac{1}{2}\frac{{}^*\partial^2h_{ik}}{\partial t^2}-D_{ij}D^j_k+D\big(D_{ik}-A_{ik}\big)+2A_{ij}A^{\cdot j}_{k\cdot}+ +\frac{1}{2}\left({}^*\nabla_iF_k-{}^*\nabla_kF_i\right)-\frac{1}{c^2}F_iF_k-c^2H_{ik}= =\varkappa U_{ik}+\lambda c^2h_{ik}. \tag{6.18}$$

The quantity H_{ik}, by definition, is

$$H_{ik}=H_{ijk\cdot}^{\cdots j}=\frac{{}^*\partial\Delta^j_{ij}}{\partial x^k}-\frac{{}^*\partial\Delta^j_{ik}}{\partial x^j}+\Delta^m_{ij}\Delta^j_{km}-\Delta^m_{ik}\Delta^j_{jm}, \tag{6.19}$$

where $\Delta^m_{jm}=\frac{{}^*\partial\ln\sqrt{h}}{\partial x^j}$.

Taking into account (6.17), (6.19), and also Zelmanov's identities (4.6), (4.7) that link F_i and A_{ik}, we reduce (6.18) to the form

$${}^*\Box h_{ik}=2\frac{{}^*\partial^2\ln\sqrt{h}}{\partial x^i\partial x^k}-\frac{2}{c^2}\left({}^*\nabla_iF_k+\frac{{}^*\partial A_{ik}}{\partial t}\right)- -\frac{4}{c^2}\left(A_{ij}A^{\cdot j}_{k\cdot}-D_{ij}D^j_k\right)-\frac{2D}{c^2}\left(D_{ik}+A_{ik}\right)+ +2\left(h^{pq}\Delta^m_{pq}\Delta_{ik,m}+\Delta^m_{ij}\Delta^j_{km}\right)- -h^{pm}\frac{{}^*\partial}{\partial x^p}\left(\frac{{}^*\partial h_{im}}{\partial x^k}+\frac{{}^*\partial h_{km}}{\partial x^i}\right)+ +\varkappa\left(\rho h_{ik}+\frac{2}{c^2}U_{ik}-\frac{U}{c^2}h_{ik}\right)+2\lambda h_{ik}, \tag{6.20}$$

where ${}^*\Box$ is the chr.inv.-d'Alembert operator, applied here to the chr.inv.-metric tensor h_{ik} (the observable metric tensor of the observer's three-dimensional space)*.

If we equate the right part of (6.20) in zero, the whole equation becomes a wave equation with respect to h_{ik}, namely

$${}^*\Box h_{ik}=\frac{1}{c^2}\frac{{}^*\partial^2h_{ik}}{\partial t^2}-h^{jm}\frac{{}^*\partial^2h_{ik}}{\partial x^j\partial x^m}. \tag{6.21}$$

In this case the spatial components of the Einstein equations describe gravitational inertial waves of the spatial metric h_{ik}, which travel at the velocity $u=c\left(1-\frac{\mathrm{w}}{c^2}\right)$ which depends on the value of the gravitational potential w. This coincides with the results recently obtained by Rabounski [48]. If $\mathrm{w}=0$, the waves travel at the velocity of light. The greater is w the smaller is u. The wave's velocity u becomes zero in the extreme case where $\mathrm{w}=c^2$ which occurs under collapse, hence under collapse gravitational waves stop — they become *standing gravitational waves*.

It is evident from the mathematical viewpoint, that reducing the right side of (6.20) to zero is a very difficult task, because the whole equation is a system of 6 nonlinear equations of the 2nd order, in which numerous variables are linked by relationships (6.13) and (6.14). Systems such as this cannot be solved analytically in general, but we can obtain solutions for various specific metrics.

Because experimental physicists. in their search for gravitational waves, propound experimental statements for detecting weak wave fields of gravitation, we are going to study a linearized form of the equation (6.20).

*Components of the chr.inv.-metric tensor h_{ik} satisfy the requirements ${}^*\nabla_jh_{ik}={}^*\nabla_jh^k_i={}^*\nabla_jh^{ik}=0$. For this reason we can apply the chr.inv.-d'Alembert operator ${}^*\Box=\frac{1}{c^2}\frac{{}^*\partial^2}{\partial t^2}-h^{ik}{}^*\nabla^*_i\nabla_k$ to it.

For (6.20) in emptiness, the linear approximation is*

$$^*\Box h_{ik} = 2\,\frac{^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^k} - \frac{2}{c^2}\left(^*\nabla_i F_k + \frac{^*\partial A_{ik}}{\partial t}\right). \qquad (6.22)$$

As a matter of fact, equation (6.22) describes weak plane gravitational inertial waves without sources, if the wave field satisfies the obvious chr.inv.-condition

$$\frac{^*\partial^2 \ln\sqrt{h}}{\partial x^i \partial x^k} = \frac{1}{c^2}\left(^*\nabla_i F_k + \frac{^*\partial A_{ik}}{\partial t}\right). \qquad (6.23)$$

In other words, the field of the observable metric tensor h_{ik} is a wave field if there are some relations between the inhomogeneity of the gravitational inertial force field, the non-stationary rotation of the space, and the volume transformations of the space element, taken in the field†. The condition (6.23) is true for the well-known metric of weak plane gravitational waves (6.12), because in the metric (6.12) we have $F_i = 0$, $A_{ik} = 0$, $\sqrt{h} = \sqrt{1-a^2-b^2} \approx 1$. Thus:

> Weak plane gravitational waves in emptiness are also weak plane gravitational inertial waves of the spatial observable metric h_{ik}.

As shown in [41], the metric (6.12) satisfies the Zelmanov chr.inv.-criterion for gravitational waves, where the wave functions are the Riemann-Christoffel tensor's physically observable components X^{ij}, Y^{ijk}, Z^{iklj}. Hence weak plane gravitational inertial waves (waves of the space curvature) can exist in emptiness, because of the Einstein equations. We have shown above that such wave gravitational fields can also exist in spaces of the sub-kind N by Petrov's classification (such spaces are curved themselves, and matter contributes only an additional component to the initial curvature). Hence such fields satisfy most of the known invariant criteria for gravitational waves.

As we showed above, on page 46, that fields of gravitational radiations cannot exist in spaces of the 1st kind by Petrov's classification. In spaces of the 1st kind $Y^{ijk} = 0$. Therefore it would be logical to express the Einstein equations in the physically observable components X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel curvature tensor, aiming to find relations between the ch.inv.-quantities X^{ij}, Y^{ijk}, Z^{iklj} and the physically observable components of the energy-momentum tensor $T_{\alpha\beta}$ of distributed matter (ρ, J^i, U_{ik}, see page 48).

In chr.inv.-components the Einstein equations become

$$\left.\begin{aligned} & Z_{mk\cdot\cdot}^{\cdot\cdot mk} = \kappa\left(\rho c^2 + U\right) - 2\lambda c^2, \\ & Y_{\cdot\cdot m}^{im\cdot} = \kappa J^i, \\ & X_{ik} - X h_{ik} + Z_{\cdot imk}^{m\cdots} = \\ & = \frac{\kappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right) + \lambda c^2 h_{ik}\,, \end{aligned}\right\} \qquad (6.24)$$

if matter is distributed arbitrarily. Here $X = h^{ik} X_{ik}$ is the trace (spur) of the tensor X_{ik}.

From here we see that the physical observable components of the Riemann-Christoffel tensor have different physical origins:

1. Quantities X^{ij} (and as well Z^{iklj}) are linked to the mass density ρ and the stress-tensor U_{ik};
2. Quantities Y^{ijk} are linked to the impulse density J^i of matter.

As we showed above, on page 46, in all the widely known metrics which satisfy both the invariant criteria and the chr.inv.-criterion for gravitational waves, we have $Y^{ijk} \neq 0$, although X^{ij} (and as well Z^{iklj}) can be zero. This fact leads us to a very important conclusion:

> Gravitational waves and gravitational inertial waves are mainly waves of the field of the Y^{ijk} physically observable component of the Riemann-Christoffel curvature tensor‡.

But this conclusion does not mean that only waves of the field Y^{ijk} can be discovered. As we will see in §7, relative accelerations of test-particles are derived from wave fields of all three observable components X^{ij}, Y^{ijk}, Z^{iklj} of the Riemann-Christoffel tensor. Our conclusion means:

> If in a space, filled with a gravitational field, $Y^{ijk} = 0$ is true, the structure of the space itself prohibits the gravitational field from being a wave.

Contracting (6.26) and taking (6.24) into account, we obtain

$$X = \frac{\kappa}{2}\left(U - \rho c^2\right) - 2\lambda c^2. \qquad (6.25)$$

In an empty space where there are no λ-fields, the trace of X^{ij} and the contracted quantity $Z_{mk\cdot\cdot}^{\cdot\cdot mk}$ are zero, as well as the contracted quantity $Y_{\cdot\cdot m}^{im\cdot}$. Thus the chr.inv.-Einstein equations (6.24) in emptiness take the form§

$$\left.\begin{aligned} & Z_{mk\cdot\cdot}^{\cdot\cdot mk} = 0\,, \qquad X = 0\,, \\ & Y_{\cdot\cdot m}^{im\cdot} = 0\,, \\ & X_{ik} + Z_{\cdot imk}^{m\cdots} = 0\,, \end{aligned}\right\} \qquad (6.26)$$

so, while the quantities X^{ik} and Z^{iklj} are connected to one another, the quantity Y^{ijk} (which, being non-zero, $Y^{ijk} \neq 0$, permits gravitational fields to be a wave) is the independent observable component of the Riemann-Christoffel tensor.

*In obtaining this formula, in the initial equation (6.20), we neglect products of the chr.inv.-quantities and of their derivatives.

†The integral of $\sqrt{h}\,dx^1 dx^2 dx^3$ is the volume of an element of the space. Here the differentials dx^i themselves and an interval, where values of the x^i change where we take the integral, do not depend on x^0 [42].

‡Quadrupole mass-detectors, in particular, solid-body detectors (the Weber pigs) can only register waves of the X^{ij} component, not waves of Y^{ijk} if its particles are at rest in the initial moment of time (see §7 and §8 for details). Thus, the Weber experimental statement is false at its base.

§As a matter of fact, equality to zero of inflected forms of a tensor does not imply that the tensor quantity itself is zero. Thus, equalities $X = 0$, $Y_{\cdot\cdot m}^{im\cdot} = 0$, $Z_{mk\cdot\cdot}^{\cdot\cdot mk} = 0$ do not imply that the quantities X^{ik}, Y^{ijk}, Z^{iklj} themselves are zero. Therefore the chr.inv.-Einstein equations in emptiness (6.24) permit gravitational waves if, of course, $Y^{ijk} \neq 0$.

7 Expressing Synge-Weber equation (the world-lines deviation equation) in the terms of physical observable quantities, and its exact solutions

In the previous paragraphs we focused our attention on general criteria, which differentiate gravitational wave fields from other gravitational fields in the General Theory of Relativity. As a result, we have found the main properties of gravitational wave fields.

We are now going to introduce a substantial criticism of the contemporary theoretical foundations of current attempts to detect gravitational waves by solid-body detectors of the resonance kind (the Weber pigs) and quadrupole mass detectors in general.

As we showed in the previous paragraphs, only gravitational fields located in spaces where the Riemann-Christoffel curvature tensor has a specific structure, permit the presence of gravitational waves. Therefore it would be reasonable to design experiments by which a physical detector could register wave changes of the four-dimensional (space-time) curvature* — the waves of the Riemann-Christoffel curvature tensor field.

Such a physical detector could be a system of two test-particles: their relative world-trajectories will necessarily undergo changes through the action of a wave of the space curvature. These systems are described by the world-lines deviation equation — the Synge equation of geodesic deviation (2.8) if these are two free particles, and the Synge-Weber equation (2.12) if the particles are connected by a force of non-gravitational nature.

We propose gravitational wave detectors of two possible kinds. The system of two free particles is known as a *detector built on free masses*. In practice such a detector consists of two freely suspended massive bodies, separated by a suitable distance. The system of two particles connected by a spring is known as a *quadrupole mass-detector*— this is a detector of the resonance kind, a typical instance of which is the Weber cylindrical pig.

To understand how a graviational wave would affect the different types of detectors we need to make specific calculations for their behaviour in gravitational wave fields. But before making the calculations, it is required to describe the behaviour of two test-particles in regular gravitational fields (of non-wave nature) in the terms of physically observable quantities (chronometric invariants). This analysis will show how different kinds of gravitational inertial waves cause relative deviation (both spatial and time displacements) of two test-particles.

We will solve this problem first for a system two free particles as described by the Synge equation (2.8) where the right side is zero. The problem for spring-connected particles, described by the Synge-Weber equation (2.12), will be solved in the same way except that there will be a non-gravitational force acting, so that the right side of the equation will be non-zero.

*It is important to note that the expected gravitational waves are waves of the *space-time* curvature, not merely of the spatial curvature of the three-dimensional space. Consequently, waves of the four-dimensional curvature must produce changes not only in the distance between test-particles in a detector, but also in the time flow for the particles.

Relative accelerations of free test-particles $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ as a whole and the quantity $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ are derived from components of the Riemann-Christoffel world-tensor, contracted with components of the particles' four-dimensional velocity vector U^β and their relative deviation vector η^γ, namely — from the quantity $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma$. To determine what effect is introduced by each observable component of the Riemann-Christoffel tensor into the spatial and time relative displacements, described by the relative displacement world-vector η^α, we consider the geodesic deviation equation (2.8), keeping the term $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ as a whole and the quantity $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ without expressing it in terms of the Christoffel symbols and their derivatives.

As well as any general covariant equation, the geodesic deviation equation (2.8) can be projected onto the observer's time line and spatial section (his three-dimensional space) as given in [42, 43] or on page 40 herein. Denoting

$$M^\alpha \equiv \frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = 0\,, \tag{7.1}$$

let us find equations which are its projection on the time line

$$\frac{M_0}{\sqrt{g_{00}}} = \frac{g_{0\alpha}}{\sqrt{g_{00}}}\,M^\alpha = \sqrt{g_{00}}\,M^0 - \frac{1}{c}\,v_i M^i = 0\,, \tag{7.2}$$

and its projection on the spatial section

$$M^i = 0\,. \tag{7.3}$$

To find the equations in expanded form we need first to find the chr.inv.-projections of them, consisting of the quantities η^α and U^α. Projections of the η^α onto the time line and spatial section are, respectively

$$\varphi \equiv \frac{\eta_0}{\sqrt{g_{00}}}\,, \qquad n^i \equiv \eta^i, \tag{7.4}$$

other components of the η^α are expressed through its physically observable components φ and n_i as follows

$$\eta^0 = \frac{\varphi + \frac{1}{c}v_k n^k}{\sqrt{g_{00}}}\,, \qquad \eta_i = -\frac{\varphi}{c}\,v_i - n_i\,. \tag{7.5}$$

The time and spatial components of the particles' world-velocity vector U^α are derived from the chr.inv.-definitions given by the theory of chronometric invariants for the space-time interval ds and the observable chr.inv.-velocity vector v^i

$$ds = c\,d\tau\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\,, \quad \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \quad \mathrm{v}^2 = h_{ik}\mathrm{v}^i\mathrm{v}^k, \tag{7.6}$$

so the required quantities U^0 and U^i are

$$U^0 = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{dt}{d\tau}\,, \qquad U^i = \frac{\mathrm{v}^i}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\,. \tag{7.7}$$

A formula for the time function $dt/d\tau$ is obtained from*

$$g_{\alpha\beta}\,U^\alpha U^\beta = g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 1\,, \tag{7.8}$$

which can be reduced to the quadratic equation

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{2\,v_i\mathrm{v}^i}{c^2\left(1-\frac{\mathrm{w}}{c^2}\right)}\frac{dt}{d\tau} + \\ + \frac{1}{\left(1-\frac{\mathrm{w}}{c^2}\right)^2}\left(\frac{1}{c^2}\,v_i v_k \mathrm{v}^i\mathrm{v}^k - 1\right) = 0\,, \tag{7.9}$$

which has two solutions

$$\left(\frac{dt}{d\tau}\right)_1 = \frac{\frac{1}{c^2}\,v_i\mathrm{v}^i+1}{1-\frac{\mathrm{w}}{c^2}}\,, \qquad \left(\frac{dt}{d\tau}\right)_2 = \frac{\frac{1}{c^2}\,v_i\mathrm{v}^i-1}{1-\frac{\mathrm{w}}{c^2}}\,. \tag{7.10}$$

The first solution is related to a space where time flows from past into future (a regular observer's space), the second solution is related to a space where time flows from future into past with respect to a regular observer's time flow (the mirror Universe [70, 71]). Taking only the first root, U^0 takes the form

$$U^0 = \frac{\frac{1}{c^2}\,v_i\mathrm{v}^i + 1}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\left(1-\frac{\mathrm{w}}{c^2}\right)}\,. \tag{7.11}$$

Substituting formulae (7.5), (7.7), (7.11) into $\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = 0$ (7.1), and expressing the components of the Riemann-Christoffel tensor $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ in terms of its physically observable components X^{ij}, Y^{ijk}, Z^{iklj}, we obtain a formula for the relative spatial oscillations of two free test-particles

$$\frac{\mathrm{D}^2\eta^i}{ds^2} = \frac{1}{c^2-\mathrm{v}^2}\left(Y^{\cdot\cdot\,i}_{mk\cdot}\mathrm{v}^k - X^i_m - \frac{1}{c^2}Z^{\cdot\cdot\,i\,\cdot}_{mk\cdot n}\mathrm{v}^k\mathrm{v}^n\right)\eta^m. \tag{7.12}$$

From this formula we see that:

> The relative spatial deviations of two free particles can be caused by all three observable components of the Riemann-Christoffel curvature tensor. Moreover, each of the components acts on the particles in a different way: (1) the field of X^{ik} acts the particles only if they are at rest with respect to the observer's space references, so the field of X^{ik} can move particles only if they are at rest at the initial moment of time; (2) the fields of Y^{ijk} and Z^{iklj} can displace the particles with respect of each to other only if they are in motion ($\mathrm{v}^i \neq 0$) — the effect of Z^{iklj} is perceptible if the particles move at speeds close to the velocity of light.

*That is the evident equality.

Thus, with measurement taken by any observer, the physically observable components of the Riemann-Christoffel curvature tensor are of 3 different kinds:

1. The component X^{ik} —of "electric kind", because it can displace even resting particles;
2. The component Y^{ijk} — of "magnetic kind", because it can displace only moving particles;
3. The component Z^{iklj} of "magnetic relativistic kind", because it causes an effect only in particles moving at relativistic speeds.

Besides the observable spatial component η^i of the relative deviation vector η^α there is also its observable time component φ, which indicates the difference between time flows measured by clocks located at each of the particles.

We then obtain the relative time deviation equation for two free test-particles

$$\sqrt{g_{00}}\frac{\mathrm{D}^2\eta^0}{ds^2} - \frac{1}{c}\,v_i\frac{\mathrm{D}^2\eta^i}{ds^2} = \\ = -\sqrt{g_{00}}R^{0\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma + \frac{1}{c}\,v_i R^{i\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma. \tag{7.13}$$

Taking (7.10) into account and substituting the formulae for U^0, η^0, U^i, η^i into (7.11), then, expressing $R^{0\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ in terms of physically observable quantities, we reduce formula (7.13) to its final form

$$\sqrt{g_{00}}\frac{\mathrm{D}^2\eta^0}{ds^2} - \frac{1}{c}\,v_i\frac{\mathrm{D}^2\eta^i}{ds^2} = \\ = \frac{1}{c^2-\mathrm{v}^2}\left[\frac{1}{c}X_{ik}\left(n^i - \frac{\varphi}{c}\mathrm{v}^k\right)\mathrm{v}^k + \frac{1}{c}Y_{imk}\mathrm{v}^i\mathrm{v}^k\eta^m\right]. \tag{7.14}$$

Looking at this formula we note one simple thing about the effect of gravitational waves on the system of two free particles:

> The time observable component of the relative deviation vector for two free particles undergoes oscillations due only to the X^{ik} and Y^{ijk} observable components of the Riemann-Christoffel curvature tensor, not its Z^{iklj} component. Moreover, the fields of both the components X^{ik} and Y^{ijk} act on the particles only if they are in motion with respect to the space references. If the particles are at rest with respect to each other and the observer ($\mathrm{v}^i = 0$), the fact that the space has a Riemannian curvature makes no difference to the time flow measured in the particles.

It should be added that if the particles are in motion with respect to the space references and the observer, the effect of X^{ik} is both linearly and quadratically dependent on the speed, whilst the effect of Y^{ijk} is only quadratically dependent on the speed.

Thus, there is no complete analogy between the physically observable components of the Riemann-Christoffel curvature tensor and Maxwell's electromagnetic field tensor.

The components X^{ik} can be interpreted "electric" only in *relative spatial displacements* of two particles. In relative time deviations between the particles (the difference between the time flow measured in the them both) both X^{ik} and Y^{ijk} act on them depending on the particles' velocity with respect to the space references and the observer, so in this case both X^{ik} and Y^{ijk} are of the "magnetic" kind. Therefore the terms "electric" and "magnetic" are only applicable relative to observable components of the Riemann-Chrstoffel curvature tensor. This terminology is strictly true in that case where the particles have only relative spatial deviations, while the time flow is the same on the both world lines.

A formula for the observable relative time deviation $\varphi = \frac{\eta_0}{\sqrt{g_{00}}}$ between two free particles can be obtained from the requirement that the scalar product $U_\alpha \eta^\alpha$ remains unchanged along geodesic trajectories, so $U_\alpha \eta^\alpha = \text{const}$ must be true along trajectories of free particles. For this reason, if the vectors U^α and η^α are orthogonal, they are orthogonal on the entire world-trajectory [17]. Formulating the orthogonality condition $U_\alpha \eta^\alpha = \text{const}$ in terms of physically observable quantities, we introduce some corrections to the results obtained in [17].

In terms of physically observable quantities the orthogonality condition $U_\alpha \eta^\alpha = \text{const}$, because it is actually the same as $U_0 \eta^0 + U_k \eta^k = \text{const}$, reduces to

$$\varphi - \frac{1}{c}\, n_i \mathrm{v}^i = \text{const} \times \sqrt{1 - \frac{\mathrm{v}^2}{c^2}}\,. \tag{7.15}$$

From this we see that the vectors U^α and η^α are orthogonal only if $\mathrm{v}^2 = c^2$, i.e. U^α is isotropic: $g_{\alpha\beta} U^\alpha U^\beta = 0$. So if U^α and η^α are orthogonal, we have the deviation equation for *two isotropic geodesics* — world-lines of light-like particles moving at the velocity of light. We defer this case for the moment and consider only the case of *two neighbour non-isotropic geodesics*. In the particular case when two particles are moving on neighbouring geodesics, and are at rest with respect to the observer and his references (only the time flow is different in the particles), formula (7.15) leads to $\varphi = \text{const}$.

This formula verifies our previous conclusion that the particles have a time deviation only if they are in motion. The greater their velocity with respect to the space reference and the observer, the greater the deviation between the time flow on both world-lines. Thus measurement of time deviations between two particles in gravitational waves and gravitational inertial waves would be easier in experiments where the particles move at high speeds. In practice such an experimental statement could be realized using light-like particles (in particular, photons). A time deviation of two photons in gravitational wave fields can manifest as changes in the frequencies of two parallel light rays (laser beams, for instance), while a spatial deviation of the photons can manifest as changes in the phases of the light rays. Calculations of these effects will be presented in future article. Here we focus our attention on particles of non-zero rest-mass $m_0 \neq 0$ (so-called mass-bearing particles), which are at rest with respect to the space references and the observer or, alternatively, moving at sub-light speeds.

In equations (7.10) and (7.12), we kept the second absolute derivative $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ of the relative deviation vector η^α as a whole, because we were concerned only with the effects introduced by the Riemannian curvatureto the relative spatial acceleration $\frac{\mathrm{D}^2\eta^i}{ds^2}$ and relative time acceleration $\frac{\mathrm{D}^2\eta^0}{ds^2}$ of two free test-particles.

But if we wish to obtain solutions to the world-lines deviation equation, we need to express the quantity $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ and also $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ in terms of the Christoffel symbols and their derivatives.

We are now going to obtain solutions to the deviation equation for geodesic lines (the Synge equation).

Taking the definition

$$\frac{\mathrm{D}\eta^\alpha}{ds} = \frac{d\eta^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\eta^\mu U^\nu \tag{7.16}$$

into account, we obtain

$$\begin{aligned}\frac{\mathrm{D}^2\eta^\alpha}{ds^2} = \frac{d^2\eta^\alpha}{ds^2} + \frac{d\Gamma^\alpha_{\mu\nu}}{ds}\eta^\mu U^\nu + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \\ + \Gamma^\alpha_{\mu\nu}\eta^\mu\frac{dU^\nu}{ds} + \Gamma^\alpha_{\rho\sigma}\Gamma^\rho_{\mu\nu}\eta^\mu U^\nu U^\sigma = 0\,.\end{aligned} \tag{7.17}$$

We write $R^{\alpha\cdots}_{\cdot\beta\gamma\delta}$ as

$$R^{\alpha\cdots}_{\cdot\beta\gamma\delta} = \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\sigma_{\beta\delta}\Gamma^\alpha_{\gamma\sigma} - \Gamma^\sigma_{\beta\gamma}\Gamma^\alpha_{\sigma\delta}\,, \tag{7.18}$$

express $\frac{dU^\alpha}{ds}$ via the geodesic equations

$$\frac{dU^\alpha}{ds} = -\Gamma^\alpha_{\mu\nu}U^\mu U^\nu, \tag{7.19}$$

and use the definition

$$\frac{d\Gamma^\alpha_{\mu\nu}}{ds} = \frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\sigma}U^\sigma. \tag{7.20}$$

Using the auxiliary formulae we obtain from (7.17) the Synge equation (the geodesic lines deviation equation) in its final form

$$\frac{d^2\eta^\alpha}{ds^2} + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma}U^\beta U^\delta\eta^\gamma = 0\,. \tag{7.21}$$

This is a differential equation of the 2nd order with respect to the quantity η^α: the equation is a system of 4 differential equations with respect to the quantities η^0 and η^i $(i = 1, 2, 3)$. The variable coefficients $\Gamma^\alpha_{\mu\nu}$ and their derivatives must be taken for that gravitational field, whose waves act on two free test-particles in our experiment. The world-quantities U^ν ($\nu = 0, 1, 2, 3$) can be found as solutions to the geodesic equations

$$\frac{dU^\nu}{ds} + \Gamma^\nu_{\mu\rho} U^\mu U^\rho = 0 \qquad (7.22)$$

only if the particles move with respect to the space references and the observer. If the particles are at rest with respect to the observer and his references, the components of their world-velocity vector U^ν are

$$U^0 = \frac{1}{\sqrt{g_{00}}}\,, \qquad U^i = 0\,, \qquad (7.23)$$

and, according to (7.13–7.15), their relative time deviation is zero, $\varphi = 0$ (the time flow measured on both geodesic lines is the same).

Current detectors used in the search for gravitational wave radiations are of such a construction that the particles therein, which detect the waves, are almost at rest with respect of each other and the observer. Experimental physicists, following Joseph Weber and his methods, think that gravitational waves can cause the rest-particles to undergo a relative displacement. With the current theory of the gravitational wave experiment, the experimental physicists limit themselves to the expected amplitude and energy of waves arriving from a proposed source of a gravitational wave field.

However, to set up the gravitational wave experiment correctly, we need to eliminate all extraneous assumptions and traditions. We merely need to obtain exact solutions to the world-lines deviation equation, applied to detectors of that kind which this experiment uses.

Detectors described by the geodesic lines deviation equation (the Synge equation), which we consider in this section, are known as “antennae built on free masses”. We shall consider such detectors first.

The detectors consist of two freely suspended masses which are at rest with respect of each other and the observer, and separated by an appreciable distance. These could be two mirrors, located in a near-to-Earth orbit, for instance. Each of the mirrors is fitted with a laser range-finder, so we can measure the distance between them with high precision.

In order to interpret the possible results of such an experiment, we need to solve the Synge equation (7.21), expressing its solutions in the terms of physically observable quantities (chronometric invariants). Following “tradition”, we solve the Synge equation for particles which are at rest with respect to each other and the observer’s space references. So we consider that case where the particles’ observable velocities are zero ($\mathrm{v}^i = 0$).

At first, because we are going to obtain solutions to the Synge equation in chr.inv.-from, we need to know the physically observable characteristics of the observer’s reference space through which we express the solutions. We find the chr.inv.-characteristics from the geodesic equations taken in the main chr.inv.-from [42]

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_i \mathrm{v}^i + \frac{m}{c^2} D_{ik} \mathrm{v}^i \mathrm{v}^k = 0\,, \qquad (7.24)$$

$$\frac{d}{d\tau}(m\mathrm{v}^i) + 2m\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\mathrm{v}^k - mF^i + m\Delta^i_{nk}\mathrm{v}^n\mathrm{v}^k = 0\,, \qquad (7.25)$$

for each of the particles (because both particles are at rest with respect to one another, their geodesic equations are the same). Here m is the particle’s relativistic mass, which, because in the case we are considering $\mathrm{v}^i = 0$, reduces to the rest-mass $m = m_0$. Then the geodesic equations take the very simple form

$$\frac{dm}{d\tau} = 0\,, \qquad (7.26)$$

$$mF^i = 0\,, \qquad (7.27)$$

so in this case the chr.inv.-vector of gravitational inertial force is $F^i = 0$: the particles are in free fall. In this case we can transform coordinates so that $g_{00} = 0$ and $\frac{\partial g_{0i}}{\partial t} = 0$ [42]. This implies that the Synge initial equation (7.19) can be solved correctly only for gravitational fields where the potential is weak $\mathrm{w} = 0$ (i. e. $g_{00} = 1$) and where the space rotation is stationary $\frac{\partial A_{ik}}{\partial t} = 0$. It should be noted that the metric of weak plane gravitational waves, the only metric used in the theory of gravitational wave experiments, satisfies these requirements.

Because $\varphi = \frac{1}{c} n_i \mathrm{v}^i$ (7.15), in the case we are considering the time observable component φ of the relative deviation vector η^α is zero $\varphi = 0$. For this reason we consider only the observable spatial component of the Synge equation (7.21).

In the accompanying reference frame (where the observer accompanies his references), according to the theory of chronometric invariants [42, 43], in the absence of gravitational fields $\mathrm{w} = 0$ and also gravitational inertial forces $F_i = 0$, we have: $\frac{d}{ds} = \frac{1}{c}\frac{d}{d\tau}$, $U^0 = \frac{1}{\sqrt{g_{00}}} = 1$, $U^i = \frac{1}{c}\mathrm{v}^i$, $\eta^0 = -g_{0i}\eta^i$, $\Gamma^i_{00} = -\frac{1}{c^2}\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 F^i = 0$, $\Gamma^i_{0k} = \frac{1}{c}\left(1 - \frac{\mathrm{w}}{c^2}\right)\Big(D^i_k + A^{\cdot i}_{k\cdot} + \frac{1}{c^2} v_k F^k\Big) = \frac{1}{c}\big(D^i_k + A^{\cdot i}_{k\cdot}\big)$. Employing now the formulae for the Synge equation (7.21) under $\mathrm{v}^i = 0$, we obtain the *Synge equation in chr.inv.-form**

$$\frac{d^2\eta^i}{d\tau^2} + 2\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\frac{d\eta^k}{d\tau} = 0\,. \qquad (7.28)$$

The quantity $\frac{d}{d\tau} = \frac{{}^*\partial}{\partial t} + \mathrm{v}^i \frac{{}^*\partial}{\partial x^i}$ [42, 43] here is

$$\frac{d}{d\tau} = \frac{\partial}{\partial t}\,, \qquad (7.29)$$

*As we mentioned, if the particles are at rest $\mathrm{v}^i = 0$, the chr.inv.-time component of the Synge equation becomes zero.

so the chr.inv.Synge -equation (7.28) takes its final form

$$\frac{\partial^2 \eta^i}{\partial t^2} + 2\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\frac{\partial \eta^k}{\partial t} = 0\,. \qquad (7.30)$$

We find the exact solution to the Synge chr.inv.-equation (7.30) in the field of weak plane gravitational waves*. In the case we are considering ($\mathrm{v}^i = 0$) we have

$$F_i = 0\,, \qquad A_{ik} = 0\,,$$
$$D_{22} = -D_{33} = \frac{1}{2}\frac{\partial a}{\partial t}\,, \quad D_{23} = \frac{1}{2}\frac{\partial b}{\partial t}\,. \qquad (7.31)$$

Substituting the requirements into the initial equation (7.30) we obtain a system of three equations

$$\frac{\partial^2 \eta^1}{\partial t^2} = 0\,, \qquad (7.32)$$

$$\frac{\partial^2 \eta^2}{\partial t^2} + \frac{\partial a}{\partial t}\frac{\partial \eta^2}{\partial t} - \frac{\partial b}{\partial t}\frac{\partial \eta^3}{\partial t} = 0\,, \qquad (7.33)$$

$$\frac{\partial^2 \eta^3}{\partial t^2} - \frac{\partial a}{\partial t}\frac{\partial \eta^3}{\partial t} - \frac{\partial b}{\partial t}\frac{\partial \eta^2}{\partial t} = 0\,. \qquad (7.34)$$

The solution of (7.32) is

$$\eta^1 = \eta^1_{(0)} + \dot{\eta}^1_{(0)} t\,, \qquad (7.35)$$

where $\eta^1_{(0)}$ is the particle's initial deviation, $\dot{\eta}^1_{(0)}$ is its initial velocity.

This system can be easy solved in two particular cases of a linear polarized wave: (1) $b = 0$, and (2) $a = 0$.

In the first case ($b = 0$) we obtain

$$\frac{\partial \eta^2}{\partial t} = C_1\, e^{-a}, \qquad \frac{\partial \eta^3}{\partial t} = C_2\, e^{+a}, \qquad (7.36)$$

where C_1 and C_2 are integration constants. Because values of a are weak, we can decompose e^{-a} into series. Then, assuming higher order terms infinitesimal, we obtain

$$\frac{\partial \eta^2}{\partial t} = C_1\,(1 - a)\,, \qquad \frac{\partial \eta^3}{\partial t} = C_2\,(1 + a)\,. \qquad (7.37)$$

Assuming also that a falling gravitational wave is monochrome, bearing a constant amplitude A and a frequency ω,

$$a = A \sin\frac{\omega}{c}\left(ct \pm x^1\right), \qquad (7.38)$$

we integrate the system (7.37). As a result we obtain

$$\eta^2 = C_1\Big[t + \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\Big] + D_1\,, \qquad (7.39)$$

$$\eta^3 = C_2\Big[t - \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\Big] + D_2\,, \qquad (7.40)$$

*Where the metric (5.7) is $ds^2 = c^2dt^2 - (dx^1)^2 - (1-a)(dx^2)^2 + 2bdx^2dx^3 - (1+a)(dx^3)^2$.

where D_1 and D_2 are integration constants. Assuming $x^1 = 0$ at the initial moment of time $t = 0$, we easily express the integration constants C_1, C_2, D_1, D_2 through the initial conditions. Finally, we obtain solutions

$$\eta^2 = \dot{\eta}^2_{(0)}\Big[t + \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\Big] + \eta^2_{(0)} - \frac{A}{\omega}\,\dot{\eta}^2_{(0)}\,, \quad (7.41)$$

$$\eta^3 = \dot{\eta}^3_{(0)}\Big[t - \frac{A}{\omega}\cos\frac{\omega}{c}\left(ct \pm x^1\right)\Big] + \eta^3_{(0)} - \frac{A}{\omega}\,\dot{\eta}^3_{(0)}\,, \quad (7.42)$$

where $\eta^2_{(0)}$, $\eta^3_{(0)}$ and $\dot{\eta}^2_{(0)}$, $\dot{\eta}^3_{(0)}$ are the initial numerical values of the relative deviation η and relative velocity $\dot{\eta}$ of the particles along the x^2 and x^3 axes, respectively.

We have now obtained the exact solutions to the Synge equation (the geodesic lines deviation equation). From the solutions we see,

> If at the initial moment of time $t = 0$, two free particles are at rest with respect to each other and the observer $\dot{\eta}^2_{(0)} = \dot{\eta}^3_{(0)} = 0$, weak plane gravitational waves of the deformation kind (waves of the Riemannian curvature) cannot force the particles to go into relative motion. If at the initial moment of time the particles are in motion, the waves augment the particles' initial motion, accelerating them.

Thus our purely mathematical analysis of detectors built on free masses leads to the final conclusion:

> Weak plane gravitational waves of the deformation kind (the Riemannian curvature's waves) cannot be detected by any antenna composed on free masses, if the masses are at rest with respect to each other and the observer.

8 Criticism of Weber's conclusions on the possibility of detecting gravitational waves by solid-body detectors of the resonance kind

Historically, the first gravitational wave detector was the quadrupole mass-detector built in 1964 by Prof. Joseph Weber with his students David Zipoy and Robert Forward at Maryland University [70]. It was an aluminium cylindrical pig weighing 1.5 tons, suspended by a steel "thread" in a vacuum camera. At the point of connection between the pig and the thread, the pig was covered by a piezoelectric quartz film linked to a highly sensitive voltmeter. Weber expected that *a falling gravitational wave should make relative displacements of the butt-ends of the cylindrical pig — extension or compression of the pig*. In other words, they expected that falling gravitational waves will deform the pig, necessarily causing a piezoelectric effect in it. Modified by Sinsky [71], the first detector gave a possibility of registering a 10^{-16}cm relative displacement of its butt-ends.

Later, Weber built a system of two pigs. That system worked through the principle of coincident frequencies of the signals registered in both pigs. The pigs had a relaxation time about 30 sec, were tuned for the frequency 10^4 rps, and were separated by 2 km. In 1967 Weber and his team registered coincident signals (to a precision within 0.2 sec) which appeared about once a month [1]. The registered relative displacements of the butt-ends in the pigs were $\sim 3\times 10^{-10}$ cm. Weber supposed that the origin of the observed signals were gravitational wave radiations.

Weber subsequently even used 6 pigs, one of which was located at Argonne National Laboratory (Illinois), the other 5 pigs located in his laboratory at Maryland University. The distance between the laboratories wss about 1000 km. The detectors were tuned for 1660 Hz — the frequency of supposed gravitational radiations excited from collapsing supernovae. During several months of observations, numerous coincident signals were registered [72]. A second cycle of the observations gave the same positive result [73]. Weber interpreted the registered signals as proof that strong gravitational radiations exist in the Galaxy. A peculiarity of those experiments was that the pigs located both in Illinois and Maryland were isolated as much as possible from external electromagnetic and seismic influences.

After Weber's pioneering experiments, experimental physicists built many similar detectors, much more sensitive than those of Weber. However, in contrast to those of Weber, not one of them registered any signals.

Therefore, using the world-lines deviation theory developed here in the terms of physically observable quantities, we are going to:

(1) investigate what in principle can be registered by a solid-body detector (a Weber pig) and

(2) compare our conclusion with that explanation given by Weber himself for his observed signals.

From the theoretical viewpoint we can conceive of a solid-body cylindrical detector as consisting of two test-particles, connected by a spring [16]. It is supposed that the system falls freely. It is also supposed that at the initial moment of time, when we start our measurements, the particles are at rest with respect to us (the observers) and each other. This is the standard problem statement, not only of Weber [16] or ourselves, but also of any other theoretical physicist.

The behaviour of two neighbouring particles in their motion along their neighbouring world-lines is described by the *world-lines deviation equation*. If the particles are not free, but connected by a non-gravitational force Φ^α (a spring, for instance), the Synge-Weber equation (2.12) applies, namely $\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}U^\beta U^\delta\eta^\gamma = \frac{1}{m_0c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}dv$. This is an inhomogeneous differential equation of the 2nd order with respect to the relative deviation vector η^α of the particles. In order to solve the world-lines deviation equation we need to write $\frac{\mathrm{D}^2\eta^\alpha}{ds^2}$ and $\frac{\mathrm{D}\Phi^\alpha}{dv}dv$ in expanded form. Because both terms contain the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, it would be reasonable to express the components of the Riemann-Christoffel tensor $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ in terms of the $\Gamma^\alpha_{\mu\nu}$ and their derivatives: in collecting similar terms some of them will cancel out (the same situation arose when we made the same calculations for the geodesic lines deviation equation).

Using formulae (7.17) and (7.18), and the quantity $\frac{dU^\alpha}{ds}$ from the world-line equation of a particle moved by a non-gravitational force Φ^α (2.11), we obtain

$$\frac{dU^\alpha}{ds} = -\Gamma^\alpha_{\rho\sigma}U^\rho U^\sigma + \frac{\Phi^\alpha}{m_0c^2}\,. \tag{8.1}$$

Expanding the formula for $\frac{\mathrm{D}\Phi^\alpha}{dv}dv$

$$\begin{aligned}\frac{\mathrm{D}\Phi^\alpha}{dv}dv &= \frac{\partial\Phi^\alpha}{\partial v}dv + \Gamma^\alpha_{\mu\nu}\Phi^\mu\frac{\partial x^\nu}{\partial v}dv = \\ &= \frac{\partial\Phi^\alpha}{\partial x^\sigma}\eta^\sigma + \Gamma^\alpha_{\mu\nu}\Phi^\mu\eta^\nu\end{aligned} \tag{8.2}$$

and substituting this into the world-lines deviation equation in its initial form (2.12), takeing into account that (8.1) and (8.2), we obtain

$$\begin{aligned}\frac{d^2\eta^\alpha}{ds^2} + 2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu + \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma}U^\beta U^\delta\eta^\gamma &= \\ = \frac{1}{m_0c^2}\frac{\partial\Phi^\alpha}{\partial x^\sigma}\eta^\sigma.\end{aligned} \tag{8.3}$$

This is the final form of the world-lines deviation equation for two test-particles connected by a spring. The quantities η^α and U^α are connected by (2.13): $\frac{\partial}{\partial s}(U_\alpha\eta^\alpha) = \frac{1}{m_0c^2}\Phi_\alpha\eta^\alpha$.

In a gravitational wave detector like Weber's, the cylindrical pig is isolated as much as possible from external influences of thermal, electromagnetic, seismic and another origins. To minimise external influences, experimental physicists place the detectors in mines located deep inside mountains or otherwise deep beneath the terrestrial surface, and cool the pigs to 2 K. Therefore particles of matter in the butt-ends and the pig in general, can be assumed at rest with respect to one another and to the observer.

Following Weber, experimental physicists expect that a falling gravitational wave will deform the pig, displacing its butt-ends with respect to each other. Relative displacements of the butt-ends of a pig are supposed to result in a piezo-electric effect which can be registered by a piezo-detector. In other words, experimental physicists expect that oscillations of the acting gravitational wave field give rise to a force in the world-lines deviation equation (the Synge-Weber equation), thereby displaceing the test-particles which were at rest at the initial moment of time. Oscillations of the acting gravitational wave field force the butt-ends of the pig to oscillate. As soon as the frequency of the pig's oscillations coincides with the falling wave's frequency, the amplitude of the pig's oscillations will increase significantly because of resonance, so the amplitude becomes measurable. Therefore the Weber detectors are said to be of the *resonance kind*.

Before ratifying the aforesaid conclusions it would be reasonable to study the world-lines deviation equation for two interacting test-particles that model a Weber pig, because this equation is the theoretical basis of all experimental attempts to register gravitational waves made by Weber and his followers during more than 30 years.

We will study this equation, proceeding from its form (8.3), because formula (8.3) gives a possibility of obtaining exact solutions to the relative deviation vector η^α; not the initial equation (2.12). Following analysis of the solutions we will come to a conclusion as to what effect a falling gravitational wave has on the detector*.

When we need to give a theoretical interpretation of experimental results, it is very important to analyse the results in the terms of physically observable quantities because such quantities can be registered in practice. For this reason we will study the behaviour of the Weber model (the system of two particles, connected by a non-gravitational force) in the terms of physically observable quantities (chronometric invariants) as we did in §7 when we solved a similar problem for the system of two free particles.

In detail, our task here is to consider commonly the world-lines equation (2.11) and the world-lines deviation equation (8.3), both written in chr.inv.-form. Note that the relationship (2.13), that is $\frac{\partial}{\partial s}(U_\alpha \eta^\alpha) = \frac{1}{m_0 c^2}\,\Phi_\alpha \eta^\alpha$, gives the exact solution for the quantity φ. The φ is the chr.inv.-time component of the relative deviation world-vector η^α with respect to which the world-lines deviation equation (8.3) is written. For this reason there is no need here to solve the chr.inv.-time projection of the world-lines deviation equation (8.3). We solve instead the relationship (2.13).

The world-lines equation (2.11) in chr.inv.-form is

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i \mathrm{v}^i + \frac{m}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k = \frac{\sigma}{c}\,, \qquad (8.4)$$

$$\frac{d}{d\tau}(m\mathrm{v}^i) + 2m\big(D^i_k + A^{\cdot i}_{k\cdot}\big)\mathrm{v}^k - mF^i + \Delta^i_{kn}\mathrm{v}^k\mathrm{v}^n = f^i, \qquad (8.5)$$

where $\sigma \equiv \frac{\Phi_0}{\sqrt{g_{00}}}$ and $f^i \equiv \Phi^i$ are chr.inv.-components of the prevailing non-gravitational force Φ^α. In the case of the Weber model where the particles are at rest with respect to the observer ($\mathrm{v}^i = 0$), the chr.inv.-equations (8.4) and (8.5) become

$$\sigma = 0\,, \qquad (8.6)$$

$$m_0 F^i = -f^i. \qquad (8.7)$$

The condition $\sigma = 0$ comes from the fact that, when a particle is at rest its relativistic mass becomes the rest-mass $m = m_0$. Thus resting particles are under the action of only the spatial observable components f^i of the non-gravitational force Φ^α, so that the f^i are of the same value as the acting gravitational inertial force F^i, but acts in the opposite direction. Looking at definition (4.1), given by the theory of physical observable quantities for the gravitational inertial force F^i, we see that in this case the non-gravitational force f^i acts on a resting particle only if at least one of the following factor holds:

1. Inhomogeneity of the gravitational potential $\frac{\partial \mathrm{w}}{\partial x^i} \neq 0$;
2. Non-stationarity of the vector of the space rotation linear velocity $\frac{\partial v_i}{\partial t} \neq 0$.

If neither factor holds, $F^i = 0$ and hence $f^i = 0$, in which case both interacting particles, which are at rest with respect to each other and the observer, behaviour like free particles: their connecting force Φ^α does not manifest. Looking at the well-known metric (5.7) that describes weak plane gravitational waves, we see there that $F^i = 0$, $A_{ik} = 0$ and hence $v_i = 0$. Therefore:

> Weak plane gravitational waves described by the metric (5.7) **cannot be registered** by a solid-body detector of the resonance kind (a Weber detector).

Writing the relationship (2.13) in chr.inv.-form, we obtain

$$\frac{d}{d\tau}\left(\frac{\varphi - \frac{1}{c}\,n_i \mathrm{v}^i}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\right) = \frac{\sigma\varphi - f_i n^i}{mc}\,, \qquad (8.8)$$

where again, $\varphi \equiv \frac{\eta_0}{\sqrt{g_{00}}}$ and $n^i \equiv \eta^i$ are chr.inv.-components of the relative deviation world-vector η^α.

From this we see that the angle between the vectors U_α and η^α is a variable depending on many factors, including the velocity v^i of the particles. At speeds close to that of light c, the angle increases. At $\mathrm{v} = c$ formula (8.8) becomes senseless: the denominator on the left side becomes zero.

If both particles are at rest, formula (8.8) becomes

$$\frac{d\varphi}{d\tau} = -\frac{f_i n^i}{m_0 c} = \frac{F_i n^i}{c}\,, \qquad (8.9)$$

so that in the case of interacting rest-particles, in contrast to free ones, there is the time observable component φ of the relative deviation vector η^α. This implies that there are not only relative spatial displacements of the particles, but also a deviation between measurements of time made by the clocks of both particles. The "time deviation" φ can be found by integrating (8.9). We obtain

$$\varphi = \frac{1}{c}\int F_i n^i + \mathrm{const}\,, \qquad (8.10)$$

so the value of the "time deviation" φ increases with time. It Note that $\frac{\varphi}{\tau} = 0$ only if the vector F^i (and hence, in this case, also f^i) is orthogonal to the vector n^i, so that $F_i n^i = -\frac{1}{m_0} f_i n^i = 0$.

*It is evident that equation (8.3) can be solved also for other forcing fields, which can be of a non-wave origin.

The integral (8.10) is the solution to the chr.inv.-time component of the world-lines deviation equation (8.3). This solution for φ itself, being a chronometric invariant, is a physically observable quantity.

We are now going to obtain solutions to the remaining three chr.inv.-equations with respect to η^α:

$$\begin{aligned}&\frac{1}{c^2}\frac{d^2\eta^i}{d\tau^2}+2\Gamma^i_{00}\frac{1}{c}\frac{d\eta^0}{d\tau}U^0+2\Gamma^i_{k0}\frac{1}{c}\frac{d\eta^k}{d\tau}U^0+\\&+\frac{\partial\Gamma^i_{00}}{\partial x^0}U^0U^0\eta^0+\frac{\partial\Gamma^i_{00}}{\partial x^k}U^0U^0\eta^k=\frac{1}{m_0c^2}\frac{\partial\Phi^i}{\partial x^\sigma}\eta^\sigma,\end{aligned}\tag{8.11}$$

— the chr.inv.-spatial components of the world-lines deviation equation (8.3), in the case of two rest-particles $ds=cd\tau$.

In the left side of (8.11) we substitute the formulae for the quantities Γ^i_{00}, Γ^i_{00}, U^0, η^0, given on page 53, and also derivatives of the quantities. Then we transform the right part of (8.11) as follows

$$\frac{\partial\Phi^i}{\partial x^\sigma}\eta^\sigma=\frac{\partial f^i}{\partial x^0}\eta^0+\frac{\partial f^i}{\partial x^k}\eta^k=\frac{\varphi}{c}\frac{^*\partial f^i}{\partial t}+\frac{^*\partial f^i}{\partial x^k}n^k,\tag{8.12}$$

where we use the definitions of the chr.inv.-derivative operators (see page 40): $\frac{^*\partial}{\partial t}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{^*\partial}{\partial x^i}=\frac{\partial}{\partial x^i}+\frac{1}{c^2}v_i\frac{^*\partial}{\partial t}$.

The initial equation (8.11) becomes

$$\begin{aligned}&\frac{d^2n^i}{d\tau^2}+2\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\frac{dn^k}{d\tau}-\frac{2}{c}\frac{d\varphi}{d\tau}F^i+\frac{2}{c^2}F_kn^kF^i-\\&-\frac{\varphi}{c}\frac{^*\partial F^i}{\partial t}-\frac{^*\partial F^i}{\partial x^k}n^k=\frac{1}{m_0}\left(\frac{\varphi}{c}\frac{^*\partial f^i}{\partial t}+\frac{^*\partial F^i}{\partial x^k}n^k\right).\end{aligned}\tag{8.13}$$

Owing to the particular conditions (8.7) and (8.9), derived from the requirement that the particles are at rest ($\mathrm{v}^i=0$), formula (8.13) becomes much more simple

$$\frac{d^2n^i}{d\tau^2}+2\left(D^i_k+A^{\cdot i}_{k\cdot}\right)\frac{dn^k}{d\tau}=0\,,\tag{8.14}$$

which is the final form for the chr.inv.-spatial deviation equation for two rest-particles, connected by a non-gravitational force.

Equation (8.14) is like the chr.inv.-spatial deviation equation for two free rest-particles (7.30) — the chr.inv.-spatial part of the Synge general covariant equation. The difference is that (8.14) contains derivatives $\frac{d}{d\tau}=\frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$, while (7.30) contains $\frac{\partial}{\partial t}$. This difference is derived from the fact that (7.30) is applicable to gravitational fields where $F_i=0$, the potential w is neglected and hence $\frac{\partial v_i}{\partial t}=0$, while (8.14) describes the relative deviation of two particles located in gravitational fields where $F_i\neq 0$.

The required condition $F_i\neq 0$ implies:

1. In this region the gravitational potential is $\mathrm{w}\neq 0$, hence, because the interval of physical observable time is $d\tau=\left(1-\frac{\mathrm{w}}{c^2}\right)dt-\frac{1}{c^2}v_idx^i$, the time flow differences at different points inside the region. In particular, if $v_i=0$, synchronization of clocks located at different points cannot be conserved. In the more general case where $v_i\neq 0$, clocks located at different points cannot be synchronized [42, 43];*
2. If the gravitational inertial force field F^i is vortical, the space rotation is non-stationary $\frac{\partial v_i}{\partial t}\neq 0$.

Let us get back to the chr.inv.-spatial equation for two particles connected by a non-gravitational force (8.14). There are quantities D_{ik} and A_{ik}, so relative accelerations of the particles can be derived from both the space deformations and rotation. In this problem statement, w implies that the gravitational potential of a distant source of gravitational radiations. So in a gravitational wave experiment we should specify the acting gravitational field as weak $\frac{\mathrm{w}}{c^2}\approx 0$, hence in the experiment the chr.inv.-gravitational inertial force vector F_i (4.1) becomes $F_i\approx\frac{\partial\mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}$. There are as well $\frac{d}{d\tau}=\frac{\partial}{\partial t}$.

We solve equation (8.14) in two cases, aiming to find what kind of gravitational field fluctuations were registered by Weber and his colleagues.

First case: $A_{ik}=0$, $D_{ik}\neq 0$.

In this case equation (8.14), with $\frac{\mathrm{w}}{c^2}\approx 0$, is the same as the chr.inv.-world-lines deviation equation for two free particles (7.30). As it was shown in §7, with solutions of equation (7.30) considered, a gravitational wave can affect the system of two free particles only if the particles are in motion at the initial moment of time. In that case a gravitational wave can only augment the initial motion of the particles. If they are at rest gravitational waves can have no effect on the particles.

*To realize the condition $\mathrm{w}\neq 0$ it is not necessary to have a wave gravitational field. In particular, $\mathrm{w}\neq 0$ is true even in stationary gravitational fields derived from island masses (like Schwarzschild's metric). Moreover, the phenomenon of different time flow in the Earth gravitational field is well-known from experimental tests of the General Theory of Relativity: a standard clock, located on the terrestrial surface, shows time which is $\sim 10^{-9}$sec different from time measured by the same standard clock, located in a balloon a few kilometers above the terrestrial surface (the difference increases with the duration of the experiment). But such corrections of time are not linked to the presence of gravitational waves.

There time corrections can also be registered, the origin of which are wave changes of the gravitational potential w. They can be interpreted as waves of the gravitational inertial force field F^i. In this case corrections to standard clocks, located at different points, should bear a relation to wave changes of w.

The presence of the space rotation $v_i\neq 0$ changes the time flow as well. Experiments, where a standard clock was moved by a jet plane around the world [49, 50, 51, 52], showed differential time flow with respect to the same standard clock located at rest at the air force base. Such difference of measured time, called the *desynchronization correction*, depends on the flight direction — with or opposite to Earth's rotation. Although such corrections are derived from the Earth rotation (the reference space rotation), in the "background" of such corrections there could also be registered additional tiny corrections derived from the rapid stationary rotation field of a massive space body, located far from the Earth.

When $A_{ik}=0$ the chr.inv.-world-lines deviation equation (8.14), describing a Weber detector, coincides with equation (7.30), and we conclude:

> A Weber detector (a solid-body detector of the resonance kind) will have no response to a falling gravitational wave of the pure deformation kind, if the particles of which the detector is composed are at rest at the initial moment of measurements (the situation assumed in the Weber experiment).

Second case: $D_{ik}=0$, $A_{ik}\neq 0$.

We assume that the space rotation has a constant angular velocity ω around the x^3 axis, while the linear velocity of this rotation is $v^i=\omega_{k\cdot}^{\cdot i}x^k \ll c$. For the background metric, following the classical approach [14, 15], we use the Minkowski line element, where the gravitational waves are superimposed as tiny corrections to it. Then the components of v^i are

$$v^1=-\omega x^2,\quad v^2=\omega x^1,\quad v^3=0\,, \tag{8.15}$$

and the space metric in its expanded form is

$$ds^2=c^2dt^2+2\omega\left(x^2dx^1-x^1dx^2\right)dt-\\-\left(dx^1\right)^2-\left(dx^2\right)^2-\left(dx^3\right)^2. \tag{8.16}$$

This metric describes the four-dimensional space of a uniformly rotating reference frame, whose rotational linear velocity is negligible with respect to c.

Components of the tensor A_{ik} are

$$A_{2\cdot}^{\cdot 1}=\omega_{2\cdot}^{\cdot 1}=-\omega\,,\quad A_{1\cdot}^{\cdot 2}=\omega_{1\cdot}^{\cdot 2}=\omega\,,\quad A_{1\cdot}^{\cdot 3}=0\,. \tag{8.17}$$

Substituting (8.17) into the chr.inv.-world-lines deviation equation (8.14) we obtain a system of deviation equations

$$\frac{\partial^2\eta^1}{\partial t^2}-2\omega\frac{\partial\eta^2}{\partial t}=0\,, \tag{8.18}$$

$$\frac{\partial^2\eta^2}{\partial t^2}+2\omega\frac{\partial\eta^1}{\partial t}=0\,, \tag{8.19}$$

$$\frac{\partial^2\eta^3}{\partial t^2}=0\,, \tag{8.20}$$

which commonly describe behaviour of two neighboring rest-particles in a uniformly rotating reference frame.

Equation (8.20) can be integrated immediately

$$\eta^3=\eta^3_{(0)}+\dot{\eta}^3_{(0)}t\,, \tag{8.21}$$

where $\eta^3_{(0)}$ and $\dot{\eta}^3_{(0)}$ are the initial values of the relative displacement and velocity of the particles along the x^3 axis.

In integrating equations (8.19) and (8.20), we introduce the notation $\frac{\partial\eta^1}{\partial t}\equiv x$ and $\frac{\partial\eta^2}{\partial t}\equiv y$. Then we have

$$\left.\begin{aligned}\dot{x}-2\omega y=0\\ \dot{y}+2\omega x=0\end{aligned}\right\}. \tag{8.22}$$

We differentiate the first equation with respect to t

$$\ddot{x}=2\omega\dot{y} \tag{8.23}$$

and substitute $\dot{y}=\ddot{x}/2\omega$ into the second one. We obtain a harmonic oscillation equation

$$\ddot{x}+4\omega^2x=0\,, \tag{8.24}$$

with respect to the relative velocity $x=\frac{\partial\eta^1}{\partial t}$ of the particles. The solution to (8.24) is

$$x=\frac{\partial\eta^1}{\partial t}=C_1\cos 2\omega t+C_2\sin 2\omega t\,, \tag{8.25}$$

where C_1 and C_2 are integration constants, which can be obtained from the initial conditions. Thus we obtain

$$\frac{\partial\eta^1}{\partial t}=\left(\frac{\partial\eta^1}{\partial t}\right)_{(0)}\cos 2\omega t+\frac{1}{2\omega}\left(\frac{\partial^2\eta^1}{\partial t^2}\right)_{(0)}\sin 2\omega t\,, \tag{8.26}$$

where terms marked with zero are the initial values of the relative velocity and acceleration of the particles. Integrating (8.26), we obtain

$$\eta^1=\frac{\dot{\eta}^1_{(0)}}{2\omega}\sin 2\omega t-\frac{\ddot{\eta}^1_{(0)}}{4\omega^2}\cos 2\omega t+B_1\,, \tag{8.27}$$

where B_1 is an integration constant. Obtaining B_1 from the initial conditions, we obtain the final formula for η^1

$$\eta^1=\frac{\dot{\eta}^1_{(0)}}{2\omega}\sin 2\omega t-\frac{\ddot{\eta}^1_{(0)}}{4\omega^2}\cos 2\omega t+\eta^1_{(0)}+\frac{\ddot{\eta}^1_{(0)}}{4\omega^2}\,. \tag{8.28}$$

In the same fashion we obtain a formula for η^2

$$\eta^2=\frac{\dot{\eta}^2_{(0)}}{2\omega}\sin 2\omega t-\frac{\ddot{\eta}^2_{(0)}}{4\omega^2}\cos 2\omega t+\eta^2_{(0)}+\frac{\ddot{\eta}^2_{(0)}}{4\omega^2}\,. \tag{8.29}$$

By the exact solutions (8.21), (8.28), (8.29), obtained for the world-lines deviation equation taken in chr.inv.-form (8.14), it follows that:

> Stationary rotations of the space cannot force two neighbouring particles to initiate relative motion, if they are at rest at the initial moment of time.

In common with the result obtained in §7, where we discussed gravitational wave detectors built on free masses, we arrive at a final conclusion for the possibilities of gravitational wave detectors:

> Behaviour of both a gravitational wave detector built on free masses and a sold-body detector (a Weber pig) are **similar**. The only difference is that a solid-body detector can register both the time observable component and spatial observable components of the relative deviation vector, while a free-mass detector can register only spatial observable deviations. Deformations and stationary rotation of the space do not affect detectors of either kind.

Thus neither deformations nor stationary rotation of the space can not induce relative motion in the butt-ends of a Weber detector, if they are at rest. However Weber and his teamregistered signals. The question therefore arises:

> What signals did Weber register, and why, during the past 30 years, have his signals remained undetected by other researchers using superior detectors of the Weber kind?

We assume that the signals registered by Weber and his team, were much more than noise, and beyond doubt. Therefore, according to our theoretical analysis of the behaviour of solid-body detectors in weak gravitational waves, we make the following suppositions:

1. Weber registered signals which were an effect made in the pig by a vortex of the gravitational inertial force field. In other words, the origin of the signals could be rapid non-stationary rotation of a distant object in the depths of space;
2. The particles of the aluminium cylindrical pig, used by Weber, had substantial thermal motions. In this case parametric oscillations could appear as an effect of a falling gravitational wave. But in order to get such a real effect, the "background thermal oscillations" should be substantial;
3. The signals were registered only by Weber and his team. Not one signal has been registered by other experimental physicist during the subsequent 30 years, using superior detectors of the Weber kind. Either Weber registered gravitational waves derived from a non-stationary rotating object in the Universe, which occurred as a unique and short-lived phenomenon, or his original detector had a substantial peculiarity that made it differ in principle from the detectors used by other scientists.

We consider Weber's theory, aiming to ascertain what he registered with his solid-body detector.

9 Criticism of Weber's theory of detecting gravitational waves

In his book in 1960 [16], Weber propounded his theoretical arguments for the detection of gravitational waves by means of a solid-body detector of the resonance kind. He built his theory on the world-lines deviation equation for two particles, connected by a non-gravitational force (a spring, for instance). This is equation (2.12), being a modification of the well-known deviation equation for two free particles deduced by Synge (2.8), is known as the Synge-Weber equation. We considered both equations in detail above.

There is no doubt that the Synge-Weber equation is valid. Our main claim here is that Weber himself, in his analysis of the equation in order to build the theory for detecting gravitational waves, introduced a substantial assumption:

Weber's assumption 1 A falling gravitational wave should produce relative displacements of the butt-ends of a cylindrical pig.

So he obtained the same principle that he introduced, precluding himself from any possibility of obtaining anything else.

This line of reasoning constitutes a vicious circle. It would be been more reasonable and honest to have solved the world-lines deviation equation. Then he would have obtained exact solutions to the equation as was done in the previous sections herein.

In detail Weber's assumption 1 leads to the fact that, having a system of two test-particles connected by a spring, the resulting distance vector between them should be [16]

$$\eta^\alpha = r^\alpha + \xi^\alpha, \qquad r^\alpha \gg \xi^\alpha, \tag{9.1}$$

where the initial distance vector r^α is the such that

$$\frac{\mathrm{D}r^\alpha}{ds} = 0\,. \tag{9.2}$$

He supposed as well that $\eta^\alpha \to r^\alpha$ in the ultimate case where the friction rises infinitely or the Riemann-Christoffel curvature tensor becomes zero $R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta} = 0$ [16].

Taking the main supposition (9.1) into account, Weber transforms the Synge-Weber equation (2.12) into

$$\frac{\mathrm{D}^2\xi^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}\,U^\beta U^\delta\left(r^\gamma + \xi^\gamma\right) = \frac{1}{m_0c^2}\,f^\alpha, \tag{9.3}$$

where f^α is the difference between non-gravitational forces of the particles' interaction. Weber assumes f^α the sum of the elasticity force $f_1^\alpha = -\mathcal{K}^\alpha_\sigma\,\xi^\sigma$ that restores the particles, and the oscillation relaxing force $f_2^\alpha = -c\,\mathcal{D}^\alpha_\sigma\frac{\mathrm{D}\xi^\sigma}{ds}$, where $\mathcal{K}^\alpha_\sigma$ and $\mathcal{D}^\alpha_\sigma$ are the elasticity and friction coefficients, respectively. Then (9.3) takes the form

$$\frac{\mathrm{D}^2\xi^\alpha}{ds^2} + \frac{1}{m_0c}\,\mathcal{D}^\alpha_\sigma\frac{\mathrm{D}\xi^\sigma}{ds} + \frac{1}{m_0c^2}\,\mathcal{K}^\alpha_\sigma\,\xi^\sigma = -R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}\,U^\beta U^\delta\left(r^\gamma + \xi^\gamma\right). \tag{9.4}$$

Weber introduced additional substantial assumptions:

Weber's assumption 2 The whole detector is in the state of free falling;

Weber's assumption 3 The reference frame in his laboratory is such that the Christoffel symbols can be assumed zero.

Because of these assumptions, and the condition $|r| \gg |\xi|$, Weber writes equation (9.4) as follows

$$\frac{d^2\xi^\alpha}{dt^2} + \frac{1}{m_0}\,\mathcal{D}^\alpha_\sigma\frac{d\xi^\sigma}{dt} + \frac{1}{m_0c^2}\,\mathcal{K}^\alpha_\sigma\,\xi^\sigma = -c^2 R^{\alpha\cdot\cdot\cdot}_{\cdot 0\sigma 0}\,r^\sigma. \tag{9.5}$$

Looking at the right side of Weber's equation (9.5) we see his fourth hidden assumption:

Weber's assumption 4 Particles located on two neighbouring world-lines in the Weber experimental statement (the butt-ends of his cylindrical pig) are at rest at the initial moment of time, so $U^i = 0$.

In §7, where we considered chr.inv.-equations of motion for two particles connected by a non-gravitational force (8.4) and (8.5), we came to the conclusion: a reference frame where interacting particles ($\Phi^\alpha \neq 0$) are at rest ($\mathrm{v}^i = 0$) cannot be in astate of free fall. Really, the free fall condition is $F^i = 0$. Equation $m_0 F^i = -f^i$ (8.7), which is the chr.inv.-form of spatial equations of motion of the interacting particles, implies that when $F^i = 0$, $f^i = 0$. Therefore:

The Weber assumption 2 is **inapplicable** to his experimental statement.

Moreover, a reference frame where the Christoffel symbols are zero can be applicable only at a point, it is unapplicable to a finite region. At the same time, in the Weber experimental statement, the detector itself is a system of two particles located at the distance η from each other. In a Riemannian space the Riemann-Christoffel curvature tensor is different from zero, so the Riemannian coherence objects (the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$) cannot be reduced to zero by coordinate transformations. We can merely choose a reference system where, at a given point P, the coherence objects are zero $(\Gamma^\alpha_{\beta\gamma})_P = 0$. Such a reference frame is known as a *geodesic reference frame* [37]. Therefore:

The Weber assumption 3 is **inapplicable** to his experimental statement.

Thus if we retain the rest-condition $U^i = 0$ and the free fall condition in the Weber equation (9.4), there must still be the non-gravitational force $\Phi^\alpha = 0$. So the Weber equation becomes the free particles deviation equation, which in chr.inv.-form is (7.30).

If we reject free fall in the Weber equation (9.4), but retain $U^i = 0$, it takes the same form as (8.14), which is not a free oscillation equation, in which case weak plane gravitational waves can act on the particles only if they are in motion at the initial moment of time.

Collecting these results we conclude that:

The Weber equation (9.4) is **incorrect**, because the free fall condition in common with the rest-condition for two neighbouring particles, connected by a non-gravitational force, lead to the requirement that this force should be zero, thus contradicting the initial conditions of the Weber experimental statement.

It is evident that in aiming to determine the sort of effects a falling gravitational wave has on a free-mass detector or a Weber detector, it would be reasonable to consider a case where the particles are in motion $U^i \neq 0$. In this case, before solving the deviation equation (2.8) for two free particles or (2.12) for two interacting particles (depending on the type of detector used), we should solve the equations of motion for free particles (2.6) or forced particles (2.11), respectively.

It should be noted that the main structure of motion is determined by the left (geometrical) side of equations of motion, while the right side introduces only an additional effect into the motion.

In my previous articles [74, 75, 76] common exact solutions to the geodesic equations and the deviation equation had been obtained in the field of weak plane gravitational waves, described by the metric (6.12). The exact solutions had been obtained in general covariant form.

The solutions to the equations of motion for a free particle, equations (2.6), in a linear polarized harmonic wave $a = A\sin\frac{\omega}{c}(ct \pm x^1)$, $b = 0$ are as follows

$$U^0 + U^1 = \varepsilon = \text{const}\,, \tag{9.6}$$

$$\begin{aligned} U^1 = -\frac{1}{4\varepsilon}\Big[\big(U^2_{(0)}\big)^2 e^{2A\sin\frac{2\omega}{c}(ct\pm x^1)} + \\ + \big(U^3_{(0)}\big)^2 e^{-2A\sin\frac{\omega}{c}(ct\pm x^1)}\Big]_a + U^1_{(0)}\,, \end{aligned} \tag{9.7}$$

$$U^2 = U^2_{(0)}\, e^{A\sin\frac{\omega}{c}(ct\pm x^1)}, \tag{9.8}$$

$$U^3 = U^3_{(0)}\, e^{A\sin\frac{\omega}{c}(ct\pm x^1)}, \tag{9.9}$$

where $U^1_{(0)}$, $U^2_{(0)}$, $U^3_{(0)}$ are the initial values of the particle's velocity along each of the spatial axes.

From the solutions two important conclusions follow:

1. A weak plane gravitational wave, falling in the x^1 direction, acts on free particles only if they have non-zero velocities in directions x^2 and x^3 orthogonal to the wave motion.
2. The presence of transverse oscillations in the plane (x^2, x^3) leads also to longitudinal oscillations in the direction x^1.

The solutions to the free-particles deviation equation (2.8) in the field of a weak plane gravitational wave are

$$\begin{aligned} \eta^1 = \frac{A\Big[\big(U^3_{(0)}\big)^2 - \big(U^2_{(0)}\big)^2\Big]}{2\varepsilon^2}\Big(\eta^1_{(0)} + \dot{\eta}^1_{(0)} t\Big) \times \\ \times \sin\frac{\omega}{c}(ct \pm x^1) + \frac{AL}{2\varepsilon\omega}\cos\frac{\omega}{c}(ct\pm x^1) + \\ + \left\{\dot{\eta}^1_{(0)} - \frac{A\Big[\big(U^3_{(0)}\big)^2 - \big(U^2_{(0)}\big)^2\Big]}{2\varepsilon^2}\,\omega\eta^1_{(0)}\right\} t + \\ + \eta^1_{(0)} - \frac{AL}{2\varepsilon}\,, \end{aligned} \tag{9.10}$$

$$\begin{aligned} \eta^2 = \dot{\eta}^2_{(0)}\Big[t + \frac{A}{\omega}\cos\frac{\omega}{c}(ct\pm x^1)\Big] + \eta^2_{(0)} - \frac{A}{\omega}\dot{\eta}^2_{(0)} - \\ - \frac{AU^2_{(0)}}{\varepsilon}\Big\{\dot{\eta}^1_{(0)}\Big[t - \frac{1}{\omega}\cos\frac{\omega}{c}(ct\pm x^1)\Big] - \\ - \Big(\eta^1_{(0)} + \dot{\eta}^1_{(0)} t\Big)\cos\frac{\omega}{c}(ct\pm x^1) + \frac{\dot{\eta}^1_{(0)}}{\omega} + \eta^1_{(0)}\Big\}, \end{aligned} \tag{9.11}$$

$$\eta^3 = \dot{\eta}^3_{(0)}\left[t-\frac{A}{\omega}\cos\frac{\omega}{c}\,(ct\pm x^1)\right]+\eta^3_{(0)}+\frac{A}{\omega}\,\dot{\eta}^3_{(0)} - {} + \frac{AU^3_{(0)}}{\varepsilon}\left\{\dot{\eta}^1_{(0)}\left[t-\frac{1}{\omega}\cos\frac{\omega}{c}(ct\pm x^1)\right] - {} - \left(\eta^1_{(0)}+\dot{\eta}^1_{(0)}\,t\right)\cos\frac{\omega}{c}\,(ct\pm x^1)-\frac{\dot{\eta}^1_{(0)}}{\omega}-\eta^1_{(0)}\right\}, \tag{9.12}$$

where

$$L = U^2_{(0)}\dot{\eta}^2_{(0)} - U^3_{(0)}\dot{\eta}^3_{(0)} = \frac{\eta^1_{(0)}}{\varepsilon}\left[\left(U^3_{(0)}\right)^2 - \left(U^2_{(0)}\right)^2\right]. \tag{9.13}$$

The solutions η^1, η^2, η^3 are the relative deviations of two free particles in directions orthogonal to the direction of the wave's motion. The deviations are actually generalizations of the solutions (7.41) and (7.42), where the particles were at rest. The only difference is that here (9.10–9.12) there are additional parts, where the particles' initial velocities $U^2_{(0)}$ and $U^3_{(0)}$ are added.

Here we see that, besides regular harmonic oscillations, the term $t\cos\frac{\omega}{c}\,(ct\pm x^1)$ describes oscillations with an amplitude that increases without bound with time. Another substantial difference is that, in contrast to solutions (7.35), (7.42), (7.43) given for rest-particles, the solutions (9.10), (9.11), (9.12) contain longitudinal oscillations — they are described by solution (9.10). Both harmonic oscillations and unbounded-rising oscillations exist there only if, at the initial moment of time, the particles are in motion along x^2 and x^3 (orthogonal to the x^1 direction of the wave's motion).

So, we come to our final conclusions on both free-mass detectors and solid-body detectors of gravitational waves:

> The greater the velocities of particles (atoms and molecules) in a gravitational wave detector (built on either free masses or of the Weber kind), the more sensitive is the detector to a falling weak plane gravitational wave. In current experiments researchers cool the Weber pigs to super low temperatures, about 2 K, aiming to minimize the inherent oscillations of the particles of which they consist. This is a counter-productive procedure by which experimental physicists actually reduce the sensitivity of the Weber detectors to practically zero. We see the same vicious drawback in current experiments with free-mass detectors, where such a detector consists of two satellites located in the same orbit near the Earth. Because the observer (a laser range-finder) is located in one of the satellites, both satellites are at rest with respect to each other and the observer. All the current experiments cannot register gravitational waves in principle. In a valid experiment for discovering gravitational waves, the particles of which the detector consists must be in as rapid motion as possible. It would be better to design a detector using two laser beams directed parallel to each other, because of the light velocity of the moving particles (photons). The indicative quantities to be observed are the light frequency and phase.

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On Geometric Probability, Holography, Shilov Boundaries and the Four Physical Coupling Constants of Nature

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By recurring to Geometric Probability methods, it is shown that the coupling constants, $\alpha_{EM}; \alpha_W; \alpha_C$ associated with Electromagnetism, Weak and the Strong (color) force are given by the *ratios of the ratios* of the measures of the Shilov *boundaries* $Q_2 = S^1 \times RP^1$; $Q_3 = S^2 \times RP^1$; S^5, respectively, with respect to the ratios of the measures $\mu[Q_5]/\mu_N[Q_5]$ associated with the $5D$ conformally compactified real Minkowski spacetime $\bar{M}_5$ that has the same topology as the Shilov boundary Q_5 of the 5 complex-dimensional poly-disc D_5. The homogeneous symmetric complex domain $D_5 = SO(5,2)/SO(5) \times SO(2)$ corresponds to the conformal relativistic *curved* 10 real-dimensional phase space $\mathcal{H}^{10}$ associated with a particle moving in the $5D$ Anti de Sitter space AdS_5. The geometric coupling constant associated to the gravitational force can also be obtained from the ratios of the measures involving Shilov boundaries. We also review our derivation of the observed vacuum energy density based on the geometry of de Sitter (Anti de Sitter) spaces.

1 The fine structure constant and Geometric Probability

Geometric Probability [21] is the study of the probabilities involved in geometric problems, e. g., the distributions of length, area, volume, etc. for geometric objects under stated conditions. One of the most famous problem is the Buffon's Needle Problem of finding the probability that a needle of length l will land on a line, given a floor with equally spaced parallel lines a distance d apart. The problem was first posed by the French naturalist Buffon in 1733. For $l < d$ the probability is

$$P = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{l|\cos(\theta)|}{d} = \frac{4l}{2\pi d} \int_0^{\pi/2} \cos(\theta) = \frac{2l}{\pi d} = \frac{2ld}{\pi d^2} \,. \tag{1}$$

Hence, the Geometric Probability is essentially the *ratio* of the areas of a rectangle of length $2d$, and width l and the area of a circle of radius d. For $l > d$, the solution is slightly more complicated [21]. The Buffon needle problem provides with a numerical experiment that determines the value of π empirically. Geometric Probability is a vast field with profound connections to Stochastic Geometry.

Feynman long ago speculated that the fine structure constant may be related to π. This is the case as Wyler found long ago [1]. We will based our derivation of the fine structure constant based on Feynman's physical interpretation of the electron's charge as the probability amplitude that an electron emits (or absorbs) a photon. The clue to evaluate this probability within the context of Geometric Probability theory is provided by the electron self-energy diagram. Using Feynman's rules, the self-energy $\Sigma(p)$ as a function of the electron's incoming (outgoing) energy-momentum p_μ is given by the integral involving the photon and electron propagator along the internal lines

$$-i\Sigma(p) = (-ie)^2 \times \times \int \frac{d^4k}{(2\pi)^4} \, \gamma^\mu \, \frac{i}{\gamma^\rho(p_\rho - k_\rho) - m} \, \frac{-ig_{\mu\nu}}{k^2} \, \gamma^\nu . \tag{2}$$

The integral is taken with respect to the values of the photon's energy-momentum k^μ. By inspection one can see that the electron self-energy is proportional to the fine structure constant $\alpha_{EM} = e^2$, the square of the probability amplitude (in natural units of $\hbar = c = 1$) and physically represents the electron's emission of a virtual photon (off-shell, $k^2 \neq 0$) of energy-momentum k_ρ at a given moment, followed by an absorption of this virtual photon at a later moment.

Based on this physical picture of the electron self-energy graph, we will evaluate the Geometric Probability that an electron emits a photon at $t = -\infty$ (infinite past) and reabsorbs it at a much later time $t = +\infty$ (infinite future). The off-shell (virtual) photon associated with the electron self-energy diagram *asymptotically* behaves on-shell at the very moment of emission ($t = -\infty$) and absorption ($t = +\infty$). However, the photon can remain off-shell in the intermediate region between the moments of emission and absorption by the electron.

The topology of the boundaries (at conformal infinity) of the past and future light-cones are spheres S^2 (the celestial sphere). This explains why the (Shilov) boundaries are essential mathematical features to understand the geometric derivation of all the coupling constants. In order to describe the physics at infinity we will recur to Penrose's ideas [10] of conformal compactifications of Minkowski spacetime by attaching the light-cones at conformal infinity. Not unlike the one-point compactification of the complex plane by adding the points at infinity leading to the Gauss-Riemann sphere. The conformal group leaves the light-cone fixed and it does not alter the causal properties of spacetime despite the rescalings of the metric. The topology of the conformal compactification of real Minkowski spacetime $\bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$ is precisely the same as the topology of the Shilov boundary Q_4 of the 4 complex-dimensional poly-disc D_4. The action of the discrete group Z_2 amounts to an antipodal identification of the future null infinity $\mathcal{I}^+$ with the past null infinity $\mathcal{I}^-$; and the antipodal identification of the past timelike infinity i^- with the future timelike infinity, i^+, where the electron emits, and absorbs the photon, respectively.

Shilov boundaries of homogeneous (symmetric spaces) complex domains, G/K [7, 8, 9] are not the same as the ordinary topological boundaries (except in some special cases). The reason being that the action of the isotropy group K of the origin is not necesarily *transitive* on the ordinary topological boundary. Shilov boundaries are the minimal subspaces of the ordinary topological boundaries which implement the Maldacena-'T Hooft-Susskind *holographic* principle [13] in the sense that the holomorphic data in the interior (bulk) of the domain is fully determined by the holomorphic data on the Shilov boundary. The latter has the property that the maximum modulus of any holomorphic function defined on a domain is attained at the Shilov boundary.

For example, the poly-disc D_4 of 4 complex dimensions is an 8 real-dim Hyperboloid of constant negative scalar curvature that can be identified with the conformal relativistic *curved* phase space associated with the electron (a particle) moving in a $4D$ Anti de Sitter space AdS_4. The poly-disc is a Hermitian symmetric homogeneous coset space associated with the $4D$ conformal group $SO(4,2)$ since $D_4 = SO(4,2)/SO(4) \times SO(2)$. Its Shilov boundary $Shilov$ $(D_4) = Q_4$ has precisely the *same* topology as the $4D$ conformally compactified real Minkowski spacetime $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$. For more details about Shilov boundaries, the conformal group, future tubes and holography we refer to the article by Gibbons [12] and [7, 16].

In order to define the Geometric Probability associated with this process of the electron's emission of a photon at i^- ($t = -\infty$), followed by an absorption at i^+ ($t = +\infty$), we must take into account the important fact that the photon is on-shell $k^2 = 0$ *asymptotically* (at $t = \pm\infty$), but it can move off-shell $k^2 \neq 0$ in the intermediate region which is represented by the *interior* of the conformally compactified real Minkowski spacetime $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$.

Denoting by $\hat{\mu}[Q_4]$ the measure-density (the measure-current) whose *flux* through the future and past celestial spheres S^2 (associated with the future/past light-cones) at timelike infinity i^+, i^-, respectively, is $V(S^2)\hat{\mu}[Q_4]$. The *net* flux through the two celestial spheres S^2 at timelike infinity $i^\pm$ requires an overall factor of 2 giving then the value of $2V(S^2)\hat{\mu}[Q_4]$. The Geometric Probability is defined by the ratio of the measures associated with the celestial spheres S^2 at i^+, i^- timelike infinity, where the photon moves on-shell, relative to the measure of the full *interior* region of $Q_4 = \bar{M}_4 = S^3 \times S^1/Z_2 = S^3 \times RP^1$, where the photon can move off-shell, as it propagates from i^- to i^+:

$$\alpha = \frac{2V(S^2)\,\hat{\mu}[Q_4]}{\mu[Q_4]}. \tag{3}$$

The ratio $(\hat{\mu}[Q_4]/\mu[Q_4]\ )$ can be re-written in terms of the ratios of the normalized measures of

$$\bar{M}_5 = Q_5 = Shilov\ [D_5] = S^4 \times S^1/Z_2 = S^4 \times RP^1, \tag{4}$$

namely, in terms of the normalized measures of the conformally compactified $5D$ Minkowski spacetime. This is achieved as follows [4]

$$\frac{\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{1}{V(S^4)}\,\frac{\mu_N[Q_5]}{\mu[Q_5]}, \tag{5}$$

resulting from the embeddings (inmersions) of $D_4 \to D_5$.

The origin of the factor $V(S^4)$ in the r. h. s of (5), as one goes from the ratio of measures in Q_4 to the ratio of the measures in Q_5, is due to the reduction from the action of the isotropy group of the origin $SO(5) \times SO(2)$ on Q_5, to the action of the isotropy group of the origin $SO(4) \times SO(2)$ on Q_4, furnishing an overall reduction factor of $V[SO(5)/SO(4)] = V(S^4)$. The 5 complex-dimensional poly-disc $D_5 = SO(5,2)/SO(5) \times SO(2)$ is the 10 real-dim Hyperboloid $\mathcal{H}^{10}$ corresponding to the conformal relativistic curved phase space of a particle moving in $5D$ Anti de Sitter Space AdS_5. This picture is also consistent with the Kaluza-Klein compactification procedure of obtaining $4D$ EM from pure Gravity in $5D$. The $\mathcal{H}^{10}$ can be embedded in the 11-dim pseudo-Euclidean $R^{9,2}$ space, with two-time like directions. This is where 11-dim lurks into our construction.

Next we turn to the Hermitian metric on D_5 constructed by Hua [8] which is $SO(5,2)$-invariant and is based on the Bergmann kernel [15] involving a crucial normalization factor of $1/V(D_5)$. However, the standard normalized measure $\mu_N[Q_5]$ based on the Poisson kernel and involving a normalization factor of $1/V(Q_5)$ is *not* invariant under the full group $SO(5,2)$. It is only invariant under the isotropy group of the origin $SO(5) \times SO(2)$. In order to construct an invariant measure on Q_5 under the full group $SO(5,2)$ one requires to introduce a crucial factor related to the Jacobian measure involving the action of the conformal group $SO(5,2)$ on the full bulk domain D_5. As explained by [4] one has:

$$\frac{\mu_N[Q_5]}{\mu[Q_5]} = \frac{1}{V(Q_5)}\,||\mathcal{J}_C^{-1}|| =$$
$$= \frac{1}{V(Q_5)}\,\sqrt{||\mathcal{J}_C^{-1}(\mathcal{J}_C^*)^{-1}||} = \frac{1}{V(Q_5)}\,\sqrt{||\mathcal{J}_R^{-1}||} =$$

$$= \frac{1}{V(Q_5)}\sqrt{\sqrt{|\det g|^{-1}}} = \frac{1}{V(Q_5)}[|\det(g)|]^{-\frac{1}{4}} = \\ = \frac{1}{V(Q_5)}[V(D_5)]^{\frac{1}{4}}, \qquad (6)$$

the z dependence of the complex Jacobian is no longer explicit because the determinant of the $SO(5,2)$ matrices is unity.

This explains very clearly the origins of the factor $[V(D_5)]^{\frac{1}{4}}$ in Wyler's formula for the fine structure constant [1]. This reduction factor of $V(Q_5)$ is in this case given by $V(D_5)^{\frac{1}{4}}$. As we shall see below, the power of $\frac{1}{4}$ is related to the inverse of the $\dim(S^4)=4$. This summarizes, briefly, the role of Bergmann kernel [15] in the construction by Hua [8], and adopted by Wyler [1], of the Hermitian metric of a bounded homogenous (symmetric) complex domain. To sum up, we must perform the reduction from $V(Q_5)\to V(Q_5)/V(D_5)^{\frac{1}{4}}$ in the construction of the normalized measure $\mu_N[Q_5]$. This approach is very different than the interpretation given by Smith [3] and later adopted by Smilga [5].

Hence, the Geometric Probability ratio becomes

$$\frac{\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{1}{V(S^4)}\frac{\mu_N[Q_5]}{\mu[Q_5]} = \\ = \frac{1}{V(S^4)}\frac{1}{V(Q_5)}\,[\,V(D_5)\,]^{\frac{1}{4}} \equiv \frac{1}{\alpha_G}. \qquad (7a)$$

This last ratio, for reasons to be explained below, is nothing but the inverse of the geometric coupling strength of gravity, $1/\alpha_G$. The relationship to the gravitational constant is based on the definition of the coupling appearing in the Einstein-Hilbert Lagrangian $(R/16\pi G)$, as follows

$$(16\pi G)(m^2_{Planck}) \equiv \alpha_{EM}\,\alpha_G = 8\pi \;\Rightarrow \\ G = \frac{1}{16\pi}\frac{8\pi}{m^2_{Planck}} = \frac{1}{2m^2_{Planck}} \;\Rightarrow \qquad (7b) \\ Gm^2_{proton} = \frac{1}{2}\left(\frac{m_{proton}}{m_{Planck}}\right)^2 \sim 5.9{\times}10^{-39},$$

and in natural units $\hbar=c=1$ yields the physical force strength of Gravity at the Planck Energy scale $1.22{\times}10^{19}$ GeV. The Planck mass is obtained by equating the Schwarzschild radius $2Gm_{Planck}$ to the Compton wavelength $1/m_{Planck}$ associated with the mass; where $m_{Planck}\sqrt{2} = 1.22{\times}10^{19}$ GeV and the proton mass is 0.938 GeV. Some authors define the Planck mass by absorbing the factor of $\sqrt{2}$ inside the definition of $m_{Planck}=1.22{\times}10^{19}$ GeV.

The role of the conformal group in Gravity in these expressions (besides the holographic bulk/boundary AdS/CFT duality correspondence [13]) stems from the MacDowell-Mansouri-Chamseddine-West formulation of Gravity based on the conformal group $SO(3,2)$ which has the same number of 10 generators as the $4D$ Poincare group. The $4D$ vielbein e^a_μ which gauges the spacetime translations is identified with the $SO(3,2)$ generator $A^{[a5]}_\mu$, up to a crucial scale factor R, given by the size of the Anti de Sitter space (de Sitter space) throat. It is known that the Poincare group is the Wigner-Inonu group contraction of the de Sitter Group $SO(4,1)$ after taking the throat size $R=\infty$. The spin-connection ω^{ab}_μ that gauges the Lorentz transformations is identified with the $SO(3,2)$ generator $A^{[ab]}_\mu$. In this fashion, the e^a_μ, ω^{ab}_μ are encoded into the $A^{[mn]}_\mu$ $SO(3,2)$ gauge fields, where m,n run over the group indices $1,2,3,4,5$. A word of caution, Gravity is a gauge theory of the full diffeomorphisms group which is infinite-dimensional and which includes the translations. Therefore, strictly speaking gravity is not a gauge theory of the Poincare group. The Ogiovetsky theorem shows that the diffeomorphisms algebra in $4D$ can be generated by an infinity of *nested* commutators involving the $GL(4,R)$ and the $4D$ Conformal Group $SO(4,2)$ generators.

In [17] we have shown why the MacDowell-Mansouri-Chamseddine-West formulation of Gravity, with a cosmological constant and a topological Gauss-Bonnet invariant term, can be obtained from an action inspired from a BF-Chern-Simons-Higgs theory based on the conformal $SO(3,2)$ group. The AdS_4 space is a natural vacuum of the theory. The vacuum energy density was derived to be the geometric-mean between the UV Planck scale and the IR throat size of de Sitter (Anti de Sitter) space. Setting the throat size to coincide with the future horizon scale (of an accelerated de Sitter Universe) given by the Hubble scale (today) R_H, the geometric mean relationship yields the observed value of the vacuum energy density $\rho \sim (L_P)^{-2}(R_H)^{-2} = (L_P)^{-4} \times (L^2_P/R^2_H) \sim 10^{-122}M^4_{Planck}$. Nottale [23] gave a different argument to explain the small value of ρ based on Scale Relativistic arguments. It was also shown in [17] why the Euclideanized AdS_{2n} spaces are $SO(2n-1,2)$ instantons solutions of a non-linear sigma model obeying a double self duality condition.

Therefore, the Geometric Probability α_{EM} for an electron to emit a photon at $t=-\infty$ and to absorb it at $t=+\infty$ agrees with the Wyler's celebrated expression for the fine structure constant

$$\alpha_{EM} = \frac{2V(S^2)\hat{\mu}[Q_4]}{\mu[Q_4]} = (8\pi)\frac{1}{V(S^4)}\frac{1}{V(Q_5)} \times \\ \times [V(D_5)]^{\frac{1}{4}} = \frac{9}{8\pi^4}\left(\frac{\pi^5}{2^4{\times}5!}\right)^{\frac{1}{4}} = \frac{1}{137.03608}, \qquad (8)$$

after one inserts the values of the volumes:

$$V(D_5) = \frac{\pi^5}{2^4{\times}5!}, \quad V(Q_5) = \frac{8\pi^3}{3}, \quad V(S^4) = \frac{8\pi^2}{3}. \qquad (9)$$

In general

$$V(D_n) = \frac{\pi^n}{2^{n-1}n!}, \quad V(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad (10a)$$

$$V(Q_n) = V(S^{n-1} \times RP^1) = V(S^{n-1}) \times V(RP^1) = \\ = \frac{2\pi^{n/2}}{\Gamma(n/2)} \times \pi = \frac{2\pi^{(n+2)/2}}{\Gamma(n/2)}. \tag{10b}$$

Objections were raised to Wyler's original expression by Robertson [2]. One of them was that the hyperboloids (discs) are not compact and whose volumes diverge since the Lobachevsky metric diverges on the boundaries of the poly-discs. Gilmore explained [2] why one requires to use the Euclideanized regularized volumes as Wyler did. Furthermore, in order to resolve the scaling problems of Wyler's expression, Gilmore showed why it is essential to use dimensionless volumes by setting the throat sizes of the Anti de Sitter hyperboloids to $r = 1$, because this is the only choice for r where all elements in the bounded domains are also coset representatives, and therefore, amount to honest group operations. Hence the so-called scaling objections against Wyler raised by Robertson were satisfactory solved by Gilmore [2].

The question as to *why* the value of α_{EM} obtained in Wyler's formula is precisely the value of α_{EM} observed at the *scale* of the Bohr radius a_B, has not been solved, to my knowledge. The Bohr radius is associated with the ground (most stable) state of the Hydrogen atom [3]. The spectrum generating group of the Hydrogen atom is well known to be the conformal group $SO(4,2)$ due to the fact that there are two conserved vectors, the angular momentum and the Runge-Lentz vector. After quantization, one has two commuting $SU(2)$ copies $SO(4) = SU(2) \times SU(2)$. Thus, it makes physical sense why the Bohr-scale should appear in this construction. Bars [14] has studied the many physical applications and relationships of many seemingly distinct models of particles, strings, branes and twistors, based on the (super) conformal groups in diverse dimensions. In particular, the relevance of two-time physics in the formulation of M, F, S theory has been advanced by Bars for some time. The Bohr radius corresponds to an energy of $137.036 \times 2 \times 13.6$ eV $\sim 3.72 \times 10^3$ eV. It is well known that the Rydberg scale, the Bohr radius, the Compton wavelength of electron, and the classical electron radius are all related to each other by a successive scaling in products of α_{EM}.

2 The fiber bundle interpretation of the Wyler formula

Having found Wyler's expression from Geometric Probability, we shall present a Fiber Bundle interpretation of the Wyler expression by starting with a Fiber bundle E over the base curved-space $D_5 = SO(5,2)/SO(5) \times SO(2)$. The subgroup $H = SO(5)$ of the isotropy group $K = SO(5) \times SO(2)$ acts on the Fibers $F = S^4$ (the internal symmetry space). Locally, and only locally, the Fiber bundle E is the product $D_5 \times S^4$. However, this is *not* true globally. On the Shilov boundary Q_5, the restriction of the Fiber bundle E to the Shilov boundary Q_5 is written by $E|_{Q_5}$ and *locally* is the product of $Q_5 \times S^4$, but this is *not* true globally. For this reason one has that the volume $V(E|_{Q_5}) \neq V(Q_5 \times S^4) = V(Q_5) \times V(S^4)$. But instead, $V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$.

This is the reasoning behind the construction of the quantity $\hat{\mu}[Q_4]/\mu[Q_4]$ that has the units of a density. Its inverse $\mu[Q_4]/\hat{\mu}[Q_4]$ is the volume associated with the restriction of the Fiber Bundle E to the Shilov boundary Q_5: $V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$.

The reason why one embeds $D_4 \to D_5$ and $Q_4 \to Q_5$ is because the space $Q_4 = S^3 \times RP^1$ is *not* large enough to implement the action of the $SO(5)$ group, the compact version of the Anti de Sitter Group $SO(3,2)$ that is required in the MacDowell-Mansouri-Chamseddine-West formulation of Gravity. However, the space $Q_5 = S^4 \times RP^1$ is large enough to implement the action of $SO(5)$ via the internal symmetry space $S^4 = SO(5)/SO(4)$. This justifies the embedding procedure of $D_4 \to D_5$. This Fiber Bundle interpretation is not very different from Smith's interpretation [3]. Following the Fiber Bundle interpretation of the volume $V(E|_{Q_5}) = V(S^4) \times (V(Q_5)/V(D_5)^{1/4})$, we will now prove why

$$2\,V(S^2) = \frac{\mu(S^1)}{\hat{\mu}(S^1)} = 8\pi\,. \tag{11}$$

The space S^1 is associated with the $U(1)$ group action and naturally encodes the $U(1)$ gauge invariance linked to Electromagnetism (EM). The result of eq-(11) is what will allow us to *define* α_{EM} as the *ratio of the ratios* of suitable measures in S^1 and Q_4, respectively,

$$\alpha_{EM} = \frac{2V(S^2)\;\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{(\mu(S^1)/\hat{\mu}(S^1))}{(\mu[Q_4]/\hat{\mu}[Q_4])}\,. \tag{12}$$

We may notice that $S^1 \equiv Q_1$ (very special case) since the circle is both the Shilov and ordinary topological boundary of the disc D_1. However, $Q_2 \equiv S^1 \times S^1/Z_2 = S^1 \times RP^1$. Once again, we will write the ratio of the measures in $Q^1 = S^1$ in terms of the ratio of the normalized measures in Q^2 via the reduction from $S^1 \times S^1/Z_2$ to S^1. This requires the embedding (inmersion) of $D_1 \to D_2$ in order to construct the measures on D_1, Q_1 as induced from the measures in D_2, Q_2 resulting from the embedding (inmersion):

$$\frac{\hat{\mu}(S^1)}{\mu(S^1)} = \frac{\hat{\mu}(Q_1)}{\mu(Q_1)} = \frac{1}{V(S^1/Z_2)}\frac{\mu_N[Q_2]}{\mu[Q_2]} = \\ = \frac{1}{V(S^1/Z_2)}\;\frac{1}{(V(Q_2)/V(D_2)}\,. \tag{13}$$

Notice that $\hat{\mu}(S^1)$ as explained before is a measure-density on S^1. Likewise, $\hat{\mu}(Q_4)$ was a measure-density on Q_4. We should not confuse these measure-densities with the normalized measures in one-higher dimension.

By inserting the values of the measures and using

$$V(S^1/Z_2) = V(RP^1) = \pi\,, \quad V(D_2) = \frac{\pi^2}{2\times 2!}\,, \qquad V(Q^2) = \frac{2\pi^2}{\Gamma(1)} = 2\pi^2, \tag{14}$$

it yields then

$$\frac{\mu(S^1)}{\hat{\mu}(S^1)} = (2\pi^2)\,(\pi)\,\frac{1}{(\pi^2/2\times 2!)} = 8\pi = 2\,V(S^2) \tag{15}$$

as claimed. Therefore, $2V(S^2) = \mu(S^1)/\hat{\mu}(S^1) = 8\pi$ is the crucial factor appearing in Wyler's formula which admits a natural Geometric probability explanation which is very different from the different interpretations provided in [3, 4, 5].

The Fiber Bundle interpretation associated with the $U(1) \sim SO(2)$ group is the following. The Fiber bundle E is defined over the curved space $D_2 = SO(2,2)/SO(2) \times SO(2)$. The subgroup $H = SO(2) \sim U(1)$ of the isotropy group $K = SO(2) \times SO(2)$ acts on the fibers identified with the symmetry space S^1 (where the $U(1)$ group acts). The Fiber bundle E locally can be written as $D_2 \times S^1$ but not globally. The restriction of the Fiber bundle E to the Shilov boundary $Q_2 = S^1 \times S^1/Z_2 = S^1 \times RP^1$ is $E|_{Q_2}$ and locally can be written as $Q_2 \times S^1$, but *not* globally. This is why the volume $V(E|_{Q_2}) \neq V(Q_2) \times V(S_1)$ but instead it equals $(V(Q_2)/V(D_2)) \times V(S^1/Z_2) = 2V(S^2) = 8\pi$.

Concluding, the Geometric Probability that an electron emits a photon at $t = -\infty$ and absorbs it at $t = +\infty$ is given by the *ratio* of the *ratios* of measures, and it agrees with Wheeler's ideas that one must normalize the couplings with respect to the geometric coupling strength of Gravity:

$$\alpha_{EM} = \frac{2V(S^2)\hat{\mu}[Q_4]}{\mu[Q_4]} = \frac{(\mu(S^1)/\hat{\mu}(S^1))}{(\mu[Q_4]/\hat{\mu}[Q_4])} = = (8\pi)\frac{1}{V(S^4)}\frac{1}{V(Q_5)}[V(D_5)]^{\frac{1}{4}} = \frac{1}{137.03608}\,. \tag{16}$$

The second important conclusion that can be *derived* from Geometric Probability theory is the general numerical values of the exponents s_n appearing in the factors $V(D_n)^{s_n}$. The normalization factor $V(Q_5)/V(D_5)^{1/4}$ in the construction of the ratio of measures $\mu_N[Q_5]/\mu[Q_5]$ involves in this case powers of the type $V(D_5)^{1/4}$. The power of $\frac{1}{4}$ is related to the inverse of the $\dim(S^4) = 4$ (the internal symmetry space $SO(5)/SO(4)$). From eq-(13) we learnt that the reduction factor of $V(Q^2)/V(D_2)$ was $V(D_2)$; i. e. the exponent is unity. The power of *unity* is related to the inverse of the $\dim(S^1/Z_2) = 1$. Thus, the arguments based on Geometric Probability leads to normalized measures by factors of $V(Q_n)/V(D_n)^{s_n}$ and whose exponents s_n are given by the *inverse* of the dimensions of the internal symmetry spaces $s_n = (\dim(S^{n-1}))^{-1}$. There is a different interpretation of these factors $V(D_n)^{s_n}$ given by Smith [3]. In general, for other homogeneous complex domains, this power is given by the inverse of the dimension of the internal symmetry space.

3 The weak and strong coupling constants from Geometric Probability

We turn now to the derivation of the other coupling constants. The Fiber Bundle picture of the previous section is essential in our construction. The Weak and the Strong geometric coupling constant strength, defined as the probability for a particle to emit and later absorb a $SU(2), SU(3)$ gauge boson, respectively, can both be obtained by using the main formula derived from Geometric Probability after one identifies the suitable homogeneous domains and their Shilov boundaries to work with. We will show why the weak and strong couplings are given by

$$\alpha_{Weak} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(\mu[Q_4]/\hat{\mu}[Q_4])} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{\alpha_G} = = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(8\pi/\alpha_{EM})}\,, \tag{17}$$

and

$$\alpha_{Color} = \frac{(\mu[S^4]/\hat{\mu}[S^4])}{(\mu[Q_4]/\hat{\mu}[Q_4]} = \frac{(\mu[S^4]/\hat{\mu}[S^4])}{\alpha_G} = = \frac{(\mu[S^4]/\hat{\mu}[S^4])}{(8\pi/\alpha_{EM})}\,. \tag{18}$$

At this point we must emphasize that we define α_{weak}, α_{color} as g_w^2, g_c^2 instead of the conventional $(g_w^2/4\pi)$, $(g_c^2/4\pi)$ definitions used in the Renormalization Group program. The Shilov boundary of (D_2) is $Q_2 = S^1 \times RP^1$ but is not large enough to accommodate the action of the isospin group $SU(2)$. One needs a Fiber Bundle over $D_3 = = SO(3,2)/SO(3) \times SO(2)$ whose subgroup $H = SO(3)$ of the isotropy group $K = SO(3) \times SO(2)$ acts on the internal symmetry space S^2 (the fibers). Since the coset space $SU(2)/U(1)$ is a double-cover of the S^2 as one goes from the $SO(3)$ action to the $SU(2)$ action one must take into account an extra factor of 2. This is the reason why one jumps to one-dimension higher from Q_2 to $Q_3 = S^2 \times RP^1$, because the coset $SU(2)/U(1)$ is a double-cover of the sphere $S^2 = SO(3)/SO(2)$ and can accommodate the action of the $SU(2)$ group.

By following the same procedure as above, i. e. by rewriting the ratio of the measures $(\hat{\mu}[Q_2]/\mu[Q_2])$ in terms of the ratio of the measures $(\mu_N[Q_3]/\mu[Q_3])$ via the embeddings of $D_2 \to D_3$, one has

$$(\hat{\mu}[Q_2]/\mu[Q_2]) = \frac{1}{V(SU(2)/U(1))}\,\frac{\mu_N[Q_3]}{\mu[Q_3]}. \tag{19}$$

Notice that because $SU(2)$ is a $2-1$ covering of the $SO(3)$, this implies that the measure

$$V(SU(2)/U(1)) = 2V(SO(3)/U(1)) = 2V(S^2) = 8\pi\,. \tag{20}$$

As indicated above, because the dimension of the internal symmetry space is $\dim(S^2) = 2$, the construction of the normalized measure $\mu_N[Q_3]$ will require a reduction of $V(Q_3)$ by a factor of $V(D_3)$ raised to the power of $(\dim(S^2))^{-1} = \frac{1}{2}$:

$$\frac{\mu_N[Q_3]}{\mu[Q_3]} = \frac{1}{V(Q_3)/V(D_3)^{1/2}} = \frac{1}{V(Q_3)}\, V(D_3)^{1/2}. \quad (21)$$

Therefore, the ratio of the measures is

$$\frac{\hat{\mu}[Q_2]}{\mu[Q_2]} = \frac{1}{2V(S^2)}\, \frac{1}{V(Q_3)}\, V(D_3)^{1/2}, \quad (22)$$

whose Fiber Bundle interpretation is that the volume of the Fiber Bundle over D_3, but restricted to the Shilov boundary Q_3, and whose structure group is $SU(2)$ (the double cover of $SO(3)$), is $V(E|_{Q_3}) = 2V(S^2) \times (V(Q_3)/V(D_3)^{1/2})$. Thus, that the Geometric probability expression is

$$\alpha_{Weak} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(\mu[Q_4]/\hat{\mu}[Q_4])} = \frac{(\mu[Q_2]/\hat{\mu}[Q_2])}{(8\pi/\alpha_{EM})} = = 2V(S^2)V(Q_3)\frac{1}{V(D_3)^{1/2}}\frac{\alpha_{EM}}{8\pi} = 0.2536, \quad (23)$$

that corresponds to the weak geometric coupling constant α_W at an energy of the order of

$$E = M = 146 \text{ GeV} \sim \sqrt{M^2_{W_+} + M^2_{W_-} + M^2_Z}\,, \quad (24)$$

after we have inserted the expressions

$$V(S^2) = 4\pi\,, \quad V(Q_3) = 4\pi^2\,, \quad V(D_3) = \frac{\pi^3}{24}\,, \quad (25a)$$

into the formula (23). The relationship to the Fermi coupling G_{Fermi} goes as follows (after indentifying the energy scale $E = M = 146$ GeV):

$$G_F \equiv \frac{\alpha_W}{M^2} \Rightarrow G_F\, m^2_{proton} = \left(\frac{\alpha_W}{M^2}\right) m^2_{proton} = = 0.2536 \times \left(\frac{m_{proton}}{146 \text{ GeV}}\right)^2 \sim 1.04{\times}10^{-5} \quad (25b)$$

in very good agreement with experimental observations.

Once more, it is unknown why the value of α_{Weak} obtained from Geometric Probability corresponds to the energy scale related to the W_+, W_-, Z_0 boson mass, after spontaneous symmetry breaking.

Finally, we shall derive the value of α_{Color} from eq-(18). Since S^4 is not large enough to accommodate the action of the color group $SU(3)$ one needs to work with one-dimension higher S^5 , that can be interpreted as the boundary of the $6D$ Ball $B_6 = SU(4)/U(3) = SU(4)/SU(3) \times U(1)$. Thus, the $SU(3)$ group is part of the isotropy group $K = SU(3) \times U(1)$ that defines the coset space B_6. In this special case the Shilov and ordinary topological boundaries of B_6 coincide with S^5 [3]. Hence, following the same procedures as above, the ratio of the measures in S^4 (boundary of B_5) can be re-written in terms of the ratio of the measures in S^5 (boundary of B_6) via the embeddings of $B_5 \to B_6$ as follows:

$$\frac{\hat{\mu}[S^4]}{\mu[S^4]} = \frac{1}{V(S^4)}\frac{\mu_N[S^5]}{\mu[S^5]} = \frac{1}{V(S^4)}\frac{1}{V(S^5)/V(B_6)^{1/4}} = = \frac{1}{V(S^4)}\frac{1}{V(S^5)}\, V(B_6)^{1/4}, \quad (26)$$

since the exponent of the reduction factor $V(B_6)^{1/4}$ is given by $(\dim(S^4))^{-1} = \frac{1}{4}$. Notice, again, that $\hat{\mu}[S^4]$ is the measure-density in S^4 and must not be confused with the normalized measures.

Therefore, one arrives at

$$\alpha_{Color} = V(S^4)\, V(S^5)\, \frac{1}{V(B_6)^{1/4}}\, \frac{\alpha_{EM}}{8\pi} = 0.6286, \quad (27)$$

that corresponds to the strong coupling constant at an energy related to the pion masses [3]:

$$E = 241 \text{ MeV} \sim \sqrt{m^2_{\pi^+} + m^2_{\pi^-} + m^2_{\pi^0}} \quad (28)$$

and where we have used the expressions:

$$V(S^4) = \frac{8\pi^2}{3}\,, \quad V(S^5) = 4\pi^3\,, \quad V(B_6) = \frac{\pi^3}{6}\,. \quad (29)$$

The pions are the known lightest quark/antiquark pairs that feel the strong interaction [3]. For a detailed analysis of volumes of compact manifolds (coset spaces) see [24].

Once again, it is unknown why the value of α_{Color} obtained from Geometric Probability (28) corresponds to the energy scale related to the masses of the three pions [3]. Masses of the fundamental particles were derived in [3] based on the definitions that mass is the probability amplitude for a particle to change direction.

To conclude, by defining the geometric coupling constants $\alpha = g^2$ as the Geometric Probability to emit (and later absorb) a gauge boson, all the three geometric coupling constants, $\alpha_{EM}; \alpha_{Weak}; \alpha_{Color}$ are given by the *ratios of the ratios* of the measures of the Shilov *boundaries* $Q_2 = S^1 \times RP^1$; $Q_3 = S^2 \times RP^1$; S^5, respectively, with respect to the ratios of the measures $\mu[Q_5]/\mu_N[Q_5]$ associated with the $5D$ conformally compactified real Minkowski spacetime $\bar{M}_5$ that has the same topology as the Shilov boundary Q_5 of the 5 complex-dimensional poly-disc D_5. The latter corresponds to the conformal relativistic *curved* 10 real-dimensional phase space $\mathcal{H}^{10}$ associated with a particle moving in the $5D$ Anti de Sitter space AdS_5. The ratios of particle masses, like the proton to electron mass ratio $m_p/m_e \sim 6\pi^5$ has also been calculated using the volumes of homogeneous bounded domains [3, 4].

It is not known whether this procedure would work for Grand Unified Theories based on the groups

$$SU(5), SO(10), E_6, E_7, E_8\,. \qquad (30)$$

Beck [6] has obtained all the Standard Model parameters by studying the numerical minima (and zeros) of certain potentials associated with the Kaneko coupled two-dim lattices based on Stochastic Quantization methods. The results above and by Smith [3] are analytical rather than being numerical [6] and it is not clear if there is any relationship between these two approaches. Noyes has proposed an iterated numerical hierarchy based on Mersenne primes $M_p = 2^p - 1$ for certain values of p = primes [18] and obtained many numerical values for the physical parameters. Pitkanen has developed methods to calculate the physical masses recurring to a p-adic hierarchy of scales based on Mersenne primes [19].

An important connection between anomaly cancellation in string theory and perfect even numbers was found in [22]. These are numbers which can be written in terms of sums of its divisors, including unity, like $6 = 1 + 2 + 3$, and are of the form $P(p) = \frac{1}{2}\, 2^p(2^p - 1)$ if, and only if, $2^p - 1$ is a Mersenne prime. Not all values of p = prime yields primes. The number $2^{11} - 1$ is not a Mersenne prime, for example. The number of generators of the anomaly free groups $SO(32)$, $E_8 \times E_8$ of the 10-dim superstring is 496 which is an even perfect number. Another important group related to the unique tadpole-free bosonic string theory is the $SO(2^{13}) = SO(8192)$ group related to the bosonic string compactified on the $E_8 \times SO(16)$ lattice. The number of generators of $SO(8192)$ is an even perfect number since $2^{13} - 1$ is a Mersenne prime. For an introduction to p-adic numbers in Physics and String theory see [20]. A lot more work needs to be done to be able to answer the question: Is all this just a mere numerical coincidence or is it design?

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On the Generalisation of Kepler's 3rd Law for the Vacuum Field of the Point-Mass

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I derive herein a general form of Kepler's 3rd Law for the general solution to Einstein's vacuum field. I also obtain stable orbits for photons in all the configurations of the point-mass. Contrary to the accepted theory, Kepler's 3rd Law is modified by General Relativity and leads to a finite angular velocity as the proper radius of the orbit goes down to zero, without the formation of a black hole. Finally, I generalise the expression for the potential function of the general solution for the point-mass in the weak field.

1 Introduction

In previous papers [1, 2] I derived the general solution for Einstein's vacuum field and showed that black holes do not exist in Einstein's universe. In those papers I also obtained expressions for Kepler's 3rd Law for the simple (i. e. non-rotating) point-mass and the simple point-charge. In this paper I obtain expressions for Kepler's 3rd Law for the rotating point-mass and the rotating point-charge. Owing to the rotation of the source of the field, Kepler's 3rd Law for the polar orbit is not the same as that for the equatorial orbit, so that stable photon orbits are also different in the polar and equatorial orbits, showing that in the rotating configurations spacetime is no longer isotropic.

The expressions I obtain readily reduce to those I have previously derived for the non-rotating configurations.

2 Definitions

I have already shown [3] that the most general static metric for the point-mass is,

$$ds^2 = A(D)dt^2 - B(D)dD^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

$$D = |r - r_0|,$$

$$A, B, C > 0,$$

where r_0 is an arbitrary real number. With respect to this metric I identify the coordinate radius, the r-parameter, the radius of curvature, and the proper radius thus:

1. *The coordinate radius is* $D = |r - r_0|$.
1. *The* r*-parameter is the variable* r.
2. *The radius of curvature is* $R = \sqrt{C(D)}$.
3. *The proper radius is* $R_p = \int \sqrt{B(D)}\, dD$.

3 The equatorial orbit

The general Kerr-Newman form in Boyer-Lindquist coordinates is,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ - \frac{\sin^2\theta}{\rho^2}\left[\left(R^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dR^2 - \rho^2 d\theta^2.$$

This can be written as,

$$ds^2 = \left(\frac{\Delta - a^2\sin^2\theta}{\xi}\right)dt^2 - \frac{\xi}{\Delta}dR^2 - \\ - \xi d\theta^2 + \left[\frac{a^2\Delta\sin^4\theta - \left(R^2 + a^2\right)^2\sin^2\theta}{\xi}\right]d\varphi^2 - \\ - \left[\frac{2a\Delta\sin^2\theta - 2a\left(R^2 + a^2\right)\sin^2\theta}{\xi}\right]dt\,d\varphi\,, \qquad (1)$$

where I have previously shown [2, 3] in the case of the rotating point-charge,

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re,\ r \in \Re,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2},$$

$$a^2 + q^2 < m^2, \quad n \in \Re^+, \quad \xi = \rho^2 = R^2 + a^2\cos^2\theta,$$

$$a = \frac{L}{m}, \quad \Delta = R^2 - \alpha R + a^2 + q^2,$$

$$0 < |r - r_0| < \infty,$$

where L is the angular momentum, and n and r_0 are arbitrary.

I have also shown previously that Kepler's 3rd Law for the simple point-mass is,

$$\omega^2 = \frac{\alpha}{2R^3}, \qquad (2)$$

where

$$\lim_{r \to r_0^{\pm}} \sqrt{C_n(r)} = R_0 = \alpha = 2m \ \forall\ r_0,$$

is a scalar invariant; and for the simple point-charge is,

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}, \qquad (3)$$

where, $\forall\, r_0$,

$$\lim_{r\to r_0^\pm}\sqrt{C_n(r)} = R_0 = \beta = m + \sqrt{m^2-q^2},\ q^2 < m^2\,,$$

is a scalar invariant.

In the case of the equatorial orbit, $\theta = \frac{\pi}{2}$ and $\dot\theta = 0$, so (1) becomes,

$$\begin{gathered} ds^2 = \left(\frac{\Delta - a^2}{\xi}\right) dt^2 - \frac{\xi}{\Delta} dR^2 + \\ + \left[\frac{a^2\Delta - \left(R^2+a^2\right)^2}{\xi}\right] d\varphi^2 - \\ - \left[\frac{2a\Delta - 2a\left(R^2+a^2\right)}{\xi}\right] dt d\varphi\,. \end{gathered} \tag{4}$$

$$R^2 = C_n(r) = \left(\left|r-r_0\right|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2-q^2}\,,\ q^2 < m^2\,,$$

$$\xi = R^2\,,\ \Delta = R^2 - \alpha R + a^2 + q^2\,,$$

$$0 < |r-r_0| < \infty\,.$$

Consider the associated Lagrangian, where the dot indicates $\partial/\partial\tau$,

$$\begin{gathered} L = \frac{1}{2}\left[\frac{\Delta - a^2}{\xi}\dot t^2 - \frac{\xi}{\Delta}\dot R^2\right] + \\ + \frac{1}{2}\left[\frac{a^2\Delta - \left(R^2+a^2\right)^2}{\xi}\right]\dot\varphi^2 - \\ - \frac{1}{2}\left[\frac{2a\Delta - 2a\left(R^2+a^2\right)}{\xi}\right]\dot t\,\dot\varphi\,. \end{gathered} \tag{5}$$

Then,

$$\begin{gathered} \frac{\partial L}{\partial R} - \frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial\dot R}\right) = 0 \Rightarrow \frac{\xi\Delta' - \xi'\left(\Delta - a^2\right)}{2\xi^2}\dot t^2 + \\ + \frac{\xi\left[a^2\Delta' - 4R\left(R^2+a^2\right)\right]}{2\xi^2}\dot\varphi^2 - \\ - \frac{\xi'\left[a^2\Delta - \left(R^2+a^2\right)^2\right]}{2\xi^2}\dot\varphi^2 - \\ - \frac{\xi\left(2a\Delta' - 4aR\right) - \xi'\left[2a\Delta - 2a\left(R^2+a^2\right)\right]}{2\xi^2}\dot t\dot\varphi + \\ + \frac{\Delta\xi' - \xi\Delta'}{2\Delta^2}\dot R^2 + \frac{\xi}{\Delta}\ddot R = 0\,. \end{gathered} \tag{6}$$

Taking $R = \text{const.}$ reduces (6) to,

$$\begin{gathered} \Big\{\xi\left[a^2\Delta' - 4R\left(R^2+a^2\right)\right] - \\ - \xi'\left[a^2\Delta - \left(R^2+a^2\right)^2\right]\Big\}\omega^2 - \\ - \left\{\xi\left(2a\Delta' - 4aR\right) - \xi'\left[2a\Delta - 2a\left(R^2+a^2\right)\right]\right\}\omega + \\ + \xi\Delta' - \xi'\left(\Delta - a^2\right) = 0\,, \end{gathered} \tag{7}$$

where $\omega = \frac{\dot\varphi}{\dot t}$. The solutions for ω are,

$$\omega = \frac{a\alpha R - 2aq^2 \pm R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4}\,.$$

In order for this to reduce to the non-rotating configurations, the plus sign must be taken so,

$$\omega = \frac{a\alpha R - 2aq^2 + R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4}\,, \tag{8}$$

$$R^2 = C_n(r) = \left(\left|r-r_0\right|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2-q^2}\,,\ q^2 < m^2\,,$$

$$\alpha = 2m\,,$$

$$0 < |r-r_0| < \infty\,.$$

Equation (8) is Kepler's 3rd Law for the equatorial plane of the rotating point-charge. I remark that the radius of curvature in the equatorial orbit is precisely that for the simple point-charge. The expression for Kepler's 3rd Law for the equatorial plane of the rotating point-mass is obtained from (8) by setting $q = 0$,

$$\omega = \frac{a\alpha R + R^2\sqrt{2\alpha R}}{a^2\alpha R - 2R^4}\,,$$

$$R^2 = C_n(r) = \left(\left|r-r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}\,,$$

$$\alpha = 2m\,,$$

$$0 < |r-r_0| < \infty\,,$$

in which case the radius of curvature in the equatorial orbit is precisely that for the simple point-mass.

Taking the near-field limit on (8) gives,

$$\lim_{r\to r_0^\pm}\omega = \frac{a\alpha\beta - 2aq^2 + \beta^2\sqrt{2\alpha\beta - 4q^2}}{a^2\alpha\beta - 2a^2q^2 - 2\beta^4}\,, \tag{9}$$

which is a scalar invariant.

When $a = 0$ and $q \neq 0$, equation (8) reduces to,

$$\omega^2 = \frac{\alpha}{2R^3} - \frac{q^2}{R^4}\,,$$

which recovers Kepler's 3rd Law (3) for the simple point-charge. If $a = q = 0$, equation (8) reduces to,

$$\omega^2 = \frac{\alpha}{2R^3}\,,$$

$$\beta = \alpha = 2m\,,$$

which recovers Kepler's 3rd Law (2) for the simple point-mass.

When $a = 0$ and $q \neq 0$, (9) reduces in the near-field limit, to

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{\alpha}{2\beta^3} - \frac{q^2}{\beta^4}\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,,$$

the scalar invariant of Kepler's 3rd Law for the simple point-charge; and when $a = q = 0$, (9) reduces to the near-field limit,

$$\lim_{r \to r_0^{\pm}} \omega^2 = \frac{1}{2\alpha^2}\,,$$

$$\alpha = 2m\,,$$

the scalar invariant for Kepler's 3rd Law for the simple point-mass, as originally obtained by Karl Schwarzschild [4] for his particular solution.

4 Photons in equatorial orbit

Setting $\theta = \frac{\pi}{2}$ in (1) and setting (1) equal to zero gives,

$$\left[a^2\Delta - \left(R^2 + a^2\right)^2\right]\omega^2 - \left[2a\Delta - 2a\left(R^2 + a^2\right)\right]\omega + \left(\Delta - a^2\right) = 0\,, \tag{10}$$

from which it follows,

$$\omega = \frac{\dot\varphi}{\dot t} = \frac{a\left(q^2 - \alpha R\right) + R^2\sqrt{R^2 - \alpha R + a^2 + q^2}}{a^2q^2 - \alpha a^2 R - a^2R^2 - R^4}\,. \tag{11}$$

Equating (8) to (11) gives,

$$\frac{a\alpha R - 2aq^2 + R^2\sqrt{2\alpha R - 4q^2}}{a^2\alpha R - 2a^2q^2 - 2R^4} = \frac{a\left(q^2 - \alpha R\right) + R^2\sqrt{R^2 - \alpha R + a^2 + q^2}}{a^2q^2 - \alpha a^2 R - a^2R^2 - R^4}\,, \tag{12}$$

$$\alpha = 2m\,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2},\ q^2 < m^2\,,$$

$$r_0 \in \Re,\ n \in \Re^+\,,$$

for the radius of curvature $R_{ph-e} = R = \sqrt{C_n(r_{ph-e})}$ of the equatorial orbit of a photon for the rotating point-charge. When $a = 0$ equation (12) reduces to,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = \frac{3\alpha + \sqrt{9\alpha^2 - 32q^2}}{4}\,,$$

recovering the stable radius of curvature for the photon orbit about the simple point-charge [2]. When $a = q = 0$, equation (12) reduces to,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = \frac{3\alpha}{2} = 3m\,, \tag{13}$$

which recovers the stable radius of curvature for the photon around the simple point-mass [1].

When $n = 1$ and $r_0 = \alpha$, equation (13) gives,

$$R_{ph-e} = \sqrt{C_n(r_{ph-e})} = r_{ph-e} = 3m\,,$$

This radius is taken incorrectly by the orthodox relativists as a measurable proper radius in the gravitational field of the simple point-mass. The actual proper radius associated with (13) is,

$$R_p = \frac{\alpha\sqrt{3}}{2} + \alpha\ln\left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right),$$

which is a scalar invariant for the photon orbit about the point-mass.

The expression for the radius of curvature of the stable photon equatorial orbit for the rotating point-mass is obtained from (12) by setting $q = 0$, thus

$$\frac{a\alpha R + R^2\sqrt{2\alpha R}}{a^2\alpha R - 2R^4} = \frac{a\alpha R - R^2\sqrt{R^2 - \alpha R + a^2}}{\alpha a^2 R + a^2R^2 + R^4}\,,$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}\,,$$

$$\alpha = 2m\,,$$

$$r_0 \in \Re,\ n \in \Re^+\,.$$

5 The polar orbit

According to (1), if $R = \sqrt{C_n(r)}$ is a function of t,

$$R = R(t, \theta) = = \sqrt{C_n(r(t))} = \left(\left|r(t) - r_0\right|^n + \beta^n\right)^{\frac{1}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta}\,,$$

so if $\dot r = 0,\ \dot R = 0$.

In the polar orbit there is no loss of generality in taking $\varphi = \text{const.},\ \dot\varphi = 0$. Then (1) becomes,

$$ds^2 = \frac{\Delta - a^2\sin^2\theta}{\xi}dt^2 - \frac{\xi}{\Delta}dR^2 - \xi d\theta^2\,, \tag{14}$$

$$R^2 = C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re,\ r \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,$$

$$a^2 + q^2 < m^2\,, \quad n \in \Re^+, \quad \xi = \rho^2 = R^2 + a^2\cos^2\theta\,,$$

$$a = \frac{L}{m}\,, \quad \Delta = R^2 - \alpha R + a^2 + q^2\,,$$

$$0 < |r - r_0| < \infty\,.$$

Consider the associated Lagrangian,

$$L = \frac{1}{2}\left[\frac{\Delta - a^2\sin^2\theta}{\xi}\dot t^2 - \frac{\xi}{\Delta}\dot R^2 - \xi\dot\theta^2\right].$$

Then,

$$\frac{\partial L}{\partial R}-\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial \dot{R}}\right)=\frac{1}{2}\left[\frac{\xi\Delta'-\xi'(\Delta-a^2\sin^2\theta)}{\xi^2}\dot{t}^2\right]+ -\frac{1}{2}\left[\frac{(\Delta\xi'-\xi\Delta')}{\Delta^2}\dot{R}^2+\xi'\dot{\theta}^2\right]+\frac{\xi}{\Delta}\ddot{R}=0\,. \tag{15}$$

If $\dot{R}=0$, then (15) yields,

$$\omega^2=\frac{\dot{\theta}^2}{\dot{t}^2}=\frac{\xi\Delta'-\xi'(\Delta-a^2\sin^2\theta)}{\xi'\xi^2}= =\frac{\alpha R^2-\alpha a^2\cos^2\theta-2q^2R}{2R\,(R^2+a^2\cos^2\theta)^2}= =\frac{\alpha C_n-\alpha a^2\cos^2\theta-2q^2\sqrt{C_n}}{2\sqrt{C_n}\,(C_n+a^2\cos^2\theta)^2}\,, \tag{16}$$

$$\beta=m+\sqrt{m^2-a^2\cos^2\theta-q^2}\,,\ a^2+q^2<m^2\,,$$
$$n\in\Re^+\ r_0\in\Re\,,$$
$$0<|r-r_0|<\infty\,.$$

Equation (16) is Kepler's 3rd Law for the polar orbit of the rotating point-charge. I remark that the angular velocity depends upon azimuth.

Let $a=0,\ q\neq 0$, then (16) reduces to,

$$\omega^2=\frac{\alpha}{2R^3}-\frac{q^2}{R^4}\,,$$

$$R^2=C_n(r)=\left(\left|r-r_0\right|^n+\beta^n\right)^{\frac{2}{n}},\ \beta=m+\sqrt{m^2-q^2},$$
$$q^2<m^2,\ r_0\in\Re,\ n\in\Re^+\,,$$
$$0<|r-r_0|<\infty,$$

which recovers Kepler's 3rd Law (3) for the simple point-charge. Setting $a=q=0$ reduces (16) to,

$$\omega^2=\frac{\alpha}{2R^3}\,,$$

$$R^2=C_n(r)=\left(\left|r-r_0\right|^n+\alpha^n\right)^{\frac{2}{n}}\,,$$
$$n\in\Re^+,\ r(0)\in\Re\,,$$
$$0<|r-r_0|<\infty,$$

which recovers Kepler's 3rd Law (2) for the simple point-mass.

Taking the near-field limit on (16),

$$\lim_{r\to r_0^{\pm}}\omega^2=\frac{\alpha\beta^2-\alpha a^2\cos^2\theta-2q^2\beta}{2\beta\,(\beta^2+a^2\cos^2\theta)^2}\,, \tag{17}$$

which is a scalar invariant, subject to azimuth, for the polar orbit of the rotating point-charge.

When $q=0,\ a\neq 0$, equation (16) reduces to,

$$\omega^2=\frac{\alpha R^2-\alpha a^2\cos^2\theta}{2R\,(R^2+a^2\cos^2\theta)^2}= =\frac{\alpha C_n-\alpha a^2\cos^2\theta}{2\sqrt{C_n}\,(C_n+a^2\cos^2\theta)^2}\,, \tag{18}$$

$$\beta=m+\sqrt{m^2-a^2\cos^2\theta}\,,\ a^2<m^2\,,$$
$$n\in\Re^+\ r_0\in\Re\,,$$
$$0<|r-r_0|<\infty\,.$$

This is Kepler's 3rd Law for the polar orbit of the rotating point-mass.

Taking the near-field limit on (18),

$$\lim_{r\to r_0^{\pm}}\omega^2=\frac{\alpha\beta^2-\alpha a^2\cos^2\theta}{2\beta\,(\beta^2+a^2\cos^2\theta)^2}\,, \tag{19}$$

which is a scalar invariant, subject to azimuth, for the polar orbit of the rotating point-mass.

Thus, ω varies with azimuth as does $R=\sqrt{C_n(r)}$. At the poles of the rotating point-charge,

$$R^2=C_n(r)=\left(|r-r_0|^n+\beta^n\right)^{\frac{2}{n}}\,,\quad \beta=m+\sqrt{m^2-a^2-q^2}\,,\quad \omega^2=\frac{\alpha R^2-\alpha a^2-2q^2R}{2R\,(R^2+a^2)^2}\,, \tag{20}$$

and at the equator,

$$R^2=C_n(r)=\left(|r-r_0|^n+\beta^n\right)^{\frac{2}{n}}\,,\quad \beta=m+\sqrt{m^2-q^2}\,,\quad \omega^2=\frac{\alpha}{2R^3}-\frac{q^2}{R^4}\,. \tag{21}$$

It is noted that at the momentary equator in a polar orbit, the radius of curvature and Kepler's 3rd Law are precisely those for the simple point-charge.

In the case of the rotating point-mass, at the poles,

$$R^2=C_n(r)=\left(|r-r_0|^n+\beta^n\right)^{\frac{2}{n}}\,,\quad \beta=m+\sqrt{m^2-a^2}\,,\quad \omega^2=\frac{\alpha R^2-\alpha a^2}{2R\,(R^2+a^2)^2}\,, \tag{22}$$

and at the equator,

$$R^2=C_n(r)=\left(|r-r_0|^n+\beta^n\right)^{\frac{2}{n}}\,,\quad \beta=2m=\alpha\,,\quad \omega^2=\frac{\alpha}{2R^3}\,. \tag{23}$$

At the momentary equator in a polar orbit the radius of curvature and Kepler's 3rd Law are precisely those for the simple point-mass.

6 Photons in the polar orbit

Setting (14) equal to zero, with $\dot{R}=0$, gives

$$\omega^2 = \frac{\Delta - a^2\sin^2\theta}{\xi^2} = \frac{R^2 - \alpha R + a^2\cos^2\theta + q^2}{\left(R^2 + a^2\cos^2\theta\right)^2}\,. \tag{24}$$

Denote the stable photon radius of curvature for a photon in polar orbit by $R_{ph-p} = \sqrt{C_n(r_{ph-p})}$. Then equating (24) to (16) gives,

$$2R^3_{ph-p} - 3\alpha R^2_{ph-p} +$$

$$+\left(2a^2\cos^2\theta + 4q^2\right)R_{ph-p} + \alpha a^2\cos^2\theta = 0\,,$$

$$R^2_{ph-p} = C_n(r_{ph-p}) = \left(|r_{ph} - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,, \tag{25}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2},\ a^2 + q^2 < m^2\,,$$

$$r_0 \in \Re, \quad n \in \Re^+\,.$$

Equation (25) gives the stable photon radius of curvature in the polar orbit. The orbit has a variable radius of curvature with azimuth.

When $a=0,\ q\neq 0$, equation (25) reduces to

$$R_{ph-p} = \sqrt{C_n(r_{ph-p})} = \frac{3\alpha + \sqrt{9\alpha^2 - 32q^2}}{4}\,, \tag{26}$$

$$C_n(r_{ph-p}) = \left(\left|r_{ph-p} - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2},\ q^2 < m^2\,,$$

$$r_0 \in \Re,\ n \in \Re^+\,,$$

which recovers the radius of curvature for the stable orbit of a photon about the simple point-charge. When $a=q=0$, (25) reduces to,

$$R_{ph-p} = \sqrt{C_n(r_{ph-p})} = \frac{3\alpha}{2}\,, \tag{27}$$

$$C_n(r_{ph-p}) = \left(\left|r_{ph-p} - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}\,,$$

$$\alpha = 2m,\ r_0 \in \Re,\ n \in \Re^+\,,$$

which recovers the curvature radius for the stable orbit of a photon about the simple point-mass. When $n=1$ and $r_0=\alpha$, equation (27) gives,

$$R_{ph-p} = \sqrt{C_n(r_{ph-p})} = r_{ph-p} = 3m\,,$$

which is the stable radius of curvature for the photon about the simple point-mass, but which is misinterpreted by the orthodox relativists as a measurable proper radius.

To obtain the stable photon radius of curvature of the polar orbit for the rotating point-mass, set $q=0$ in (25),

$$2R^3_{ph-p} - 3\alpha R^2_{ph-p} + 2a^2\cos^2\theta R_{ph-p} +$$

$$+\ \alpha a^2\cos^2\theta = 0\,,$$

$$R^2_{ph-p} = C_n(r_{ph-p}) = \left(|r_{ph} - r_0|^n + \beta^n\right)^{\frac{2}{n}}\,, \tag{28}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta},\ a^2 < m^2\,,$$

$$r_0 \in \Re, \quad n \in \Re^+\,.$$

7 Potential functions in the weak field

In the case of the rotating point-charge,

$$g_{00} = \frac{\Delta - a^2\sin^2\theta}{\rho^2}\,, \tag{29}$$

$$\Delta = C_n(r) - \alpha\sqrt{C_n(r)} + a^2 + q^2\,,$$

$$\rho^2 = C_n(r) + a^2\cos^2\theta\,.$$

The potential Φ for a general metric is given by,

$$g_{00} = (1-\Phi)^2 = 1 - 2\Phi + \Phi^2\,.$$

In the weak field,

$$g_{00} \approx 1 - 2\Phi\,.$$

Now (29) gives,

$$g_{00} = \frac{C_n(r) - \alpha\sqrt{C_n(r)} + a^2\cos^2\theta + q^2}{C_n(r) + a^2\cos^2\theta} =$$

$$= 1 - \frac{\alpha\sqrt{C_n(r)} - q^2}{C_n(r) + a^2\cos^2\theta}\,,$$

so the potential is,

$$\Phi = \frac{\alpha\sqrt{C_n(r)} - q^2}{2\left(C_n(r) + a^2\cos^2\theta\right)}\,, \tag{30}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,$$

$$a^2 + q^2 < m^2\,, \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,.$$

The potential therefore depends upon azimuth.

The potential for the rotating point-mass is obtained from (30) by setting $q = 0$,

$$\Phi = \frac{\alpha\sqrt{C_n(r)}}{2\left(C_n(r) + a^2\cos^2\theta\right)}\,, \tag{31}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,,$$

$$a^2 < m^2\,, \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,.$$

If $a = 0$ the potential for the simple point-charge is recovered from (30),

$$\Phi = \frac{\alpha}{2\sqrt{C_n(r)}} - \frac{q^2}{2C_n(r)}\,, \tag{32}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re\,,$$

$$\beta = m + \sqrt{m^2 - q^2}\,,$$

$$q^2 < m^2\,, \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,,$$

and if $a = q = 0$ the potential for the simple point-mass is recovered,

$$\Phi = \frac{\alpha}{2\sqrt{C_n(r)}}\,, \tag{33}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}, \quad r_0 \in \Re \quad n \in \Re^+\,,$$

$$0 < |r - r_0| < \infty\,.$$

According to (30), orbit in the equatorial gives equations (32) for the simple point-charge. According to (31), orbit in the equatorial gives equations (33) for the simple point-mass. For orbits in the polar, equations (32) and (33) are momentarily realised at the equator for a test particle orbiting the rotating point-charge and the rotating point-mass respectively. Thus, the effects of rotation of the source of the field do not manifest for a test particle in an equatorial orbit.

Taking the near-field limit on (30) gives,

$$\lim_{r \to r_0^{\pm}} \Phi = \frac{\alpha\beta - q^2}{2\left(\beta^2 + a^2\cos^2\theta\right)}\,, \tag{34}$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta - q^2}\,,$$

$$a^2 + q^2 < m^2\,.$$

The potential approaches a finite limit with azimuth. The limiting values for the simpler configurations are easily obtained from (34) in the obvious way.

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On the Vacuum Field of a Sphere of Incompressible Fluid

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The vacuum field of the point-mass is an unrealistic idealization which does not occur in Nature — Nature does not make material points. A more realistic model must therefore encompass the extended nature of a real object. This problem has also been solved for a particular case by K. Schwarzschild in his neglected paper on the gravitational field of a sphere of incompressible fluid. I revive Schwarzschild's solution and generalise it. The black hole is necessarily precluded. A body cannot undergo gravitational collapse to a material point.

1 Introduction

In my previous papers [1, 2] concerning the general solution for the point-mass I showed that the black hole is not consistent with General Relativity and owes its existence to a faulty analysis of the Hilbert [3] solution. In this paper I shall show that, along with the black hole, gravitational collapse to a point-mass is also untenable. This was evident to Karl Schwarzschild who, immediately following his derivation of his exact solution for the mass-point [4], derived a particular solution for an extended body in the form of a sphere of incompressible, homogeneous fluid [5]. This is also an idealization, and so too has its shortcomings, but represents a somewhat more plausable end result of gravitational collapse.

The notion that Nature makes material points, i. e. masses without extension, I view as an oxymoron. It is evident that there has been a confounding of a mathematical point with a material object which just cannot be rationally sustained. Einstein [6, 7] objected to the introduction of singularities in the field but could offer no viable alternative, even though Schwarzschild's extended body solution was readily at his hand.

The point-mass and the singularity are equivalent. Abrams [8] has remarked that singularities associated with a spacetime manifold are not uniquely determined until a boundary is correctly attached to it. In the case of the point-mass the source of the gravitational field is identified with a singularity in the manifold. The fact that the vacuum field for the point-mass is singular at a boundary on the manifold indicates that the point-mass does not occur in Nature. Oddly, the conventional view is that it embodies the material point. However, there exists no observational or experimental data supporting the idea of a point-mass or point-charge. I can see no way an electron, for instance, could be compressed into a material point-charge, which must occur if the point-mass is to be admitted. The idea of electron compression is meaningless, and therefore so is the point-mass. Eddington [9] has remarked in similar fashion concerning the electron, and relativistic degeneracy in general.

I regard the point-mass as a mathematical artifice and consider it in the fashion of a centre-of-mass, and therefore not as a physical object. In Newton's theory of gravitation, $r = 0$ is singular, and equivalently in Einstein's theory, the proper radius $R_p(r_0) \equiv 0$ is singular, as I have previously shown. Both theories therefore share the non-physical nature of the idealized case of the point-mass.

To obtain a model for a star and for the gravitational collapse thereof, it follows that the solution to Einstein's field equations must be built upon some manifold without boundary. In more recent years Stavroulakis [10, 11, 12] has argued the inappropriateness of the solutions on a manifold with boundary on both physical and mathematical grounds, and has derived a stationary solution from which he has concluded that gravitational collapse to a material point is impossible.

Utilizing Schwarzschild's particular solution I shall extend his result to a general solution for a sphere of incompressible fluid.

2 The general solution for Schwarzschild's incompressible sphere of fluid

At the surface of the sphere the required solution must maintain a smooth transition from the field outside the sphere to the field inside the sphere. Therefore, the metric for the interior and the metric for the exterior must attain the same value for the radius of curvature at the surface of the sphere. Furthermore, owing to the extended nature of the sphere, the exterior metric must take the form of the metric for the point-mass, but with a modified invariant containing the factors giving rise to the field, reflecting the non-pointlike nature of the source, thereby treating the source as a mass concentrated at the centre-of-mass of the sphere, just as in Newton's theory. Schwarzschild has achieved this in his particular case. He obtained the following metric for the field inside his sphere,

$$ds^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 dt^2 - \\ - \frac{3}{\kappa\rho_0}d\chi^2 - \frac{3\sin^2\chi}{\kappa\rho_0}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{1}$$

$$\sin\chi = \sqrt{\frac{\kappa\rho_0}{3}}\,\eta^{\frac{1}{3}}, \quad \eta = r^3 + \rho\,,$$

$$\rho = \left(\frac{\kappa\rho_0}{3}\right)^{\frac{-3}{2}}\left[\frac{3}{2}\sin^3\chi_a - \frac{9}{4}\cos\chi_a\left(\chi_a - \frac{1}{2}\sin 2\chi_a\right)\right],$$

$$\kappa = 8\pi k^2\,,$$

$$0 \leqslant \chi \leqslant \chi_a < \frac{\pi}{2},$$

where ρ_0 is the constant density of the fluid, k^2 Gauss' gravitational constant, and the subscript a denotes values at the surface of the sphere. Metric (1) is non-singular.

Schwarzschild's particular metric outside the sphere is,

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1}dR^2 - \\ - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2}$$

$$R^3 = r^3 + \rho\,, \quad \alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3\chi_a\,,$$

$$0 \leqslant \chi_a < \frac{\pi}{2}\,,$$

$$r_a \leqslant r < \infty\,.$$

Metric (2) is non-singular for an extended body.

In the case of the simple point-mass (i. e. non-rotating, no charge) I have shown elsewhere [13] that the general solution is,

$$ds^2 = \left[\frac{(\sqrt{C_n} - \alpha)}{\sqrt{C_n}}\right]dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n} - \alpha)}\right]\frac{C_n'^{\,2}}{4C_n}dr^2 - \\ - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{3}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{2}{n}}\,,\ \alpha = 2m\,,$$

$$n \in \Re^+\,,\ r \in \Re,\ r_0 \in \Re\,,$$

$$0 < |r - r_0| < \infty\,,$$

where n and r_0 are arbitrary.

Now Schwarzschild fixed his solution for $r_0 = 0$. I note that his equations, rendered herein as equations (1) and (2), can be easily generalised to an arbitrary $r_0 \in \Re$ and arbitrary $\chi_0 \in \Re$ by replacing his r and χ by $|r - r_0|$ and $|\chi - \chi_0|$ respectively. Furthermore, equation (3) must be modified to account for the extended configuration of the gravitating mass. Consequently, equation (1) becomes,

$$ds^2 = \left[\frac{3\cos|\chi_a - \chi_0| - \cos|\chi - \chi_0|}{2}\right]^2 dt^2 - \\ - \frac{3}{\kappa\rho_0}d\chi^2 - \frac{3\sin^2|\chi - \chi_0|}{\kappa\rho_0}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{4}$$

$$\sin|\chi - \chi_0| = \sqrt{\frac{\kappa\rho_0}{3}}\eta^{\frac{1}{3}}, \quad \eta = \left|r - r_0\right|^3 + \rho\,,$$

$$\rho = \left(\frac{\kappa\rho_0}{3}\right)^{\frac{-3}{2}}\left\{\frac{3}{2}\sin^3|\chi_a - \chi_0| - \right. \\ \left. - \frac{9}{4}\cos|\chi_a - \chi_0|\left[|\chi_a - \chi_0| - \frac{1}{2}\sin 2\,|\chi_a - \chi_0|\right]\right\},$$

$$\kappa = 8\pi k^2\,,\ r_0 \in \Re\,,\ r \in \Re\,,\ \chi_a \in \Re\,,\ \chi_0 \in \Re\,,$$

$$0 \leqslant |\chi - \chi_0| \leqslant |\chi_a - \chi_0| < \frac{\pi}{2}\,,$$

and outside the sphere, equation (2) becomes,

$$ds^2 = \left(1 - \frac{\alpha}{R}\right)dt^2 - \left(1 - \frac{\alpha}{R}\right)^{-1}dR^2 - \\ - R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{5}$$

$$R^3 = \left|r - r_0\right|^3 + \rho, \quad \alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0|\,,$$

$$n \in \Re^+\,,\ r_0 \in \Re\,,\ r \in \Re\,,\ \chi_0 \in \Re\,,\ \chi_a \in \Re\,,$$

$$0 \leqslant |\chi_a - \chi_0| < \frac{\pi}{2}\,,$$

$$|r_a - r_0| \leqslant |r - r_0| < \infty\,,$$

and outside the sphere, equation (3) becomes,

$$ds^2 = \left[\frac{(\sqrt{C_n} - \alpha)}{\sqrt{C_n}}\right]dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n} - \alpha)}\right]\frac{C_n'^{\,2}}{4C_n}dr^2 - \\ - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{6}$$

$$C_n(r) = \left(\left|r - r_0\right|^n + \epsilon^n\right)^{\frac{2}{n}}\,,$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0|\,,$$

$$\epsilon = \sqrt{\frac{3}{\kappa\rho_0}}\left\{\frac{3}{2}\sin^3|\chi_a - \chi_0| - \right. \\ \left. - \frac{9}{4}\cos|\chi_a - \chi_0|\left[|\chi_a - \chi_0| - \frac{1}{2}\sin 2\,|\chi_a - \chi_0|\right]\right\}^{\frac{1}{3}},$$

$$r_0 \in \Re\,,\ r \in \Re\,,\ n \in \Re^+\,,\ \chi_0 \in \Re\,,\ \chi_a \in \Re\,,$$

$$|r_a - r_0| \leqslant |r - r_0| < \infty\,.$$

The general solution for the interior of the incompressible Schwarzschild sphere is given by (4), and (6) gives the general solution on the exterior of the sphere.

Consider the general form for a static metric for the gravitational field [13],

$$ds^2 = A(D)dt^2 - B(D)dD^2 - C(D)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,,$$

$$D = |r - r_0|\,,$$

$$A, B, C > 0 \;\forall\; r \neq r_0\,.$$

With respect to this metric I identify the real r-parameter, the radius of curvature, and the proper radius thus:

1. *The real r-parameter is the variable r.*
2. *The radius of curvature is $R_c = \sqrt{C(D)}$.*
3. *The proper radius is $R_p = \int \sqrt{B(D)}\, dD$.*

According to the foregoing, the proper radius of the sphere of incompressible fluid determined from *inside* the sphere is, from (4),

$$R_p = \int_{\chi_0}^{\chi_a} \sqrt{\frac{3}{\kappa\rho_0}} \frac{(\chi - \chi_0)}{|\chi - \chi_0|} d\chi = \sqrt{\frac{3}{\kappa\rho_0}}\, |\chi_a - \chi_0|\,. \qquad (7)$$

The proper radius of the sphere cannot be determined from *outside* the sphere. According to (6) the proper radius to a spacetime event outside the sphere is,

$$\begin{aligned} R_p &= \int \sqrt{\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}}\, \frac{C_n'}{2\sqrt{C_n}}\, dr = \\ &= K + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} + \\ &+ \alpha \ln\left|\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}\right|, \end{aligned} \qquad (8)$$

$$K = \text{const.}$$

At the surface of the sphere the proper radius from outside has some value R_{p_a}, for some value r_a of the parameter r. Therefore, at the surface of the sphere,

$$\begin{aligned} R_{p_a} &= K + \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} + \\ &+ \alpha \ln\left|\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}\right|. \end{aligned}$$

Solving for K,

$$\begin{aligned} K &= R_{p_a} - \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} - \\ &- \alpha \ln\left|\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}\right|. \end{aligned}$$

Substituting into (8) for K gives for the proper radius from outside the sphere,

$$\begin{aligned} R_p(r) &= R_{p_a} + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} - \\ &- \sqrt{\sqrt{C_n(r_a)}\left(\sqrt{C_n(r_a)} - \alpha\right)} + \\ &+ \alpha \ln\left|\frac{\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}}{\sqrt{\sqrt{C_n(r_a)}} + \sqrt{\sqrt{C_n(r_a)} - \alpha}}\right|. \end{aligned} \qquad (9)$$

Then by (9), for $|r - r_0| \geqslant |r_a - r_0|$

$$|r - r_0| \to |r_a - r_0| \Rightarrow R_p \to R_{p_a}^{+}\,,$$

but R_{p_a} cannot be determined.

According to (4) the radius of curvature $R_c = P_a$ at the surface of the sphere is,

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}} \sin|\chi_a - \chi_0|\,. \qquad (10)$$

Furthermore, inside the sphere,

$$\frac{G}{R_p} \leqslant 2\pi\,,$$

and

$$\lim_{\chi \to \chi_0^{\pm}} \frac{G}{R_p} = 2\pi\,,$$

where $G = 2\pi R_c$ is the circumference of a great circle.

But outside the sphere,

$$\frac{G}{R_p} \geqslant 2\pi\,,$$

with the equality only when $R_p \to \infty$.

The radius of curvature of (6) at the surface of the sphere is $\sqrt{C_n(r_a)}$ so,

$$\sqrt{C_n(r_a)} = \left(|r_a - r_0|^n + \epsilon^n\right)^{\frac{1}{n}}. \qquad (11a)$$

At the surface of the sphere the measured circumference G_a of a great circle is,

$$G_a = 2\pi P_a = 2\pi\sqrt{C_n(r_a)}\,.$$

Therefore, at the surface of the sphere equations (10) and (11a) are equal,

$$\left(|r_a - r_0|^n + \epsilon^n\right)^{\frac{1}{n}} = \sqrt{\frac{3}{\kappa\rho_0}} \sin|\chi_a - \chi_0|\,, \qquad (11b)$$

and so,

$$|r_a - r_0| = \left[\left(\frac{3}{\kappa\rho_0}\right)^{\frac{n}{2}} \sin^n|\chi_a - \chi_0| - \epsilon^n\right]^{\frac{1}{n}}. \qquad (11c)$$

The variable r is just a *parameter* for the radial quantities R_p and R_c associated with (6). Similarly, χ is also a *parameter* for the radial quantities R_p and R_c associated with (4). I remark that r_0 and χ_0 are both *arbitrary*, and *independent* of one another, and that $|r - r_0|$ and $|\chi - \chi_0|$ do not of themselves denote radii in any direct way. The arbitrary values of the parameter "origins", r_0 and χ_0, are simply boundary points on r and χ respectively. Indeed, by (7), $R_p(\chi_0) \equiv 0$, and by (9), $R_p(r_a) \equiv R_{p_a}$, irrespective of the values of r_0, r_a, and χ_0. The centre-of-mass of the sphere of fluid is always located precisely at $R_p(\chi_0) \equiv 0$. Furthermore, $R_p(r)$ for $|r - r_0| < |r_a - r_0|$ has no meaning since inside the sphere (4) describes the geometry, not (6).

According to (11b), equation (9) can be written as,

$$R_p(r) = R_{p_a} + \sqrt{\sqrt{C_n(r)}\left(\sqrt{C_n(r)} - \alpha\right)} -$$

$$- \sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0|\left(\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0| - \alpha\right)} + \tag{12}$$

$$+ \alpha \ln \left| \frac{\sqrt{\sqrt{C_n(r)}} + \sqrt{\sqrt{C_n(r)} - \alpha}}{\sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0|} + \sqrt{\sqrt{\frac{3}{\kappa\rho_0}}\sin|\chi_a - \chi_0| - \alpha}} \right| ,$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0| \,.$$

Note that in (4), $|\chi - \chi_0|$ cannot grow up to $\frac{\pi}{2}$, so that Schwarzschild's sphere does not constitute the whole spherical space, which has a radius of curvature of $\sqrt{\frac{3}{\kappa\rho_0}}$. From (4) and (6),

$$\frac{\alpha}{P_a} = \sin^2|\chi_a - \chi_0| \,, \quad \alpha = \frac{\kappa\rho_0}{3} P_a^3 \,. \tag{13}$$

The volume of the sphere is,

$$V = \left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}} \int_{\chi_0}^{\chi_a} \sin^2|\chi - \chi_0| \frac{(\chi - \chi_0)}{|\chi - \chi_0|} d\chi \times$$

$$\times \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi =$$

$$= 2\pi \left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}} \left(|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right) ,$$

so the mass of the sphere is,

$$M = \rho_0 V = \frac{3}{4k^2}\sqrt{\frac{3}{\kappa\rho_0}} \left(|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right) .$$

Schwarzschild [5] has also shown that the velocity of light inside his sphere of incompressible fluid is given by,

$$v_c = \frac{2}{3\cos\chi_a - \cos\chi} \,,$$

which generalises to,

$$v_c = \frac{2}{3\cos|\chi_a - \chi_0| - \cos|\chi - \chi_0|} \,. \tag{14}$$

At the centre $\chi = \chi_0$, so v_c reaches a maximum value there of,

$$v_c = \frac{2}{3\cos|\chi_a - \chi_0| - 1} \,,$$

Equation (14) is singular when $\cos|\chi_a - \chi_0| = \frac{1}{3}$, which means that there is a lower bound on the possible radii of curvature for spheres of incompressible, homogeneous fluid, which is, by (13) and (6),

$$P_{a\ (min)} = \frac{9}{8}\alpha = \sqrt{\frac{8}{3\kappa\rho_0}} \,, \tag{15a}$$

and consequently, by equation (11a),

$$|r_a - r_0|_{(min)} = \left[\left(\frac{9\alpha}{8}\right)^n - \epsilon^n\right]^{\frac{1}{n}} = \left[\left(\frac{8}{3\kappa\rho_0}\right)^{\frac{n}{2}} - \epsilon^n\right]^{\frac{1}{n}} , \tag{15b}$$

from which it is clear that a body cannot collapse to a material point.

From (13), a sphere of given gravitational mass $\frac{\alpha}{k^2}$, cannot have a radius of curvature, determined from outside, smaller than,

$$P_{a\ (min)} = \alpha \,,$$

so

$$|r_a - r_0|_{(min)} = [\alpha^n - \epsilon^n]^{\frac{1}{n}} \,,$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0| \,.$$

3 Kepler's 3rd Law for the sphere of incompressible fluid

There is no loss of generality in considering only the equatorial plane, $\theta = \frac{\pi}{2}$. Equation (6) then leads to the Lagrangian,

$$L = \frac{1}{2}\left[\left(\frac{\sqrt{C} - \alpha}{\sqrt{C}}\right)\dot{t}^2 - \left(\frac{\sqrt{C}}{\sqrt{C} - \alpha}\right)\left(\dot{\sqrt{C}}\right)^2 - C\dot{\varphi}^2\right] ,$$

where the dot indicates $\partial/\partial\tau$.

Let $R = \sqrt{C_n(r)}$. Then,

$$\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = \frac{R}{R-\alpha}\,\ddot{R} + \frac{\alpha}{2R^2}\,\dot{t}^2 - \\ - \frac{\alpha}{2\,(R-\alpha)}\,\dot{R}^2 - R\,\dot{\varphi}^2 = 0\,.$$

Now let $R = \text{const}$. Then,

$$\frac{\alpha}{2R^2}\,\dot{t}^2 = R\,\dot{\varphi}^2\,,$$

so

$$\omega^2 = \frac{\alpha}{2R^3} = \frac{\alpha}{2C^{\frac{3}{2}}} = \frac{\alpha}{2\left(\left|r - r_0\right|^n + \epsilon^n\right)^{\frac{3}{n}}}\,. \tag{16}$$

Equation (16) is Kepler's 3rd Law for the sphere of incompressible fluid.

Taking the near-field limit gives,

$$\omega_a^2 = \lim_{|r-r_0|\to|r_a-r_0|^+} \omega^2 = \frac{\alpha}{2\left(\left|r_a - r_0\right|^n + \epsilon^n\right)^{\frac{3}{n}}}\,.$$

According to (11b) and (10) this becomes,

$$\omega_a^2 = \frac{\alpha}{2\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\sin^3\left|\chi_a - \chi_0\right|} = \frac{\alpha}{2P_a^3}\,.$$

Finally, using (13),

$$\omega_a = \frac{\sin^3\left|\chi_a - \chi_0\right|}{\alpha\sqrt{2}}\,, \tag{17}$$

$$\alpha = \sqrt{\frac{3}{\kappa\rho_0}}\,\sin^3\left|\chi_a - \chi_0\right|\,.$$

In contrast, the limiting value of ω for the simple point-mass [4] is,

$$\omega_0 = \frac{1}{\alpha\sqrt{2}}\,,$$

$$\alpha = 2m\,.$$

When P_a is minimum, (17) becomes,

$$\omega_a^2 = \frac{16}{27\alpha}\,, \tag{18}$$

$$\alpha = \frac{16}{27}\sqrt{\frac{6}{\kappa\rho_0}}\,.$$

Clearly, equation (17) is an invariant,

$$\omega_a = \sqrt{\frac{\kappa\rho_0}{6}}\,.$$

4 Passive and active mass

The relationship between passive and active mass manifests, owing to the difference established by Schwazschild, between what he called "substantial mass" (passive mass) and the gravitational (i.e. active) mass. He showed that the former is larger than the latter.

Schwarzschild has shown that the substantial mass M is given by,

$$M = 2\pi\rho_0\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\left(\chi_a - \frac{1}{2}\sin 2\,\chi_a\right),$$

$$0 \leqslant \chi_a < \frac{\pi}{2}\,,$$

and the gravitational mass is,

$$m = \frac{\alpha c^2}{2G} = \frac{1}{2}\sqrt{\frac{3}{\kappa\rho_0}}\,\sin^3\chi_a = \frac{\kappa\rho_0}{6}P_a^3 = \frac{4\pi}{3}P_a^3\rho_0\,,$$

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}}\,\sin\chi_a\,,$$

$$0 \leqslant \chi_a < \frac{\pi}{2}\,.$$

I have generalised Schwarzschild's result to,

$$M = 2\pi\rho_0\left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}}\left(\left|\chi_a - \chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a - \chi_0\right|\right),$$

$$m = \frac{\alpha c^2}{2G} = \frac{1}{2}\sqrt{\frac{3}{\kappa\rho_0}}\,\sin^3\left|\chi_a - \chi\rho_0\right| = \\ = \frac{\kappa\rho_0}{6}P_a^3 = \frac{4\pi}{3}P_a^3\rho_0\,, \tag{19}$$

$$P_a = \sqrt{\frac{3}{\kappa\rho_0}}\,\sin\left|\chi_a - \chi_0\right|\,,$$

$$0 \leqslant \left|\chi_a - \chi_0\right| < \frac{\pi}{2}\,,$$

where G is Newton's gravitational constant. Equation (19) is only formally the same as that for the Euclidean sphere, because the radius of curvature P_a is not a Euclidean quantity, and cannot be measured in the gravitational field.

The ratio between the gravitational mass and the substantial mass is,

$$\frac{m}{M} = \frac{2\sin^3\left|\chi_a - \chi_0\right|}{3\left(\left|\chi_a - \chi_0\right| - \frac{1}{2}\sin 2\left|\chi_a - \chi_0\right|\right)}\,.$$

Schwarzschild has shown that the naturally measured fall velocity of a test particle, falling from rest at infinity down to the surface of the sphere of incompressible fluid is,

$$v_a = \sin\chi_a\,,$$

which I generalise to,

$$v_a = \sin\left|\chi_a - \chi_0\right| .$$

The quantity v_a is the escape velocity.

Therefore, as the escape velocity increases, the ratio $\frac{m}{M}$ decreases, owing to the increase in the mass concentration.

In the case of the fictitious point-mass,

$$\lim_{|\chi_a - \chi_0| \to 0} \left(\frac{m}{M}\right) = 1 .$$

However, according to equation (14), for an incompressible sphere of fluid,

$$\cos\left|\chi_a - \chi_0\right|_{min} = \frac{1}{3} ,$$

so

$$\left(\frac{m}{M}\right)_{max} \approx 0.609 .$$

Finally,

$$\text{as } \left|\chi_a - \chi_0\right| \to \frac{\pi}{2}, \ \frac{m}{M} \to \frac{4}{3\pi} .$$

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

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Power as the Cause of Motion and a New Foundation of Classical Mechanics

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Laws of motion are derived based on power rather than on force. I show how power extends the law of inertia to include curvilinear motion and I also show that the law of action-reaction can be expressed in terms of the mutual time rate of change of kinetic energies instead of mutual forces. I then compare the laws of motion based on power to Newton's Laws of Motion and I investigate the relation of power to Leibniz's notion of vis viva. I also discuss briefly how the metaphysics of power as the cause of motion can be grounded in a modern version of occasionalism for the purpose of establishing an alternative foundation of mechanics. The laws of motion derived in this paper along with the metaphysical foundation proposed come in defense of the hypotheses that time emerges as an ordered progression of now and that gravitation is the effect of energy transfer between an unobservable substance and all matter in the Universe.

1 Introduction

This paper's central aim is the derivation of laws of motion based on the notion of power rather than on the classical notion of force. Although the derivation of laws of motion is traditionally a subject of mechanics, several references are made herein to the history and philosophy of science. This is necessary because this paper deals primarily with the foundations of mechanics. Specifically, the hypothesis that power is the cause of motion, as contrasted to the Newtonian hypothesis according to which force is the cause of motion, leads to a major revision of the foundations of Classical Mechanics.

Most contemporary philosophers of science focus on the foundational problems of General Relativity and Quantum Mechanics and, unlike their seventeenth-century counterparts, think of Classical Mechanics as unproblematic. Butterfield mentions two errors found in this view that correspond to what he calls the matter-in-motion picture and the particle-in-motion picture [1]. According to the matter-in-motion picture, for example, bodies are collections of particles separated by voids, can move in vacuum and interact with each other, whilst their motion is completely determined by Newton's Second Law. This view has become a part of an "educated layperson's" common sense nowadays but according to Butterfield it is problematic: it does not offer, amongst other things, any explanation of the mechanism(s) of the assumed interactions but resorts to concepts such as forces acting across an intervening void ("action-at-a-distance").

The failure of modern theories to provide solutions to the foundational problems of Classical Mechanics is partly due to the fact that alternative rigid foundations have not been proposed but issues seem to have been further perplexed. Quantum uncertainty and the four-dimensional space-time of relativity have taken the place of the determinism and of the unobservable absolute space and universal time of Classical Mechanics. Mysterious action-at-a-distance still prevails in the quantum world and attempts to quantize gravity and unite Quantum Mechanics with General Relativity have failed to this date. In presenting an alternative system of laws of motion based on power, I aim primarily in the investigation of a new foundation, which offers an alternative approach for solutions to some of the unsolved problems of Classical Mechanics.

In a similar way to the matter-in-motion picture, the notion of force has also become part of an "educated layperson's" common sense, thanks to the empirical support the laws of mechanics have enjoyed over the past 300 years. It is well known, however, that Newton was heavily criticized for his use of the notion of force in an effort to ground his physics on his metaphysics and there is still considerable interest in the metaphysics of his *Principia*. In *Science and Hypothesis*, Poincaré writes [2]:

> When are two forces equal? We are told that it is when they give the same acceleration to the same mass, or when acting in opposite directions they are in equilibrium. This definition is a sham.

In *Principles of Dynamics*, Donald T. Greenwood offers an introduction to the issues raised by Newton's concept of force [3]:

> The concept of force as a fundamental quantity in the study of mechanics has been criticized by various scientists and philosophers of science from shortly after Newton's enunciation of the laws of motion until the present time. Briefly, the idea of a force, and a field force in particular, was considered to be an

> intellectual construction, which has no real existence. It is merely another name for the product of mass and acceleration, which occurs in the mathematics of solving a problem. *Furthermore, the idea of force as a cause of motion should be discarded since the assumed cause and effect relationship cannot be proven*. (Italics added)

The questions raised from Newton's definition of force and postulation of absolute space are well known to the philosophers of science and will be further discussed in sections 4, 5 and 6. In the following two sections, 2 and 3, I will show that using the notion of power as *a priori* principle, laws of motion can be derived with remarkably different definitions of inertia and action-reaction. I will then argue in section 4, where I discuss the relation of this alternative system of laws to Newton's, that the existence of a more general principle of motion is even acknowledged by Newton, in his own writings. In section 5, the relation of the notion of power to Leibniz's notion of vis viva is examined. Then, in section 6, I discuss how the metaphysics of power can be grounded in a modern version of occasionalism for the purpose of establishing an alternative foundation of Classical Mechanics. I argue that the alternative foundation proposed, along with an appropriate space-time structure, support a new hypothesis about time and about the nature of gravitation.

2 The axiom of motion

I begin the derivation of the laws of motion by stating the axiom of motion, an expression relating the velocity and the time rate of change of momentum of a particle, to a scalar quantity called the time rate of change of kinetic energy, also known as (instantaneous) power. The status of this axiom is assumed here to be that of *a priori* truth as opposed to a self-evident or empirical principle.

Axiom of Motion: The time rate of change of the kinetic energy of a particle is equal to the scalar product of its velocity and time rate of change of its momentum.

Denoting the kinetic energy by E_k and the momentum by $\mathbf{p}$, the axiom of motion can be expressed as follows:

$$\frac{dE_k}{dt} = \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{r}}{dt}\,, \tag{1}$$

where $\mathbf{r}$ is the position vector of the particle. The momentum $\mathbf{p}$ is defined as

$$\mathbf{p} = m\frac{d\mathbf{r}}{dt}\,. \tag{2}$$

If the mass m of the particle is independent of time t and position $\mathbf{r}$, then by combining equations (1) and (2), the time rate of change of the kinetic energy E_k can be written as follows:

$$\frac{dE_k}{dt} = m\frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt}\,. \tag{3}$$

Corollary I: The kinetic energy of a particle with a constant mass m is given by

$$E_k = \frac{1}{2}\,m\mathbf{v}\cdot\mathbf{v}\,, \tag{4}$$

where $\mathbf{v}$ is defined as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}\,. \tag{5}$$

Proof: From equation (3) we obtain

$$\frac{dE_k}{dt} = m\frac{d^2\mathbf{r}}{dt^2}\cdot\frac{d\mathbf{r}}{dt} = m\frac{d\mathbf{r}}{dt}\cdot\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = m\frac{d}{dt}\left(\frac{1}{2}\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}\right),$$

which yields

$$E_k = \frac{1}{2}\,m\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt} = \frac{1}{2}\,m\mathbf{v}\cdot\mathbf{v}\,. \tag{6}$$

The axiom of motion is the only principle required for deriving the laws of motion, as it will be shown in the next section.

3 The laws of motion

Law of Inertia: If the time rate of change of the kinetic energy of a particle is zero, the particle will continue in its state of motion.

Proof: If the time rate of change of the kinetic energy of a particle is zero, then from equation (3) we obtain

$$m\frac{d^2\mathbf{r}}{dt^2}\cdot\frac{d\mathbf{r}}{dt} = 0\,. \tag{7}$$

Assuming m remains constant, the following satisfy equation (7)

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_0\,, \tag{8}$$

$$\frac{d\mathbf{r}}{dt} = 0\,, \tag{9}$$

$$\frac{d^2\mathbf{r}}{dt^2}\cdot\mathbf{v} = 0\,, \tag{10}$$

where $\mathbf{v}_0$ is a constant. Thus, solutions to equation (7) include motion with a constant velocity $\mathbf{v}_0$, given by equation (8), or a state of rest, given by equation (9) and in both these cases the time rate of change of kinetic energy is zero. These are trivial solutions to equation (7) arising when either the velocity or the acceleration of the particle, are null vectors. Yet, these two trivial solutions result in the simplest kinematic states possible and the only two states allowed when there are no forces acting on a particle according to Newton's First Law. However, if power is postulated as the cause of motion there is another trivial solution, that of uniform circular motion, as it will be shown below.

General solutions to equation (10) include all curvilinear paths with a constant kinetic energy E_k. The requirement of a constant kinetic energy could have been included in the statement of the law of inertia but this is obviously redundant since, if the time rate of change of the kinetic energy is zero then kinetic energy is constant. Clearly, the states of motion resulting from (8) and (9) are trivial solutions to (10) with zero velocity and zero acceleration, respectively. From equations (5), (6) and (10) we obtain:

$$\frac{d^2\mathbf{r}}{dt^2} \cdot \frac{d\mathbf{r}}{dt} = 0 \Leftrightarrow \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{v} \cdot \mathbf{v} = \frac{2E_k}{m} = k\,, \qquad (11)$$

where k is a constant equal to twice the kinetic energy per unit mass. Thus, all motion paths that satisfy equation (10) also satisfy the following equation

$$\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = k\,, \qquad (12)$$

which is equivalent to the statement that the magnitude of velocity, or the speed, must be constant. In the case of motion in a plane, **v** can be expressed in polar coordinates as follows:

$$\mathbf{v} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\theta}\,. \qquad (13)$$

From equations (12) and (13) we obtain:

$$\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = k^2. \qquad (14)$$

A trivial solution to equation (14) is uniform circular motion given by

$$\mathbf{r}(t) = r\,\hat{\mathbf{r}}(t)\,, \qquad (15)$$

where r is a constant radius and the unit radial vector $\hat{\mathbf{r}}$ rotates at a constant rate $d\theta/dt$. In the context of this law of inertia, if a particle is in uniform circular motion and the time rate of change of its kinetic energy remains zero, the state of uniform circular motion will be maintained. Notice that no claim of any sort is made herein that zero power is the cause of uniform circular motion. Obviously, a zero of something cannot be the real cause of anything. The only claim made is that if a particle is in uniform circular motion -or in any other curvilinear path that satisfies equation (12) — and power, the postulated cause of motion, remains zero then the particle will continue in its state of motion. I would like to stretch this point because, as it will be discussed further in chapter 4, the laws of motion presented in this paper can be considered as an alternative to Newton's Laws of Motion. Thus, one should refrain from evaluating these laws in the context of Newtonian mechanics, since the two systems of laws are grounded in different metaphysics. The question then of how a particle is set on a uniform circular motion in the first place is a metaphysical one and it will be placed in its proper context in chapter 6.

Non-trivial solutions to equation (14) include motion in a plane where the magnitude of the velocity **v** remains constant up to sign changes. Such motion possibilities are virtually unlimited, including for instance motion in eight-shaped figures and cycloid paths. However, some of these paths may represent physical possibilities and others may not. Uniform circular motion is a physical possibility in both micro and macro scales and this has been confirmed empirically. The choice of specific curvilinear motions over others as an effect of inertia, if power is postulated to be the cause of motion, is the subject of metaphysics discussed in section 6. The law of inertia presented in this section is a statement that the state of such motions is maintained in the absence of a cause, if power is postulated to be the cause of motion. However, the law does not provide a justification for the existence or preference of certain states of motions over others in the absence of a cause of motion.

General solutions to equation (12) in three-dimensional Euclidean space include motion along any curve. It is known from differential geometry that if a curve is regular, then there exists a reparametrization such that the curve has unit speed [4]. Thus, a particle can be made to move with constant speed along any curve in space using proper arc-length reparametrization resulting in constant kinetic energy and as a consequence, zero power.

The law of inertia is a statement about the tendency of particles to maintain their state of motion when the time rate of change in their kinetic energy is zero and this tendency is called *inertia*. Again, the law of inertia was derived based on the metaphysical hypothesis that power is the cause of motion. A consequence from such hypothesis is that the set of "cause-free" paths now includes all paths where the kinetic energy remains constant, instead of just uniform rectilinear motion and the state of rest defined in Newtonian mechanics. As it will be discussed in section 4.1, from an empirical viewpoint it is irrelevant whether one considers just rectilinear or curvilinear motion as an effect of inertia, since no experiment can be devised to prove that in the case of a freely moving particle. This is because, there is always a cause present affecting the motion of all particles. In the case of Newtonian mechanics, this cause is a gravity force and in the case of the laws of motion discussed in this paper there is always a power cause acting and giving rise to gravitational effects as it will be discussed in chapter 6.

Corollary II: If the time rate of change of the kinetic energy of a particle is zero, linear momentum is conserved.

Proof: As a direct consequence of the law of inertia, if the time rate of change of kinetic energy is zero and the velocity is denoted by **v**, then from equations (1) and (5) we obtain

$$\frac{d\mathbf{p}}{dt} \cdot \mathbf{v} = 0\,. \qquad (16)$$

By using equation (2) and since **v** is not the null vector in general, we obtain from equation (16) the result:

$$\frac{d(m\mathbf{v})}{dt} = 0 \Rightarrow (m\mathbf{v})_2 - (m\mathbf{v})_1 = 0 \Rightarrow \Rightarrow (m\mathbf{v})_2 = (m\mathbf{v})_1 = m\mathbf{v} = \text{const}\,. \tag{17}$$

Equation (17) is the mathematical statement of the theorem of the conservation of linear momentum [5].

Law of Interaction: To every action there is an equal and opposite reaction; that is, in an isolated system of two particles acting upon each other, the mutual time rate of change of kinetic energies are equal in magnitude and opposite in sign.

Proof: We denote the two interacting particles as m_1 and m_2. Furthermore, we denote m_1 as the agent causing the action in the system. The total kinetic energy of the interacting system of particles is the sum of the kinetic energies of the two particles:

$$E_k = E_{k_1} + E_{k_2}\,. \tag{18}$$

From equations (1), (5) and (18) we obtain

$$\frac{dE_k}{dt} = \frac{d\mathbf{p_1}}{dt}\cdot\mathbf{v_1} + \frac{d\mathbf{p_2}}{dt}\cdot\mathbf{v_2}\,, \tag{19}$$

where $\mathbf{v_1}$ and $\mathbf{v_2}$ are the velocities of the two particles with momentum $\mathbf{p_1}$ and $\mathbf{p_2}$, respectively.

Next, we consider the mutual time rate of change of kinetic energy imposed by the particles upon each other. The time rate of change of kinetic energy of particle m_2, denoted as E_{k_2}, is equal to the action imposed on it by particle m_1, denoted as $E_{k_{12}}$ and given by

$$\frac{dE_{k_2}}{dt} = \frac{d\mathbf{p_2}}{dt}\cdot\mathbf{v_2} = \frac{dE_{k_{12}}}{dt}\,. \tag{20}$$

The time rate of change of the kinetic energy of particle m_1 is equal to the sum of the time rate of change of the kinetic energy of the system due to its action as an agent and that imposed on it by particle m_2 in the form of a reaction and denoted as $E_{k_{21}}$

$$\frac{dE_{k_1}}{dt} = \frac{dE_k}{dt} + \frac{dE_{k_{21}}}{dt} = \frac{d\mathbf{p_1}}{dt}\cdot\mathbf{v_1}\,. \tag{21}$$

By combining equations (19), (20) and (21), we obtain the result:

$$\frac{dE_{k_{12}}}{dt} = -\frac{dE_{k_{21}}}{dt}\,. \tag{22}$$

Equation (22) is the mathematical statement of the law of interaction. According to the law, the reaction on a horse pulling on a cart, — to use Newton's example in the Principia — is equal to the action applied by the horse on the cart. In general, part of the action produced by the horse is used to change its own state of motion and the remaining to change that of the cart. In the case where the total action of the horse is reacted by the cart, from equation (21) it may be seen that dE_k/dt is equal to zero and the state of motion does not change. Then, in this special case, action is equal to reaction *by definition*. This can serve the purpose of clearing any confusion that may arise when the action by the horse on the cart is thought to be equal to the total action produced by the horse, a statement that is not true in the most general case.

The philosophical issues arising from the law of interaction will be discussed in more detail in section 4.

Corollary III: In an isolated system of two particles acting upon each other and both having velocity $\mathbf{v}$, the mutual time rate of change of momentum vectors are equal in magnitude and opposite in direction.

Proof: By denoting the mutual momentum vectors by $\mathbf{p_{12}}$ and $\mathbf{p_{21}}$, from equations (1), (5) and (22) we obtain

$$\frac{d\mathbf{p_{12}}}{dt}\cdot\mathbf{v} = -\frac{d\mathbf{p_{21}}}{dt}\cdot\mathbf{v} \Leftrightarrow \left(\frac{d\mathbf{p_{12}}}{dt} + \frac{d\mathbf{p_{21}}}{dt}\right)\cdot\mathbf{v} = 0\,. \tag{23}$$

Since $\mathbf{v}$ is not in general a null vector, we obtain the result:

$$\frac{d\mathbf{p_{12}}}{dt} = -\frac{d\mathbf{p_{21}}}{dt}\,. \tag{24}$$

In the case where $\mathbf{v}$ is orthogonal to the sum of the mutual time rate of change of the momentum vectors of the two particles, then equation (23) will still hold. However, in this case, the mutual time rate of change of momentum vectors will not in general be equal in magnitude and opposite in direction.

The axiom of motion of section 2, together with the law of inertia and the law of interaction, combined further with the axiom of conservation of energy of isolated systems, provide a framework for deriving the differential equations of motion of particles and by extension of rigid bodies in dynamical motion. Next, I will examine the relation of the laws of motion presented in this section to Newton's Laws of Motion.

4 Power versus force

Newton stated his laws of motion in *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), first published in 1686 [6]. The Principia was revised by Newton in 1713 and 1726. Using modern terminology, the laws can be stated as follows [3]:

First Law: Every body continuous in its state of rest, or of uniform motion in a straight line, unless compelled to change that state by forces acting upon it.

Second Law: The time rate of change of linear momentum of a body is proportional to the force acting upon it and occurs in the direction in which the force acts.

Third Law: To every action there is an equal and opposite reaction and thus, the mutual forces of two bodies acting upon each other are equal in magnitude and opposite in direction.

4.1 Newton's First Law: A priori truth or an experimental fact?

Newton's First Law can be deduced from the law of inertia stated in section 3 and specifically from equations (8) and (9), or from corollary II. According to the law of inertia, when the time rate of change of the momentum of a particle is zero, then that particle will either remain at rest or move in a straight line with constant velocity $\mathbf{v}_0$.

It is interesting to recall Newton's comments in Principia that follow the First Law [6]:

> Projectiles continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continuously drawn aside from rectilinear motion, do not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

The first part of Newton's comments regarding the projectile motion is problematic from an empirical perspective. No experiment can be devised where a projectile will move in the absence of gravity. Thus, there can be no cause free motion experiments in the context of Newtonian mechanics in order to observe what the resulting motion would be if the cause were to be removed. Therefore, it seems that Newton was referring to a thought experiment than to a well-established empirical fact. Furthermore, in the remaining part of Newton's comment regarding the First Law, things become even more interesting as he attempts to draw conclusions regarding the validity of the First Law from the motion of rotating bodies, such as spinning disks and planets. This is obviously a peculiar attempt for a connection between the rectilinear motion the First Law deals exclusively with, and rotational motion in the absence of a resisting medium. It appears that Newton's attempt to provide conclusive empirical support of the First Law is fraught with difficulties simply because no experiments can be devised from which the First Law can be inferred from the phenomena and rendered general by induction. This fact turns out to conflict with Newton's statement in the general scholium in book III of the Principia [6]:

> In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive forces of bodies, and the laws of motion and gravitation, were discovered.

The First Law and specifically the statement that bodies remain at rest or move uniformly in a straight line unless a force acts upon them, does not comply with the rules of the (experimental) philosophy Newton claims to abide with. The First Law does not deal with circular orbits, even if such orbits were employed by Newton as an example in his attempt to justify it. The First Law is actually an axiom, which must be accepted without proof, and not a statement derived via the use of inductive methodology. This is again due to the fact that no experiment can be devised on our planet for the purpose of observing what the motion of a projectile would be when there is no force acting upon it. According to Newton's Law of Universal Gravitation, gravity forces act upon a body unless it is set in motion in a region of space sufficiently far away from the influence of other bodies. Is then Newton alluding to the possibility of the existence of a more general First Law similar to the law of inertia of section 3? Let us recall what Poincaré said [2]:

> *The Principle of Inertia*. — A body under the action of no force can only move uniformly in a straight line. Is this a truth imposed on the mind *à priori*? If this be so, how is it that the Greeks have ignored it? How could they have believed that motion ceases with the cause of motion? Or, again, that every body, if there is nothing to prevent it, will move in a circle, the noblest of all forms of motion? If it be said that the velocity of a body cannot change, or there is no reason for it to change, may we not just as legitimately maintain that the position of a body cannot change, or that the curvature of its path cannot change, without the agency of an external cause? Is, then, the principle of inertia, which is not an *à priori* truth, an experimental fact? Have there ever been experiments on bodies acted on by no forces? And, if so, how did we know that no forces were acting?

Poincaré continues with his discussion of the principle of inertia by stating that

> Newton's First Law could be the consequence of a more general principle, of which the principle of inertia is only a particular case.

In turn, I argue that the axiom of motion, equation (1), can serve the role of this more general principle and Newton's First Law is indeed a special case of a more general law of inertia, such as the one derived in section 3.

Thus, I essentially argue that Newton's First Law makes reference to phenomena that are just two possibilities within a broader range of possibilities mandated by a more general principle of inertia, such as the law of inertia of section 3. As I will demonstrate in the proceedings, the same holds true with Newton's Third Law. There, matters are even clearer regarding my argument that Newton's laws are just a special case of the laws presented in section 3.

4.2 Newton's Second Law: The metaphysical cause of motion

The mathematical expression of Newton's Second Law, after a suitable choice of units is made is the following [3]:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) \,. \tag{25}$$

With the Second Law, Newton defines force as the cause of motion and equates it to the time rate of change of momentum. The laws of motion presented in section 3, based on the axiom of motion, challenge the notion that the Newtonian force is the cause of motion and the metaphysical foundation of mechanics. However, in these laws of motion, the metaphysics of force are replaced by those of the time rate of change of kinetic energy, also known as *power*. In a way analogous to Newton's Second Law, the axiom of motion stated in section 2 can be expressed as follows

$$P = \frac{d(E_k)}{dt} \,, \tag{26}$$

where P is the (instantaneous) power and E_k the kinetic energy of a particle.

> When we say force is the cause of motion, we are talking metaphysics...

writes Poincaré in *Science and Hypothesis* [2]. This statement made by Poincaré also applies when the time rate of change of kinetic energy, or power, is defined as the cause of motion. Whether using force or power, the physics of the associated laws of motion must be grounded in some metaphysics and this is done in section 6. It is important to understand that the particular choice of a quantity to assume the role of the cause of motion becomes the link between the empirical world of physics and the metaphysics of what exists and is real. Thus, although one can choose either force or power as the basis of stating laws of motion, the metaphysical foundations of such laws will turn out to be profoundly different. Newton used his notion of force to ground his physics in the metaphysics of absolute space and time. In section 6, I will discuss how the notion of power grounds the physics of the laws of motion of section 3 in the metaphysics of a modern version of Cartesian occasionalism and a dual space-time account. It turns out that the view of the world implied by such metaphysics is very different from the Newtonian or Leibnizian ones.

Besides the difference in metaphysics, the alternative to Newton's second law given by equation (26) offers an advantage in resolving some philosophical issues regarding the foundations of Classical Mechanics and in particular the need to consider fictitious forces when applying Newton's Second Law in non-inertial reference frames. In the case of observers at rest in accelerated reference frames in either rectilinear or uniform circular motion, the time rate of change of kinetic energy is zero and thus no additional fictitious power cause is needed to explain the state of motion. Again, this is only true if power is defined as the cause of motion. If force is defined as the cause of motion then in both non-inertial reference frames mentioned fictitious causes must be considered. Specifically, in the case of rectilinear motion, observers at rest in an accelerated frame must assume inertial fictitious forces acting and in the case of observers at rest in a uniformly rotating reference frame, centrifugal forces acting must be assumed.

The same conclusion holds in the case of fictitious Coriolis forces acting on freely moving particles in rotating reference frames. Since such fictitious forces are always orthogonal to the velocity of a particle in motion, for rotating observers it turns out that the time rate of change of kinetic energy of the particle is equal to zero, as obtained by equation (1). The same result is true for observers at rest since in that case the time rate of change of momentum of a freely moving particle is zero. Fictitious forces need to be considered regardless of whether force or power is defined as the cause of motion when a force analysis is carried out. However, when power is defined as the cause of motion, there are no philosophical issues arising from the need to consider fictitious causes of motion in non-inertial reference frames and this is the point just made. Thus, the transition from force to power as the cause of motion leads to a compatibility with the epistemological principle which states that every phenomenon is to receive the same interpretation from any given moving coordinate system. This epistemological principle also plays an important role in the axiomatic foundation of the theory of relativity [7].

4.3 Newton's Third Law: a special case of a more general action-reaction law?

Newton's Third Law may be deduced from the law of interaction of section 3 and in particular from equation (24) of corollary III. In the scholium following the Laws of Motion, Newton attempts to provide additional support for the Third Law through a host of observations related to various modes of mechanical interaction between bodies. From the closing comments in the scholium, some interesting conclusions can be drawn [6]:

> ...But to treat of mechanics is not my present business. I was aiming to show by those examples the greater extent and certainty of the third Law of Motion. *For if we estimate the action of the agent from the product of its force and velocity* and likewise the reaction of the impediment from the product of the velocities of its several parts, and the forces of resistance arising from friction, cohesion, weight, and acceleration of those parts, the action and reaction in the use of all sorts of machines will be found always equal to one another. And so far as the action is propagated by the intervening instruments, and at last impressed upon the resisting body, the ultimate action will be always contrary to the reaction. (Italics added)

It is clear that Newton was well aware of the product of velocity and force being a measure of action and of reaction, as defined in the law of interaction of section 3. Newton actually made use of the law of interaction in his scholium above to justify some particular situations where his Third

Law of action-reaction does not apply directly. But why is it the case that Newton stated his Third Law in terms of forces and not in terms of the product of force and velocity he mentions in his scholium quoted above? Why does it appear that a more general law was used to justify some particular situations Newton's Third Law does not directly apply to, but the latter was stated as a law of mechanics? The answer can be found in the attempt to model gravity in Newtonian mechanics as the effect of mutual attraction caused by central forces acting at a distance. The Third Law had to be stated in terms of the mutual action-reaction forces being equal in magnitude and opposite in direction to justify the particular form of Newton's Law of Universal Gravitation. But again, the Third Law fails the requirement set forth by the rules of the experimental philosophy of Newton, for it being deduced from the phenomena; it is just another axiom that must be accepted without proof. Forces acting on different bodies, and especially celestial ones, cannot be experimentally determined to be equal. Only forces acting on the same body can be determined to be equal by experiment.

I have shown that even Newton himself made both indirect and direct use of the notion of power in an attempt to provide a general justification of his Third Law. Can we simply assume that Newton was unaware that there is a single principle that could serve as the basis of a system of laws of mechanics that are in a certain way more general than his laws? I suspect that he was aware of it. But the consequences from stating laws based on this principle of motion would be devastating on the metaphysics of force. If force were to be just an intellectual construction and not the cause of motion, then Newton's whole system of the world was at stake. Motion then would have to be explained based on some other metaphysics, such as Cartesian occasionalism for example and the notion that all causes are due to God, or Spinoza's doctrine that everything is a mode of God [8], or even Leibniz's notion of a living force.

5 Power versus vis viva

Leibniz rejected the doctrine of Cartesian occasionalism and Newtonian substantivalism but his efforts to ground his relationism on the metaphysics of a living force were also met with difficulties. Leibniz realized that for motion to be real, it must be grounded on something that is not mere relation, something absolute and unobservable that serves as its cause [8]. Leibniz stated his laws of motion in his unpublished during his lifetime work *Dynamica de Potentia et Legibus Naturae Corporeae* in which he attempted to explain the world in terms of the conservation laws of vis viva and momentum of colliding bodies.

The laws of inertia and interaction of section 3 were derived from the axiom of motion of section 2. The latter is related to the living force, or vis viva, defined by Leibniz as being a real metaphysical property of a substance. Leibniz measure of vis viva is the quantity $m\mathrm{v}^2$, in contrast to the Cartesian definition of the *quantity of motion* being equal to size multiplied by speed, and later redefined by Newton as being equal to the product of mass and velocity. In turn, the axiom of motion stated in section 2 is related to the time rate of change of vis viva, the quantity Leibniz argued is conserved and a real metaphysical property of a substance, in an effort to support his relational account of space-time.

Leibniz's definition of vis viva as a real metaphysical property of a substance is fraught with difficulties. Roberts has argued that, in his later communications with Samuel Clarke, who was a defender of Newton's substantivalism, Leibniz seems to commit to a richer space-time structure that can support absolute velocities [9]. Roberts' work has cast light into a little known, or maybe misinterpreted, aspect of Leibniz's metaphysics. Specifically, into Leibniz's efforts to come up with laws of motion based on vis viva being a measure of force, while at the same time his relationism implies a space-time structure that is a well-founded phenomenon. This might be an indication of Leibniz's later realization that relationism fails unless absolute velocities are supported by a richer space-time structure than what is commonly referred to as Leibnizian space-time. In section 6, I define an account of space-time that can support relationism and absolute velocities in an attempt to ground the physics of the axiom and laws of motion in the metaphysics of power.

Along these lines, in a similar way to the link between the Newtonian force and momentum, the former being the time rate of change of the latter, I argue that vis viva is actually a quantity of motion and power, its time rate of change, is the cause of motion. In this way the similarities between the laws of conservation of momentum and vis viva become evident, because they are both defined as quantities of motion. In essence, I argue, the time rate of change of vis viva is the real metaphysical cause of motion. Of course, such a switch in the definitions is not compatible with Leibniz's metaphysics. This is because the time rate of change of the kinetic energy of a body moving with constant linear velocity, or even in uniform circular motion, is zero. A zero of something cannot assume the role of a real metaphysical property of a substance and the cause of motion in a Leibnizian world. Despite these metaphysical difficulties I will deal with in more detail in the next section, on the physics side it is clear that the laws of motion of section 3 were derived from a quantity that is proportional to the time derivative of vis viva. Thus, they have a direct link to Leibniz's Laws of Motion [8]. Specifically, Leibniz's laws of conservation of vis viva and momentum can be derived from the laws of inertia and interaction of section 3, respectively, but the details are left out.

6 The metaphysics of power

Before I discuss the metaphysics of power and specifically the notion that power is the cause of motion, I will briefly review the philosophical debate about the ontology of space-time. I argue that the space-time debate and the debate about the cause of motion are closely related in the sense that an answer to the former provides an answer to the latter. Thus, I essentially argue that the space-time debate is not a mere philosophical one and its resolution will have a decisive impact on which laws of motion and gravitation are assigned the status of "laws of nature" as opposed to that of mere heuristics.

6.1 The space-time debate

The publication of Newton's Principia in 1686 was the cause of the start of one of the most interesting debates in the history of the philosophy of science, dealing mainly with the ontology of space-time. Leibniz ignited the debate by arguing that Newton's substantival space-time, the notion that space and time exist independently of material things and their spatiotemporal relations, was not a well-founded phenomenon. Leibniz confronted Newtonian substantivalists with his relationism, based on which space is defined as the set of (possible) relations among material things and the only well-defined quantities of motion are relative ones [10]. Newton just grounded his physics in the metaphysics of force and absolute space and time. For Newton, the only well-defined quantities of motion are the absolute ones, like absolute position, velocity and acceleration. Substantivalism and relationism then appear in modern literature as two completely different accounts of space-time.

The key issue regarding the space-time debate, which is still alive by the way, is whether it does really make sense to speak of *either* a substantival *or* a relational account of space-time. Since diametrically opposite views of this kind have only led to sharp conflict and irreconcilable differences, maybe it would make sense to investigate whether both a substantival and relational space-time is a possibility. This two-level approach seems not to have been considered seriously because it implies a superfluous world. However, both Newtonian substantivalism and Leibnizian relationism are fraught with difficulties. On one hand, the metaphysics of Newtonian force require the postulation of unobservables, like absolute space. On the other hand, in Leibniz's relationism, for motion to be real, it must be grounded in something that is not mere relation, something absolute and unobservable that serves as its cause, what Leibniz called a vis viva [9]. The differences seem to reconcile when a two-level, or if I may call it a dual, space-time account is postulated and I will throw in here the term *substantival relationism*.

6.2 From cause-free motion to gravitation

The hypothesis about the duality of space-time just put forward is next examined in the context of gravitation and its observable effects, i. e. the motion of celestial bodies and free-falling particles. This step is of great importance since any laws of motion must account for all observable phenomena including those that are attributed to gravitation. Newton accomplished the step of grounding the physics of the Laws of Motion to his metaphysics of substantival space and universal time, by assuming that the cause of gravitation was also some type of force. Next, in what was a remarkable achievement in the history of science, he derived the famous Law of Universal Gravitation (LUG). In a similar way, I argue that power is the cause of gravitation in order to maintain a compatibility with the axiom and laws of motion of sections 2 and 3, respectively. Thus, the time rate of change of a potential energy function $E_p(r)$ is the cause of gravitation and equation (1), the axiom of motion, becomes

$$\frac{dE_k}{dt} = \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{r}}{dt} = -\frac{dE_p}{dt}\,. \tag{27}$$

The law of conservation of mechanical energy can be derived from equation (27) as follows:

$$\begin{gathered}\frac{dE_k}{dt} = -\frac{dE_p}{dt} \;\Leftrightarrow\; \frac{d}{dt}(E_k + E_p) = 0 \;\Leftrightarrow \\ \Leftrightarrow\; E_k + E_p = \text{const}\,.\end{gathered} \tag{28}$$

The Law of Universal Gravitation may be restated as follows:

Law of Universal Gravitation: All particles move in such a way as for the time rate of change of their kinetic energy to be equal the time rate of change of their potential energy.

In fact, I argue that Newton's Law of Universal Gravitation is a statement about the form of the potential function $E_p(r)$ in equation (27) and thus it can assume a variety of interpretations regarding mechanisms giving rise to it. If we postulate that energy transfer affects all particles in motion, in accordance with equation (27), this can support the hypothesis that gravitation is the result of energy transfer between all bodies in motion with some substance. Substantival space-time can serve the role of this substance and can facilitate the energy transfer to and from all bodies in motion and in such a way that all spatiotemporal quantities evolve according to certain rules giving rise to the well-known potential function $E_p(r)$ first discovered by Newton.

Since the above metaphysics are compatible with the concept of a mechanical universe, one could then postulate the existence of some type of mechanism that facilitates the transfer of energy between all bodies in motion and substantival space-time. This mechanism must be part of the substance level, whereas at the phenomenal level its effect is the observed motions. According to this dual scheme, at the phenomenal level the only well-founded quantities of motion are relative ones and space-time is relational, whereas, at the substance level, the only well-defined quantities of motion are the absolute ones and the space-time is substantival.

6.3 A new foundation of mechanics

The hypothesis just made, attributing gravitation to energy transfer between all bodies in motion and substantival space-time requires that at every instance something must accomplish this task and bring about the perceived effects. I will relate this to occasionalism in the following way: according to Nicolas Malebranche and other seventeenth-century Cartesian occasionalists, what we actually call causes are really no more than *occasions* on which, in accordance with his own laws, God acts to bring about the effect [11]. If one were to replace the notion of God by the notion of a mechanism, then a modern (or mechanical) occasionalist could assert that what we actually call causes are no more than occasions on which a mechanism acts to bring about the effect. In this sense we immediately resolve two more issues: first, time emerges as an ordered progression of instances, or nows, on which the mechanism acts to bring about the effect. Then, the matter-in-motion picture [1] is better illuminated by asserting that all motion and interactions of material bodies are facilitated by a mechanism that operates based on its own rules rather than taking place due to forces or based on rules inherent in the bodies themselves.

The concept of time as a collection of nows is in fact similar to that found in Barbour [12]. The main difference with the view I express here is that time emerges due to the actions of a mechanism hidden in substantival space-time in an orderly fashion and has a direction, i. e. there is an arrow of time. More importantly, the universal clock of Newton is now part of the mechanism that resides in substantival space-time but at the phenomenal level time and motion cannot be separated because there is no motion without time and no time without motion, i. e. time and motion are inextricably related.

What I argue essentially is that gravitation has an external cause to the phenomenal level and space-time is a substance of some kind that facilitates the energy transfer required for the manifestation of gravitational effects. These ideas may not be completely new. What is new here is the derivation of a system of laws of motion based on the notion of power. Power allows grounding the physics that all phenomena are caused by energy transfer, including those attributed to gravitation, to the metaphysics of substantival space-time being a giant mechanism and a substance. Since the times of the Greeks, Anaximander of Miletus (c. 650 BCE) expressed the view that

> The apeiron, from which the elements are formed, is something that is different (from the elements).

Then, Newton argued that all motion must be referenced to an absolute, unobservable space. Even in general relativity space-time retains its substantival account and it exists independently of the events occurring in it [10]. Baker has argued that the space-time of general relativity must be a substance and attempts to support this claim of his based on the observed expansion of the Universe [13]. Baker's argument about the requirement of a carrier of gravitational energy from its source to a detector, if it is to be compelling, must apply to all forms of energy transfer traditionally assumed to take place in vacuum. But such generalization can be further coupled with the hypothesis that some causes are external to the world of observable phenomena. In Wiithrich there are references made to the hypothesis that gravity forces have an external cause in an attempt to explain the failure in quantizing the field equations of general relativity [14]. Thus, arguments have already been made in favor of the hypothesis that space-time is some kind of a substance and that any causal connections attributed to gravitation are apparent. Usually, arguments leading to such provocative hypotheses are treated at the level of epistemological skepticism but as McCabe argues the hypothesis, for instance, that our universe is part of a computer simulation implementation generates empirical predictions and it is therefore a falsifiable hypothesis [15]. One question that arises from this discussion is the following: does the existence of external causes imply that our world is some type of virtual reality? My own answer to this important question is both yes and no. Yes, because according to the hypothesis there are external causes to the world of perceived phenomena and thus part of another world. No, because a cause being external and unobservable does not preclude it being part of an all-encompassing entity, which we can call Universe. Therefore, the answer to the question seems to depend on how one defines *Universe*. But the presence of external causes to the world of observable phenomena must not be rejected *a priori* on the basis that it leads to the provocative virtual reality hypothesis and experimental physics must pursue seriously its falsification or corroboration. Although such task is highly challenging, the state-of-the-art in precision instrumentation has reached levels that allow the initiation of a program of this nature.

7 Summary

The axiom and laws of motion presented in sections 2 and 3, respectively, are:

Axiom of Motion: The time rate of change of the kinetic energy of a particle is the scalar product of its velocity and time rate of change of its momentum.

Law of Inertia: If the time rate of change of the kinetic energy of a particle is zero, the particle will continue in its state of motion.

Law of Interaction: To every action there is an equal and opposite reaction; that is, in an isolated system of two particles acting upon each other, the mutual time rate of change of kinetic energies are equal in magnitude and opposite in sign.

A restatement of the Law of Universal Gravitation was presented in section 6 as follows:

Law of Universal Gravitation: All particles move in such a way as for the time rate of change of their kinetic energy to be equal to the time rate of change of their potential energy.

In section 4, I argued that the above laws of motion are, in a certain sense, more general than Newton's, and that this claim is even supported by Newton's own writings, especially in the case of the Third Law. Furthermore, in section 5, I discussed the relation of the axiom and laws of motion to Leibniz's laws of the conservation of vis viva and momentum. I argued that kinetic energy can be defined as a quantity of motion and its time derivative as the cause of motion, in a similar way to the Newtonian force being the time derivative of momentum and a postulated cause of motion.

In section 6, I discussed how the axiom and laws of motion of sections 2 and 3, combined further with a modified version of Cartesian occasionalism and a dual space-time account form an alternative foundation of classical mechanics in the context of a mechanical Universe. Specifically, I proposed a substantival-relational account of space-time and a mechanism residing in the substance level whose actions coordinate all motion and interactions. I argued that the proposed foundation supports the hypothesis about gravitation being the effect of energy transfer between all bodies in motion and substantival space-time and I stated a version of the Law of Universal Gravitation which is compatible with the hypothesis that power is the cause of motion. These metaphysics also provide solutions to some foundational problems of Classical Mechanics, such as the matter-in-motion picture and the emergence and direction of time. Finally, I briefly referred to the ramifications on the nature of our physical reality when the cause of gravitation is considered part of an unobservable substance. I argued that the soundness of the virtual reality or computer simulation hypothesis depends on how Universe is defined. The fact that such hypothesis about the nature of our reality is provocative should not be an excuse for rejecting *a priori* external causes of motion and gravitation. Theoretical physicists ought to seriously investigate new models incorporating such assumptions about the nature of our physical reality and experimental physicists should pursue their falsification.

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Hydrodynamic Covariant Symplectic Structure from Bilinear Hamiltonian Functions

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Starting from generic bilinear Hamiltonians, constructed by covariant vector, bivector or tensor fields, it is possible to derive a general symplectic structure which leads to holonomic and anholonomic formulations of Hamilton equations of motion directly related to a hydrodynamic picture. This feature is gauge free and it seems a deep link common to all interactions, electromagnetism and gravity included. This scheme could lead toward a full canonical quantization.

1 Introduction

It is well known that a self-consistent quantum field theory of space-time (quantum gravity) has not been achieved, up to now, using standard quantization approaches. Specifically, the request of general coordinate invariance (one of the main features of General Relativity) gives rise to unescapable troubles in understanding the dynamics of gravitational field. In fact, for a physical (non-gravitational) field, one has to assign initially the field amplitudes and their first time derivatives, in order to determine the time development of such a field considered as a dynamical entity. In General Relativity, these quantities are not useful for dynamical determination since the metric field $g_{\alpha\beta}$ can evolve at any time simply by a general coordinate transformation. No change of physical observables is the consequence of such an operation since it is nothing else but a relabelling under which the theory is invariant. This apparent "shortcoming" (from the quantum field theory point of view) means that it is necessary a separation of metric degrees of freedom into a part related to the true dynamical information and a part related only to the coordinate system. From this viewpoint, General Relativity is similar to classical Electromagnetism: the coordinate invariance plays a role analogous to the electromagnetic gauge invariance and in both cases (Lorentz and gauge invariance) introduces redundant variables in order to insure the maintenance of transformation properties. However, difficulties come out as soon as one try to disentangle dynamical from gauge variables. This operation is extremely clear in Electromagnetism while it is not in General Relativity due to its intrinsic non-linearity. A determination of independent dynamical modes of gravitational field can be achieved when the theory is cast into a canonical form involving the minimal number of degrees of freedom which specify the state of the system. The canonical formalism is essential in quantization program since it leads directly to Poisson bracket relations among conjugate variables. In order to realize it in any fundamental theory, one needs first order field equations in time derivatives (Hamilton-like equations) and a $(3+1)$-form of dynamics where time has been unambiguously singled out. In General Relativity, the program has been pursued using the first order Palatini approach [6], where metric $g_{\alpha\beta}$ is taken into account independently of affinity connections $\Gamma^{\gamma}_{\alpha\beta}$ (this fact gives rise to first order field equations) and the so called ADM formalism [7] where $(3+1)$-dimensional notation has led to the definition of gravitational Hamiltonian and time as a conjugate pair of variables. However, the genuine fundament of General Relativity, the covariance of all coordinates without the distinction among space and time, is impaired and, despite of innumerable efforts, the full quantization of gravity has not been achieved up to now. The main problems are related to the lack of a well-definite Hilbert space and a quantum concept of measure for $g_{\alpha\beta}$. An extreme consequence of this lack of full quantization for gravity could be related to the dynamical variables: very likely, the true variables could not be directly related to metric but to something else as, for example, the connection $\Gamma^{\gamma}_{\alpha\beta}$. Despite of this lack, a covariant symplectic structure can be identified also in the framework of General Relativity and then also this theory could be equipped with the same features of other fundamental theories. This statement does not still mean that the identification of a symplectic structure immediately leads to a full quantization but it could be a useful hint toward it.

The aim of this paper is to show that a prominent role in the identification of a covariant symplectic structure is played by bilinear Hamiltonians which have to be conserved. In fact, taking into account generic Hamiltonian invariants, constructed by covariant vectors, bivectors or tensors, it is possible to show that a symplectic structure can be achieved in any case. By specifying the nature of such vector fields (or, in general, tensor invariants), it gives rise to intrinsically symplectic structure which is always related to Hamilton-like equations (and a Hamilton-Jacobi-like approach is always found). This works for curvature invariants, Maxwell theory and so on. In any case, the only basic assumption is that conservation laws (in Hamiltonian sense) have to be identified in the framework of the theory.

The layout of the paper is the following. In Sec.II, we give the generalities on the symplectic structure and the canonical description of mechanics. Sec.III is devoted to the discussion of symplectic structures which are also generally covariant. We show that a covariant analogue of Hamilton equations can be derived from covariant vector (or tensor) fields in holonomic and anholonomic coordinates. In Sec. IV, the covariant symplectic structure is casted into the hydrodynamic picture leading to the recovery of the covariant Hamilton equations. Sec.V is devoted to applications, discussion and conclusions.

2 Generalities on the Symplectic Structure and the Canonical description

In order to build every fundamental theory of physics, it is worth selecting the symplectic structure of the manifold on which such a theory is formulated. This goal is achieved if suitable symplectic conjugate variables and even-dimensional vector spaces are chosen. Furthermore, we need an antisymmetric, covariant tensor which is non-degenerate.

We are dealing with a symplectic structure if the couple

$$\{\mathbf{E_{2n}}, \mathbf{w}\}, \tag{1}$$

is defined, where $\mathbf{E_{2n}}$ is a vector space and the tensor $\mathbf{w}$ on $\mathbf{E_{2n}}$ associates scalar functions to pairs of vectors, that is

$$[\mathbf{x}, \mathbf{y}] = \mathbf{w}(\mathbf{x}, \mathbf{y}), \tag{2}$$

which is the *antiscalar* product. Such an operation satisfies the following properties

$$[\mathbf{x}, \mathbf{y}] = -[\mathbf{y}, \mathbf{x}] \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbf{E_{2n}} \tag{3}$$

$$[\mathbf{x}, \mathbf{y}+\mathbf{z}] = [\mathbf{x}, \mathbf{y}] + [\mathbf{xz}] \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{E_{2n}}, \tag{4}$$

$$\mathrm{a}[\mathbf{x}, \mathbf{y}] = [\mathrm{a}\mathbf{x}, \mathbf{y}] \qquad \forall \mathrm{a} \in R, \quad \mathbf{x}, \mathbf{y} \in \mathbf{E_{2n}} \tag{5}$$

$$[\mathbf{x}, \mathbf{y}] = 0 \qquad \forall y \in \mathbf{E_{2n}} \;\Rightarrow\; \mathbf{x} = 0 \tag{6}$$

$$[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] + [\mathbf{y}, [\mathbf{z}, \mathbf{x}]] + [\mathbf{z}, [\mathbf{x}, \mathbf{y}]] = 0\,. \tag{7}$$

The last one is the Jacobi cyclic identity.

If $\{\mathbf{e}_i\}$ is a vector basis in $\mathbf{E_{2n}}$, the antiscalar product is completely singled out by the matrix elements

$$w_{ij} = [\mathbf{e}_i, \mathbf{e}_j]\,, \tag{8}$$

where $\mathbf{w}$ is an antisymmetric matrix with determinant different from zero. Every antiscalar product between two vectors can be expressed as

$$[\mathbf{x}, \mathbf{y}] = w_{ij}\, x^i y^j, \tag{9}$$

where x^i and y^j are the vector components in the given basis.

The form of the matrix $\mathbf{w}$ and the relation (9) become considerably simpler if a canonical basis is taken into account for $\mathbf{w}$. Since $\mathbf{w}$ is an antisymmetric non-degenerate tensor, it is always possible to represent it through the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{10}$$

where I is a $(n \times n)$ unit matrix. Every basis where $\mathbf{w}$ can be represented through the form (10) is a *symplectic basis*. In other words, the symplectic bases are the canonical bases for any antisymmetric non-degenerate tensor $\mathbf{w}$ and can be characterized by the following conditions:

$$[\mathbf{e}_i, \mathbf{e}_j] = 0\,, \quad [\mathbf{e}_{n+i}, \mathbf{e}_{n+j}] = 0\,, \quad [\mathbf{e}_i, \mathbf{e}_{n+j}] = \delta_{ij}\,, \tag{11}$$

which have to be verified for every pair of values i and j ranging from 1 to n.

Finally, the expression of the antiscalar product between two vectors, in a symplectic basis, is

$$[\mathbf{x}, \mathbf{y}] = \sum_{i=1}^{n} \left(x^{n+i} y^i - x^i y^{n+i}\right), \tag{12}$$

and a symplectic transformation in $\mathbf{E_{2n}}$ leaves invariant the antiscalar product

$$\mathbf{S}[\mathbf{x}, \mathbf{y}] = [\mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{y})] = [x, y]. \tag{13}$$

It is easy to see that standard Quantum Mechanics satisfies such properties and so it is endowed with a symplectic structure.

On the other hand a standard canonical description can be sketched as follows. For example, the relativistic Lagrangian of a charged particle interacting with a vector field $A(q; s)$ is

$$\mathcal{L}(q, u; s) = \frac{mu^2}{2} - eu \cdot A(q; s), \tag{14}$$

where the scalar product is defined as

$$z \cdot w = z_\mu w^\mu = \eta_{\mu\nu} z^\mu w^\nu, \tag{15}$$

and the signature of the Minkowski spacetime is the usual one with

$$z_\mu = \eta_{\mu\nu} z^\nu, \qquad \widehat{\eta} = \mathrm{diag}(1, -1, -1, -1). \tag{16}$$

Furthermore, the contravariant vector u^μ with components $u = \left(u^0, u^1, u^2, u^3\right)$ is the four-velocity

$$u^\mu = \frac{\mathrm{d}q^\mu}{\mathrm{d}s}\,. \tag{17}$$

The canonical conjugate momentum π^μ is defined as

$$\pi^\mu = \eta^{\mu\nu}\frac{\partial \mathcal{L}}{\partial u^\nu} = mu^\mu - eA^\mu\,, \tag{18}$$

so that the relativistic Hamiltonian can be written in the form

$$\mathcal{H}(q,\pi;s) = \pi\cdot u - \mathcal{L}(q,u;s). \tag{19}$$

Suppose now that we wish to use any other coordinate system x^α as Cartesian, curvilinear, accelerated or rotating one. Then the coordinates q^μ are functions of the x^α, which can be written explicitly as

$$q^\mu = q^\mu(x^\alpha). \tag{20}$$

The four-vector of particle velocity u^μ is transformed according to the expression

$$u^\mu = \frac{\partial q^\mu}{\partial x^\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = \frac{\partial q^\mu}{\partial x^\alpha}v^\alpha, \tag{21}$$

where

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,. \tag{22}$$

is the transformed four-velocity expressed in terms of the new coordinates. The vector field A^μ is also transformed as a vector

$$\mathcal{A}^\mu = \frac{\partial x^\mu}{\partial q^\alpha}A^\alpha. \tag{23}$$

In the new coordinate system x^α the Lagrangian (14) becomes

$$\mathcal{L}(x,v;s) = g_{\mu\nu}\left[\frac{m}{2}v^\mu v^\nu - ev^\mu\mathcal{A}^\nu(x;s)\right], \tag{24}$$

where

$$g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial q^\mu}{\partial x^\alpha}\frac{\partial q^\nu}{\partial x^\beta}. \tag{25}$$

The Lagrange equations can be written in the usual form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\mathcal{L}}{\partial v^\lambda}\right) - \frac{\partial\mathcal{L}}{\partial x^\lambda} = 0\,. \tag{26}$$

In the case of a free particle (no interaction with an external vector field), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(g_{\lambda\mu}v^\mu) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}v^\mu v^\nu = 0\,. \tag{27}$$

Specifying the covariant velocity v_λ as

$$v_\lambda = g_{\lambda\mu}v^\mu, \tag{28}$$

and using the well-known identity for connections $\Gamma^\alpha_{\mu\nu}$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\alpha_{\lambda\mu}g_{\alpha\nu} + \Gamma^\alpha_{\lambda\nu}g_{\alpha\mu}\,, \tag{29}$$

we obtain

$$\frac{\mathrm{D}v_\lambda}{\mathrm{D}s} = \frac{\mathrm{d}v_\lambda}{\mathrm{d}s} - \Gamma^\mu_{\lambda\nu}v^\nu v_\mu = 0\,. \tag{30}$$

Here $\mathrm{D}v_\lambda/\mathrm{D}s$ denotes the covariant derivative of the covariant velocity v_λ along the curve $x^\nu(s)$. Using Eqs. (28) and (29) and the fact that the affine connection $\Gamma^\lambda_{\mu\nu}$ is symmetric in the indices μ and ν, we obtain the equation of motion for the contravariant vector v^λ

$$\frac{\mathrm{D}v^\lambda}{\mathrm{D}s} = \frac{\mathrm{d}v^\lambda}{\mathrm{d}s} + \Gamma^\lambda_{\mu\nu}v^\mu v^\nu = 0\,. \tag{31}$$

Before we pass over to the Hamiltonian description, let us note that the generalized momentum p_μ is defined as

$$p_\mu = \frac{\partial\mathcal{L}}{\partial v^\mu} = mg_{\mu\nu}v^\nu, \tag{32}$$

while, from Lagrange equations of motion, we obtain

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}s} = \frac{\partial\mathcal{L}}{\partial x^\mu}. \tag{33}$$

The transformation from $(x^\mu, v^\mu; s)$ to $(x^\mu, p_\mu; s)$ can be accomplished by means of a Legendre transformation, and instead of the Lagrangian (24), we consider the Hamilton function

$$\mathcal{H}(x,p;s) = p_\mu v^\mu - \mathcal{L}(x,v;s). \tag{34}$$

The differential of the Hamiltonian in terms of x, p and s is given by

$$\mathrm{d}\mathcal{H} = \frac{\partial\mathcal{H}}{\partial x^\mu}\mathrm{d}x^\mu + \frac{\partial\mathcal{H}}{\partial p_\mu}\mathrm{d}p_\mu + \frac{\partial\mathcal{H}}{\partial s}\mathrm{d}s. \tag{35}$$

On the other hand, from Eq.(34), we have

$$\mathrm{d}\mathcal{H} = v^\mu\mathrm{d}p_\mu + p_\mu\mathrm{d}v^\mu - \frac{\partial\mathcal{L}}{\partial v^\mu}\mathrm{d}v^\mu - \frac{\partial\mathcal{L}}{\partial x^\mu}\mathrm{d}x^\mu - \frac{\partial\mathcal{L}}{\partial s}\mathrm{d}s. \tag{36}$$

Taking into account the defining Eq.(32), the second and the third term on the right-hand-side of Eq.(36) cancel out. Eq.(33) can be further used to cast Eq.(36) into the form

$$\mathrm{d}\mathcal{H} = v^\mu\mathrm{d}p_\mu - \frac{\mathrm{d}p_\mu}{\mathrm{d}s}\mathrm{d}x^\mu - \frac{\partial\mathcal{L}}{\partial s}\mathrm{d}s\,, \tag{37}$$

Comparison between Eqs.(35) and (37) yields the Hamilton equations of motion

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{\partial\mathcal{H}}{\partial p_\mu}\,, \qquad \frac{\mathrm{d}p_\mu}{\mathrm{d}s} = -\frac{\partial\mathcal{H}}{\partial x^\mu}\,, \tag{38}$$

where the Hamiltonian is given by

$$\mathcal{H}(x,p;s) = \frac{g^{\mu\nu}}{2m}p_\mu p_\nu + \frac{e}{m}p_\mu\mathcal{A}^\mu. \tag{39}$$

In the case of a free particle, the Hamilton equations can be written explicitly as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{g^{\mu\nu}}{m}p_\nu\,, \qquad \frac{\mathrm{d}p_\lambda}{\mathrm{d}s} = -\frac{1}{2m}\frac{\partial g^{\mu\nu}}{\partial x^\lambda}p_\mu p_\nu\,. \tag{40}$$

To obtain the equations of motion we need the expression

$$\frac{\partial g^{\mu\nu}}{\partial x^\lambda} = -\Gamma^\mu_{\lambda\alpha}g^{\alpha\nu} - \Gamma^\nu_{\lambda\alpha}g^{\alpha\mu}, \tag{41}$$

which can be derived from the obvious identity

$$\frac{\partial}{\partial x^\lambda}(g^{\mu\alpha}g_{\alpha\nu}) = 0\,, \tag{42}$$

and Eq.(29). From the second of Eqs. (40), we obtain

$$\frac{\mathrm{D}p_\lambda}{\mathrm{D}s} = \frac{\mathrm{d}p_\lambda}{\mathrm{d}s} - \Gamma^\mu_{\lambda\nu}v^\nu p_\mu = 0\,, \tag{43}$$

similar to equation (30). Differentiating the first of the Hamilton equations (40) with respect to s and taking into account equations (41) and (43), we again arrive to the equation for the geodesics (31).

Let us now show that on a generic curved (torsion-free) manifolds the Poisson brackets are conserved. To achieve this result, we need the following identities

$$g^{\mu\nu} = g^{\nu\mu} = \eta^{\alpha\beta}\frac{\partial x^\mu}{\partial q^\alpha}\frac{\partial x^\nu}{\partial q^\beta}\,, \tag{44}$$

$$\frac{\partial^2 x^\lambda}{\partial q^\alpha \partial q^\beta} = -\Gamma^\lambda_{\mu\nu}\frac{\partial x^\mu}{\partial q^\alpha}\frac{\partial x^\nu}{\partial q^\beta}\,, \tag{45}$$

■ To prove (45), we differentiate the obvious identity

$$\frac{\partial x^\lambda}{\partial q^\rho}\frac{\partial q^\rho}{\partial x^\nu} = \delta^\lambda_\nu\,. \tag{46}$$

As a result, we find

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial q^\rho}\frac{\partial^2 q^\rho}{\partial x^\mu \partial x^\nu} = -\frac{\partial q^\rho}{\partial x^\nu}\frac{\partial q^\sigma}{\partial x^\mu}\frac{\partial^2 x^\lambda}{\partial q^\rho \partial q^\sigma}.\,\blacksquare \tag{47}$$

The next step is to calculate the fundamental Poisson brackets in terms of the variables (x^μ, p_ν), initially defined using the canonical variables (q^μ, π_ν) according to the relation

$$[U,V] = \frac{\partial U}{\partial q^\mu}\frac{\partial V}{\partial \pi_\mu} - \frac{\partial V}{\partial q^\mu}\frac{\partial U}{\partial \pi_\mu}, \tag{48}$$

where $U(q^\mu, \pi_\nu)$ and $V(q^\mu, \pi_\nu)$ are arbitrary functions. Making use of Eqs.(18) and (21), we know that the variables

$$q^\mu \quad \Leftrightarrow \quad \pi_\mu = mu_\mu = m\eta_{\mu\nu}u^\nu = m\eta_{\mu\nu}\frac{\partial q^\nu}{\partial x^\alpha}v^\alpha, \tag{49}$$

form a canonical conjugate pair. Using Eq.(32), we would like to check whether the variables

$$x^\mu \quad \Leftrightarrow \quad p_\mu = mg_{\mu\nu}v^\nu = g_{\mu\nu}\eta^{\alpha\lambda}\pi_\lambda\frac{\partial x^\nu}{\partial q^\alpha}, \tag{50}$$

form a canonical conjugate pair. We have

$$\begin{aligned}[U,V] = \left[\frac{\partial U}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial q^\mu} + \frac{\partial U}{\partial p_\sigma}\eta^{\beta\lambda}\pi_\lambda\frac{\partial}{\partial q^\mu}\left(g_{\sigma\nu}\frac{\partial x^\nu}{\partial q^\beta}\right)\right] \times \\ \times \frac{\partial V}{\partial p_\alpha}g_{\alpha\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho} - \\ -\left[\frac{\partial V}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial q^\mu} + \frac{\partial V}{\partial p_\sigma}\eta^{\beta\lambda}\pi_\lambda\frac{\partial}{\partial q^\mu}\left(g_{\sigma\nu}\frac{\partial x^\nu}{\partial q^\beta}\right)\right] \times \\ \times \frac{\partial U}{\partial p_\alpha}g_{\alpha\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho}.\end{aligned} \tag{51}$$

The first and the third term on the right-hand-side of Eq.(51) can be similarly manipulated as follows

$$\begin{aligned}\textbf{I}\text{-st term} = \frac{\partial U}{\partial x^\alpha}\frac{\partial V}{\partial p_\beta}g_{\beta\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho}\frac{\partial x^\alpha}{\partial q^\mu} = \\ = g_{\beta\chi}g^{\chi\alpha}\frac{\partial U}{\partial x^\alpha}\frac{\partial V}{\partial p_\beta} = \frac{\partial U}{\partial x^\alpha}\frac{\partial V}{\partial p_\alpha}\,,\end{aligned} \tag{52}$$

$$\textbf{III}\text{-rd term} = -\frac{\partial V}{\partial x^\alpha}\frac{\partial U}{\partial p_\alpha}\,. \tag{53}$$

Next, we manipulate the second term on the right-hand-side of Eq.(51). We obtain

$$\begin{aligned}\textbf{II}\text{-nd term} = \frac{\partial U}{\partial p_\sigma}\frac{\partial V}{\partial p_\alpha}g_{\alpha\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho}\eta^{\beta\lambda}\pi_\lambda \times \\ \times\left[g_{\sigma\nu}\frac{\partial^2 x^\nu}{\partial q^\mu \partial q^\beta} + \frac{\partial x^\nu}{\partial q^\beta}\frac{\partial g_{\sigma\nu}}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial q^\mu}\right] = \\ = \frac{\partial U}{\partial p_\sigma}\frac{\partial V}{\partial p_\alpha}g_{\alpha\chi}\eta^{\rho\mu}\frac{\partial x^\chi}{\partial q^\rho}\eta^{\beta\lambda}\pi_\lambda \times \\ \times\left[-g_{\sigma\nu}\Gamma^\nu_{\gamma\delta}\frac{\partial x^\gamma}{\partial q^\mu}\frac{\partial x^\delta}{\partial q^\beta} + \frac{\partial x^\delta}{\partial q^\beta}\frac{\partial x^\gamma}{\partial q^\mu}\left(\Gamma^\nu_{\gamma\sigma}g_{\nu\delta} + \Gamma^\nu_{\gamma\delta}g_{\nu\sigma}\right)\right] = \\ = \frac{\partial U}{\partial p_\sigma}\frac{\partial V}{\partial p_\alpha}g_{\alpha\chi}g^{\chi\gamma}\eta^{\beta\lambda}\pi_\lambda\frac{\partial x^\delta}{\partial q^\beta}g_{\nu\delta}\Gamma^\nu_{\gamma\sigma} = \\ = \frac{\partial U}{\partial p_\sigma}\frac{\partial V}{\partial p_\beta}g_{\mu\nu}\eta^{\alpha\lambda}\pi_\lambda\frac{\partial x^\nu}{\partial q^\alpha}\Gamma^\mu_{\beta\sigma} = \Gamma^\lambda_{\mu\nu}p_\lambda\frac{\partial U}{\partial p_\nu}\frac{\partial V}{\partial p_\mu}.\end{aligned} \tag{54}$$

The fourth term is similar to the second one but with U and V interchanged

$$\textbf{IV}\text{-th term} = -\Gamma^\lambda_{\mu\nu}p_\lambda\frac{\partial U}{\partial p_\mu}\frac{\partial V}{\partial p_\nu}\,. \tag{55}$$

In the absence of torsion, the affine connection $\Gamma^\lambda_{\mu\nu}$ is symmetric with respect to the lower indices, so that the second and the fourth term on the right-hand-side of Eq.(51) cancel each other. Therefore,

$$[U,V] = \frac{\partial U}{\partial x^\mu}\frac{\partial V}{\partial p_\mu} - \frac{\partial V}{\partial x^\mu}\frac{\partial U}{\partial p_\mu}\,, \tag{56}$$

which means that the fundamental Poisson brackets are conserved. On the other hand, this implies that the variables $\{x^\mu, p_\nu\}$ are a canonical conjugate pair.

As a final remark, we have to say that considering a generic metric $g_{\alpha\beta}$ and a connection $\Gamma^{\alpha}_{\mu\nu}$ is related to the fact that we are passing from a Minkowski-flat spacetime (local inertial reference frame) to an accelerated reference frame (curved spacetime). In what follows, we want to show that a generic bilinear Hamiltonian invariant, which is conformally conserved, gives always rise to a canonical symplectic structure. The specific theory is assigned by the vector (or tensor) fields which define the Hamiltonian invariant.

3 A symplectic structure compatible with general covariance

The above considerations can be linked together leading to a more general scheme where a covariant symplectic structure is achieved. Summarizing, the main points which we need are: (i) an even-dimensional vector space $\mathbf{E_{2n}}$ equipped with an antiscalar product satisfying the algebra (3)-(7); (ii) generic vector fields defined on such a space which have to satisfy the Poisson brackets; (iii) first-order equations of motion which can be read as Hamilton-like equations; (iv) general covariance which has to be preserved.

Such a program can be pursued by taking into account covariant and contravariant vector fields. In fact, it is possible to construct the Hamiltonian invariant

$$\mathcal{H} = V^{\alpha}V_{\alpha}\,, \tag{57}$$

which is a scalar quantity satisfying the relation

$$\delta\mathcal{H} = \delta(V^{\alpha}V_{\alpha}) = 0\,, \tag{58}$$

being δ a spurious variation due to the transport. It is worth stressing that the vectors V^{α} and V_{α} are not specified and the following considerations are completely general. Eq.(57) is a so called "*already parameterized*" invariant which can constitute the "density" of a parameterized action principle where the time coordinate is not distinguished *a priori* from the other coordinates [8, 9].

Let us now take into account the intrinsic variation of V^{α}. On a generic curved manifold, we have

$$DV^{\alpha} = dV^{\alpha} - \delta V^{\alpha} = \partial_{\beta}V^{\alpha}dx^{\beta} - \delta V^{\alpha}, \tag{59}$$

where D is the intrinsic variation, d the total variation and δ the spurious variation due to the transport on the curved manifold. The spurious variation has a very important meaning since, in General Relativity, if such a variation for a given quantity is equal to zero, this means that the quantity is conserved. From the definition of covariant derivative, applied to the contravariant vector, we have

$$DV^{\alpha} = \partial_{\beta}V^{\alpha}dx^{\beta} + \Gamma^{\alpha}_{\sigma\beta}V^{\sigma}dx^{\beta}, \tag{60}$$

and

$$\nabla_{\beta}V^{\alpha} = \partial_{\beta}V^{\alpha} + \Gamma^{\alpha}_{\sigma\beta}V^{\sigma}, \tag{61}$$

and then

$$\delta V^{\alpha} = -\Gamma^{\alpha}_{\sigma\beta}V^{\sigma}dx^{\beta}. \tag{62}$$

Analogously, for the covariant derivative applied to the covariant vector,

$$DV_{\alpha} = dV_{\alpha} - \delta V_{\alpha} = \partial_{\beta}V_{\alpha}dx^{\beta} - \delta V_{\alpha}\,, \tag{63}$$

and then

$$DV_{\alpha} = \partial_{\beta}V_{\alpha}dx^{\beta} - \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}dx^{\beta}, \tag{64}$$

and

$$\nabla_{\beta}V_{\alpha} = \partial_{\beta}V_{\alpha} - \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}\,. \tag{65}$$

The spurious variation is now

$$\delta V_{\alpha} = \Gamma^{\sigma}_{\alpha\beta}V_{\sigma}dx^{\beta}. \tag{66}$$

Developing the variation (58), we have

$$\delta\mathcal{H} = V_{\alpha}\delta V^{\alpha} + V^{\alpha}\delta V_{\alpha}\,, \tag{67}$$

and

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = V_{\alpha}\frac{\delta V^{\alpha}}{dx^{\beta}} + V^{\alpha}\frac{\delta V_{\alpha}}{dx^{\beta}}, \tag{68}$$

which becomes

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = \frac{\delta V^{\alpha}}{dx^{\beta}}\frac{\partial\mathcal{H}}{\partial V^{\alpha}} + \frac{\delta V_{\alpha}}{dx^{\beta}}\frac{\partial\mathcal{H}}{\partial V_{\alpha}}, \tag{69}$$

being

$$\frac{\partial\mathcal{H}}{\partial V^{\alpha}} = V_{\alpha}, \qquad \frac{\partial\mathcal{H}}{\partial V_{\alpha}} = V^{\alpha}. \tag{70}$$

From Eqs.(62) and (66), it is

$$\frac{\delta V^{\alpha}}{dx^{\beta}} = -\Gamma^{\alpha}_{\sigma\beta}V^{\sigma} = -\Gamma^{\alpha}_{\sigma\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right), \tag{71}$$

$$\frac{\delta V_{\alpha}}{dx^{\beta}} = \Gamma^{\sigma}_{\alpha\beta}V_{\sigma} = \Gamma^{\sigma}_{\alpha\beta}\left(\frac{\partial\mathcal{H}}{\partial V^{\sigma}}\right), \tag{72}$$

and substituting into Eq.(69), we have

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = -\Gamma^{\alpha}_{\sigma\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\alpha}}\right) + \Gamma^{\sigma}_{\alpha\beta}\left(\frac{\partial\mathcal{H}}{\partial V_{\alpha}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\sigma}}\right), \tag{73}$$

and then, since α and σ are mute indexes, the expression

$$\frac{\delta\mathcal{H}}{dx^{\beta}} = \left(\Gamma^{\alpha}_{\sigma\beta} - \Gamma^{\alpha}_{\sigma\beta}\right)\left(\frac{\partial\mathcal{H}}{\partial V_{\sigma}}\right)\left(\frac{\partial\mathcal{H}}{\partial V^{\alpha}}\right) \equiv 0\,, \tag{74}$$

is identically equal to zero. In other words, $\mathcal{H}$ is absolutely conserved, and this is very important since the analogy with a canonical Hamiltonian structure
is straightforward. In fact, if, as above,

$$\mathcal{H} = \mathcal{H}(p,q) \tag{75}$$

is a classical generic Hamiltonian function, expressed in the canonical phase-space variables $\{p,q\}$, the total variation (in a vector space $\mathbf{E_{2n}}$ whose dimensions are generically given by p_i and q_j with $i, j = 1, ..., n$) is

$$d\mathcal{H} = \frac{\partial\mathcal{H}}{\partial q}dq + \frac{\partial\mathcal{H}}{\partial p}dp\,, \tag{76}$$

and

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial q}\dot{q} + \frac{\partial\mathcal{H}}{\partial p}\dot{p} = $$
$$= \frac{\partial\mathcal{H}}{\partial q}\frac{\partial\mathcal{H}}{\partial p} - \frac{\partial\mathcal{H}}{\partial p}\frac{\partial\mathcal{H}}{\partial q} \equiv 0\,, \tag{77}$$

thanks to the Hamilton canonical equations

$$\dot{q} = \frac{\partial\mathcal{H}}{\partial p}\,, \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial q}\,. \tag{78}$$

Such a canonical approach holds also in our covariant case if we operate the substitutions

$$V^\alpha \longleftrightarrow p \qquad\qquad V_\alpha \longleftrightarrow q \tag{79}$$

and the canonical equations are

$$\frac{\delta V^\alpha}{dx^\beta} = -\Gamma^\alpha_{\sigma\beta}\left(\frac{\partial\mathcal{H}}{\partial V_\sigma}\right) \quad\longleftrightarrow\quad \frac{dp}{dt} = -\frac{\partial\mathcal{H}}{\partial q}\,, \tag{80}$$

$$\frac{\delta V_\alpha}{dx^\beta} = \Gamma^\sigma_{\alpha\beta}\left(\frac{\partial\mathcal{H}}{\partial V^\sigma}\right) \quad\longleftrightarrow\quad \frac{dq}{dt} = \frac{\partial\mathcal{H}}{\partial p}\,. \tag{81}$$

In other words, starting from the (Hamiltonian) invariant (57), we have recovered a covariant canonical symplectic structure. The variation (67) may be seen as the generating function $\mathcal{G}$ of canonical transformations where the generators of $q-$, $p-$ and $t-$changes are dealt under the same standard.

At this point, some important remarks have to be done. The covariant and contravariant vector fields can be also of different nature so that the above fundamental Hamiltonian invariant can be generalized as

$$\mathcal{H} = W^\alpha V_\alpha\,, \tag{82}$$

or, considering scalar smooth and regular functions, as

$$\mathcal{H} = f(W^\alpha V_\alpha), \tag{83}$$

or, in general

$$\mathcal{H} = f\big(W^\alpha V_\alpha, B^{\alpha\beta}C_{\alpha\beta}, B^{\alpha\beta}V_\alpha V'_\beta, \ldots\big), \tag{84}$$

where the invariant can be constructed by covariant vectors, bivectors and tensors. Clearly, as above, the identifications

$$W^\alpha \longleftrightarrow p \qquad\qquad V_\alpha \longleftrightarrow q \tag{85}$$

hold and the canonical equations are

$$\frac{\delta W^\alpha}{dx^\beta} = -\Gamma^\alpha_{\sigma\beta}\left(\frac{\partial\mathcal{H}}{\partial V_\sigma}\right), \quad \frac{\delta V_\alpha}{dx^\beta} = \Gamma^\sigma_{\alpha\beta}\left(\frac{\partial\mathcal{H}}{\partial W^\sigma}\right). \tag{86}$$

Finally, conservation laws are given by

$$\frac{\delta\mathcal{H}}{dx^\beta} = \left(\Gamma^\alpha_{\sigma\beta} - \Gamma^\alpha_{\sigma\beta}\right)\left(\frac{\partial\mathcal{H}}{\partial V_\sigma}\right)\left(\frac{\partial\mathcal{H}}{\partial W^\alpha}\right) \equiv 0\,. \tag{87}$$

In our picture, this means that the canonical symplectic structure is assigned in the way in which covariant and contravariant vector fields are related. However, if the Hamiltonian invariant is constructed by bivectors and tensors, equations (86) and (87) have to be generalized but the structure is the same. It is worth noticing that we never used the metric field but only connections in our derivations.

These considerations can be made independent of the reference frame if we define a suitable system of unitary vectors by which we can pass from holonomic to anholonomic description and viceversa. We can define the reference frame on the event manifold $\mathcal{M}$ as vector fields $e_{(k)}$ in event space and dual forms $e^{(k)}$ such that vector fields $e_{(k)}$ define an orthogonal frame at each point and

$$e^{(k)}\big(e_{(l)}\big) = \delta^{(k)}_{(l)}\,. \tag{88}$$

If these vectors are unitary, in a Riemannian 4-spacetime are the standard *vierbiens* [5].

If we do not limit this definition of reference frame by orthogonality, we can introduce a *coordinate reference frame* $(\partial_\alpha, ds^\alpha)$ based on vector fields tangent to line $x^\alpha = \text{const}$. Both reference frames are linked by the relations

$$e_{(k)} = e^\alpha_{(k)}\partial_\alpha; \qquad e^{(k)} = e^{(k)}_\alpha dx^\alpha. \tag{89}$$

From now on, Greek indices will indicate holonomic coordinates while Latin indices between brackets, the anholonomic coordinates (*vierbien* indices in 4-spacetimes). We can prove the existence of a reference frame using the orthogonalization procedure at every point of spacetime. From the same procedure, we get that coordinates of frame smoothly depend on the point. The statement about the existence of a global reference frame follows from this. A smooth field on time-like vectors of each frame defines congruence of lines that are tangent to this field. We say that each line is a world line of an observer or a *local reference frame*. Therefore a reference frame is a set of local reference frames. The *Lorentz transformation* can be defined as a transformation of a reference frame

$$x'^\alpha = f\big(x^0, x^1, x^2, x^3, \ldots, x^n\big), \tag{90}$$

$$e'^\alpha_{(k)} = A^\alpha_\beta B^{(l)}_{(k)} e^\beta_{(l)}\,, \tag{91}$$

where

$$A^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x'^\beta}, \qquad \delta_{(i)(l)}B^{(i)}_{(j)}B^{(l)}_{(k)} = \delta_{(j)(k)}\,. \tag{92}$$

We call the transformation A^α_β the holonomic part and transformation $B^{(l)}_{(k)}$ the anholonomic part.

A vector field V has two types of coordinates: *holonomic coordinates* V^α relative to a coordinate reference frame and *anholonomic coordinates* $V^{(k)}$ relative to a reference frame. For these two kinds of coordinates, the relation

$$V^{(k)} = e^{(k)}_\alpha V^\alpha \,, \tag{93}$$

holds. We can study parallel transport of vector fields using any form of coordinates. Because equations (90) and (91) are linear transformations, we expect that parallel transport in anholonomic coordinates has the same form as in holonomic coordinates. Hence we write

$$DV^\alpha = dV^\alpha + \Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma , \tag{94}$$

$$DV^{(k)} = dV^{(k)} + \Gamma^{(k)}_{(l)(p)} V^{(l)} dx^{(p)} . \tag{95}$$

Because DV^α is also a tensor, we get

$$\Gamma^{(k)}_{(l)(p)} = e^\alpha_{(l)} e^\beta_{(p)} e^{(k)}_\gamma \Gamma^\gamma_{\alpha\beta} + e^\alpha_{(l)} e^\beta_{(p)} \frac{\partial e^{(k)}_\alpha}{\partial x^\beta} . \tag{96}$$

Eq.(96) shows the similarity between holonomic and anholonomic coordinates. Let us introduce the symbol $\partial_{(k)}$ for the derivative along the vector field $e_{(k)}$

$$\partial_{(k)} = e^\alpha_{(k)} \partial_\alpha \,. \tag{97}$$

Then Eq.(96) takes the form

$$\Gamma^{(k)}_{(l)(p)} = e^\alpha_{(l)} e^\beta_{(p)} e^{(k)}_\gamma \Gamma^\gamma_{\alpha\beta} + e^\alpha_{(l)} \partial_{(p)} e^{(k)}_\alpha \,. \tag{98}$$

Therefore, when we move from holonomic coordinates to anholonomic ones, the connection also transforms the way similarly to when we move from one coordinate system to another. This leads us to the model of anholonomic coordinates. The vector field $e_{(k)}$ generates lines defined by the differential equations

$$e^\alpha_{(l)} \frac{\partial \tau}{\partial x^\alpha} = \delta^{(k)}_{(l)} \,, \tag{99}$$

or the symbolic system

$$\frac{\partial \tau}{\partial x^{(l)}} = \delta^{(k)}_{(l)} \,. \tag{100}$$

Keeping in mind the symbolic system (100), we denote the functional τ as $x^{(k)}$ and call it the anholonomic coordinate. We call the regular coordinate holonomic. Then we can find derivatives and get

$$\frac{\partial x^{(k)}}{\partial x^\alpha} = \delta^{(k)}_\alpha \,. \tag{101}$$

The necessary and sufficient conditions to complete the integrability of system (101) are

$$\omega^{(i)}_{(k)(l)} = e^\alpha_{(k)} e^\beta_{(l)} \left(\frac{\partial e^{(i)}_\alpha}{\partial x^\beta} - \frac{\partial e^{(i)}_\beta}{\partial x^\alpha} \right) = 0 \,, \tag{102}$$

where we introduced the anholonomic object $\omega^{(i)}_{(k)(l)}$.

Therefore each reference frame has n vector fields

$$\partial_{(k)} = \frac{\partial}{\partial x^{(k)}} = e^\alpha_{(k)} \partial_\alpha \,, \tag{103}$$

which have the commutators

$$\begin{aligned}\left[\partial_{(i)}, \partial_{(j)}\right] &= \left(e^\alpha_{(i)} \partial_\alpha e^\beta_{(j)} - e^\alpha_{(j)} \partial_\alpha e^\beta_{(i)} \right) e^{(m)}_\beta \partial_{(m)} = \\ &= e^\alpha_{(i)} e^\beta_{(j)} \left(-\partial_\alpha e^{(m)}_\beta + \partial_\beta e^{(m)}_\beta \right) \partial_{(m)} = \omega^{(m)}_{(i)(j)} \partial_{(m)} .\end{aligned} \tag{104}$$

For the same reason, we introduce the forms

$$dx^{(k)} = e^{(k)} = e^{(k)}_\beta dx^\beta , \tag{105}$$

and a differential of this form is

$$\begin{aligned} d^2 x^{(k)} &= d\left(e^{(k)}_\alpha dx^\alpha \right) = \left(\partial_\beta e^{(k)}_\alpha - \partial_\alpha e^{(k)}_\beta \right) dx^\alpha \wedge dx^\beta = \\ &= -\omega^{(m)}_{(k)(l)} dx^{(k)} \wedge dx^{(l)} .\end{aligned} \tag{106}$$

Therefore when $\omega^{(i)}_{(k)(l)} \neq 0$, the differential $dx^{(k)}$ is not an exact differential and the system (101), in general, cannot be integrated. However, we can consider meaningful objects which model the solution. We can study how the functions $x^{(i)}$ changes along different lines. The functions $x^{(i)}$ is a natural parameter along a flow line of vector fields $e_{(i)}$. It is defined along any line.

All the above results can be immediately achieved in holonomic and anholonomic formalism considering the equation

$$\mathcal{H} = W^\alpha V_\alpha = W^{(k)} V_{(k)} \,, \tag{107}$$

and the analogous ones. This means that the results are independent of the reference frame and the symplectic covariant structure always holds.

4 The hydrodynamic picture

In order to further check the validity of the above approach, we can prove that it is always consistent with the hydrodynamic picture (see also [10] for details on hydrodynamic covariant formalism).

Let us define a phase space density $f(x, p; s)$ which evolves according to the Liouville equation

$$\frac{\partial f}{\partial s} + \frac{1}{m} \frac{\partial}{\partial x^\mu} \left(g^{\mu\nu} p_\nu f \right) - \frac{1}{2m} \frac{\partial}{\partial p_\lambda} \left(\frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu p_\nu f \right) = 0 \,. \tag{108}$$

Next we define the density $\varrho(x; s)$, the covariant current velocity $v_\mu(x; s)$ and the covariant stress tensor $\mathcal{P}_{\mu\nu}(x; s)$ according to the relations

$$\varrho(x; s) = mn \int \mathrm{d}^4 p f(x, p; s) \,, \tag{109}$$

$$\varrho(x;s)v_\mu(x;s) = n\int \mathrm{d}^4 p p_\mu f(x,p;s)\,, \tag{110}$$

$$\mathcal{P}_{\mu\nu}(x;s) = \frac{n}{m}\int \mathrm{d}^4 p p_\mu p_\nu f(x,p;s)\,. \tag{111}$$

It can be verified, by direct substitution, that a solution to the Liouville Eq.(108) of the form

$$f(x,p;s) = \frac{1}{mn}\varrho(x;s)\delta^4[p_\mu - m v_\mu(x;s)]\,, \tag{112}$$

leads to the equation of continuity

$$\frac{\partial\varrho}{\partial s} + \frac{\partial}{\partial x^\mu}(g^{\mu\nu}v_\nu\varrho) = 0\,, \tag{113}$$

and to the equation for balance of momentum

$$\frac{\partial}{\partial s}(\varrho v_\mu) + \frac{\partial}{\partial x^\lambda}\left(g^{\lambda\alpha}\mathcal{P}_{\alpha\mu}\right) + \frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}\mathcal{P}_{\alpha\beta} = 0\,. \tag{114}$$

Taking into account the fact that for the particular solution (112), the stress tensor, as defined by Eq.(111), is given by the expression

$$\mathcal{P}_{\mu\nu}(x;s) = \varrho v_\mu v_\nu\,, \tag{115}$$

we obtain the final form of the hydrodynamic equations

$$\frac{\partial\varrho}{\partial s} + \frac{\partial}{\partial x^\mu}(\varrho v^\mu) = 0\,, \tag{116}$$

$$\frac{\partial v_\mu}{\partial s} + v^\lambda\left(\frac{\partial v_\mu}{\partial x^\lambda} - \Gamma^\nu_{\mu\lambda}v_\nu\right) = \frac{\partial v_\mu}{\partial s} + v^\lambda\nabla_\lambda v_\mu = 0\,. \tag{117}$$

It is straightforward to see that, through the substitution $v_\mu \to V_\mu$, Eq.(72) is immediately recovered along a geodesic, that is our covariant symplectic structure is consistent with a hydrodynamic picture. It is worth noting that if $\frac{\partial v_\mu}{\partial s}$ in Eq.(117), the motion is not geodesic. The meaning of this term different from zero is that an extra force is acting on the system.

5 Applications, Discussion and Conclusions

Many applications of the previous results can be achieved specifying the nature of vector (or tensor) fields which define the Hamiltonian conserved invariant $\mathcal{H}$. Considerations in General Relativity and Electromagnetism are particularly interesting at this point. Let us take into account the Riemann tensor $R^\rho_{\sigma\mu\nu}$. It comes out when a given vector V^ρ is transported along a closed path on a generic curved manifold. It is

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma, \tag{118}$$

where ∇_μ is the covariant derivative. We are assuming a Riemannian $\mathbf{V}_n$ manifold as standard in General Relativity. If connection is not symmetric, an additive torsion field comes out from the parallel transport.

Clearly, the Riemann tensor results from the commutation of covariant derivatives and it can be expressed as the sum of two commutators

$$R^\rho_{\sigma\mu\nu} = \partial_{[\mu}, \Gamma^\rho_{\nu]\sigma} + \Gamma^\rho_{\lambda[\mu}, \Gamma^\lambda_{\nu]\sigma}\,. \tag{119}$$

Furthermore, (anti) commutation relations and cyclic identities (in particular Bianchi's identities) hold for the Riemann tensor [5].

All these straightforward considerations suggest the presence of a symplectic structure whose elements are covariant and contravariant vector fields, V^α and V_α, satisfying the properties (3)-(7). In this case, the dimensions of vector space $\mathbf{E}_{2n}$ are assigned by V^α and V_α. It is important to notice that such properties imply the connections (Christoffel symbols) and not the metric tensor.

As we said, the invariant (57) is a generic conserved quantity specified by the choice of V^α and V_α. If

$$V^\alpha = \frac{dx^\alpha}{ds}\,, \tag{120}$$

is a 4-velocity, with $\alpha = 0, 1, 2, 3$, immediately, from Eq.(80), we obtain the equation of geodesics of General Relativity,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0\,. \tag{121}$$

On the other hand, being

$$\delta V^\alpha = R^\alpha_{\beta\mu\nu}V^\beta dx_1^\mu dx_2^\nu\,, \tag{122}$$

the result of the transport along a closed path, it is easy to recover the geodesic deviation considering the geodesic (121) and the infinitesimal variation ξ^α with respect to it, i. e.

$$\frac{d^2(x^\alpha + \xi^\alpha)}{ds^2} + \Gamma^\alpha_{\mu\nu}(x+\xi)\frac{d(x^\mu + \xi^\mu)}{ds}\frac{d(x^\nu + \xi^\nu)}{ds} = 0, \tag{123}$$

which gives, through Eq.(119),

$$\frac{d^2\xi^\alpha}{ds^2} = R^\alpha_{\mu\lambda\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\xi^\lambda. \tag{124}$$

Clearly the symplectic structure is due to the fact that the Riemann tensor is derived from covariant derivatives either as

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma, \tag{125}$$

or

$$[\nabla_\mu, \nabla_\nu]V_\rho = R^\sigma_{\mu\nu\rho}V_\sigma\,. \tag{126}$$

In other words, fundamental equations of General Relativity are recovered from our covariant symplectic formalism.

Another interesting choice allows to recover the standard Electromagnetism. If $V^\alpha = A^\alpha$, where A^α is the vector potential and the Hamiltonian invariant is

$$\mathcal{H} = A^\alpha A_\alpha\,, \tag{127}$$

it is straightforward, following the above procedure, to obtain, from the covariant Hamilton equations, the electromagnetic tensor field

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \nabla_{[\alpha} A_{\beta]}\,, \tag{128}$$

and the Maxwell equations (in a generic empty curved space-time)

$$\nabla^\alpha F_{\alpha\beta} = 0\,, \qquad \nabla_{[\alpha} F_{\lambda\beta]} = 0\,. \tag{129}$$

The standard Lorentz gauge is

$$\nabla^\alpha A_\alpha = 0\,, \tag{130}$$

and electromagnetic wave equation is easily recovered.

In summary, a covariant, symplectic structure can be found for every Hamiltonian invariant which can be constructed by covariant vectors, bivectors and tensor fields. In fact, any theory of physics has to be endowed with a symplectic structure in order to be formulated at a fundamental level.

We pointed out that curvature invariants of General Relativity can show such a feature and, furthermore, they can be recovered from Hamiltonian invariants opportunely defined. Another interesting remark deserves the fact that, starting from such invariants, covariant and contravariant vector fields can be read as the configurations q^i and the momenta p_i of classical Hamiltonian dynamics so then the Hamilton-like equations of motion are recovered from the application of covariant derivative to both these vector fields. Besides, the approach can be formulated in a holonomic and anholonomic representations, once vector fields (or tensors in general) are represented in *vierbien* or coordinate–frames. This feature is essential to be sure that general covariance and symplectic structure are conserved in any case.

Specifying the nature of vector fields, we select the particular theory. For example, if the vector field is the 4-velocity, we obtain geodesic motion and geodesic deviation. If the vector is the vector potential of Electromagnetism, Maxwell equations and Lorentz gauge are recovered. The scheme is independent of the nature of vector field and, in our opinion, it is a strong hint toward a unifying view of basic interactions, gravity included.

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Entangled States and Quantum Causality Threshold in the General Theory of Relativity

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This article shows, Synge-Weber's classical problem statement about two particles interacting by a signal can be reduced to the case where the same particle is located in two different points A and B of the basic space-time in the same moment of time, so the states A and B are entangled. This particle, being actual two particles in the entangled states A and B, can interact with itself radiating a photon (signal) in the point A and absorbing it in the point B. That is our goal, to introduce entangled states into General Relativity. Under specific physical conditions the entangled particles in General Relativity can reach a state where neither particle A nor particle B can be the cause of future events. We call this specific state Quantum Causality Threshold.

1 Disentangled and entangled particles in General Relativity. Problem statement

In his article of 2000, dedicated to the 100th anniversary of the discovery of quanta, Belavkin [1] generalizes definitions assumed *de facto* in Quantum Mechanics for entangled and disentangled particles. He writes:

> "The only distinction of the classical theory from quantum is that the prior mixed states cannot be dynamically achieved from pure initial states without a procedure of either statistical or chaotic mixing. In quantum theory, however, the mixed, or decoherent states can be dynamically induced on a subsystem from the initial pure disentangled states of a composed system simply by a unitary transformation.
>
> Motivated by Eintein-Podolsky-Rosen paper, in 1935 Schrödinger published a three part essay* on *The Present Situation in Quantum Mechanics*. He turns to EPR paradox and analyses completeness of the description by the wave function for the entangled parts of the system. (The word *entangled* was introduced by Schrödinger for the description of nonseparable states.) He notes that if one has pure states $\psi(\sigma)$ and $\chi(\upsilon)$ for each of two completely separated bodies, one has maximal knowledge, $\psi_1(\sigma,\upsilon) = \psi(\sigma)\chi(\upsilon)$, for two taken together. But the converse is not true for the entangled bodies, described by a non-separable wave function $\psi_1(\sigma,\upsilon) \neq \psi(\sigma)\chi(\upsilon)$: Maximal knowledge of a total system does not necessary imply maximal knowledge of all its parts, not even when these are completely separated one from another, and at the time can not influence one another at all."

In other word, because Quantum Mechanics considers particles as stochastic clouds, there can be entangled particles — particles whose states are entangled, they build a whole system so that if the state of one particle changes the state of the other particles changes immediately as they are far located one from the other.

In particular, because of the permission for entangled states, Quantum Mechanics permits quantum teleportation — the experimentally discovered phenomenon. The term "quantum teleportation" had been introduced into theory in 1993 [2]. First experiment teleporting massless particles (quantum teleportation of photons) was done five years later, in 1998 [3]. Experiments teleporting mass-bearing particles (atoms as a whole) were done in 2004 by two independent groups of scientists: quantum teleportation of the ion of Calcium atom [4] and of the ion of Beryllium atom [5].

There are many followers who continue experiments with quantum teleportation, see [6–16] for instance.

It should be noted, the experimental statement on quantum teleportation has two channels in which information (the quantum state) transfers between two entangled particles: "teleportation channel" where information is transferred instantly, and "synchronization channel" — classical channel where information is transferred in regular way at the light speed or lower of it (the classical channel is targeted to inform the receiving particle about the initial state of the first one). After teleportation the state of the first particle destroys, so there is data transfer (not data copying).

General Relativity draws another picture of data transfer: the particles are considered as point-masses or waves, not stochastic clouds. This statement is true for both mass-bearing particles and massless ones (photons). Data transfer between any two particles is realized as well by point-mass particles, so in General Relativity this process is not of stochastic origin.

In the classical problem statement accepted in General Relativity [17, 18, 19], two mass-bearing particles are con-

*Schrödinger E. *Naturwissenschaften*, 1935, Band 23, 807–812, 823–828, 844–849.

sidered which are moved along neighbour world-lines, a signal is transferred between them by a photon. One of the particles radiates the photon at the other, where the photon is absorbed realizing data transfer between the particles. Of course, the signal can as well be carried by a mass-bearing particle.

If there are two free mass-bering particles, they fall freely along neighbour geodesic lines in a gravitational field. This classical problem has been developed in Synge's book [20] where he has deduced the geodesic lines deviation equation (Synge's equation, 1950's). If these are two particles connected by a non-gravitational force (for instance, by a spring), they are moved along neighbour non-geodesic world-lines. This classical statement has been developed a few years later by Weber [21], who has obtained the world-lines deviation equation (Synge-Weber's equation).

Anyway in this classical problem of General Relativity two interacting particles moved along both neighbour geodesic and non-geodesic world-lines are *disentangled*. This happens, because of two reasons:

1. In this problem statement a signal moves between two interacting particles at the velocity no faster than light, so their states are absolutely separated — these are *disentangled states*;
2. Any particle, being considered in General Relativity's space-time, has its own four-dimensional trajectory (world-line) which is the set of the particle's states from its birth to decay. Two different particles can not occupy the same world-line, so they are in absolutely separated states — they are *disentangled* particles.

The second reason is much stronger than the first one. In particular, the second reason leads to the fact that, in General Relativity, *entangled* are only neighbour states of the same particle along its own world-line — its own states separated in time, not in the three-dimensional space. No two different particles could be entangled. Any two different particles, both mass-bearing and massless ones, are *disentangled* in General Relativity.

On the other hand, experiments on teleportation evident that *entanglement* is really an existing state that happens with particles if they reach specific physical conditions. This is the fact, that should be taken into account by General Relativity.

Therefore our task in this research is to introduce entangled states into General Relativity. Of course, because of the above reasons, two particles can not be in entangled state if they are located in the basic space-time of General Relativity — the four-dimensional pseudo-Riemannian space with sign-alternating label $(+---)$ or $(-+++)$. Its metric is strictly non-degenerated as of any space of Riemannian space family, namely — there the determinant $g = \det \| g_{\alpha\beta} \|$ of the fundamental metric tensor $g_{\alpha\beta}$ is strictly negative $g < 0$. We expand the Synge-Weber problem statement, considering it in a *generalized space-time* whose metric can become degenerated $g = 0$ under specific physical conditions. This space is one of Smarandache geometry spaces [22–28], because its geometry is partially Riemannian, partially not.

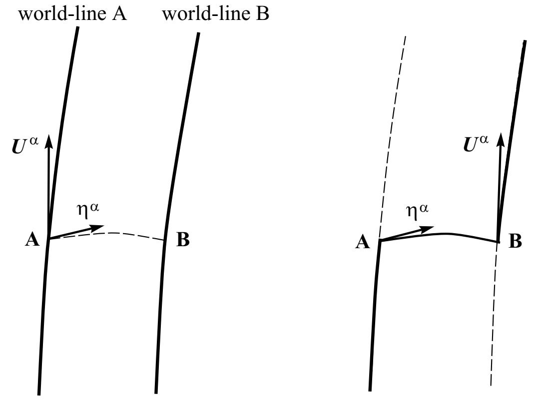

Fig. 1: Synge-Weber's statement. Fig. 2: The advanced statement.

As it was shown in [29, 30] (Borissova and Rabounski, 2001), when General Relativity's basic space-time degenerates physical conditions can imply *observable teleportation* of both a mass-bearing and massless particle — its instant displacement from one point of the space to another, although it moves no faster than light in the degenerated space-time area, outside the basic space-time. In the generalized space-time the Synge-Weber problem statement about two particles interacting by a signal (see Fig. 1) can be reduced to the case where the same particle is located in two different points A and B of the basic space-time in the same moment of time, so the states A and B are entangled (see Fig. 2). This particle, being actual two particles in the entangled states A and B, can interact with itself radiating a photon (signal) in the point A and absorbing it in the point B. That is our goal, to introduce entangled states into General Relativity.

Moreover, as we will see, under specific physical conditions the entangled particles in General Relativity can reach a state where neither particle A nor particle B can be the cause of future events. We call this specific state *Quantum Causality Threshold*.

2 Introducing entangled states into General Relativity

In the classical problem statement, Synge [20] considered two free-particles (Fig. 1) moving along neighbour geodesic world-lines $\Gamma(v)$ and $\Gamma(v+dv)$, where v is a parameter along the direction orthogonal to the geodesics (it is taken in the plane normal to the geodesics). There is $v = \text{const}$ along each the geodesic line.

Motion of the particles is determined by the well-known geodesic equation

$$\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} U^\mu \frac{dx^\nu}{ds} = 0\,, \tag{1}$$

which is the actual fact that the absolute differential $\mathrm{D}U^\alpha = dU^\alpha + \Gamma^\alpha_{\mu\nu} U^\mu dx^\nu$ of a tangential vector U^α (the velocity world-vector $U^\alpha = \frac{dx^\alpha}{ds}$, in this case), transferred along that geodesic line to where it is tangential, is zero. Here s is an invariant parameter along the geodesic (we assume it the space-time interval), and $\Gamma^\alpha_{\mu\nu}$ are Christoffel's symbols of the 2nd kind. Greek $\alpha = 0, 1, 2, 3$ sign for four-dimensional (space-time) indices.

The parameter v is different for the neighbour geodesics, the difference is dv. Therefore, in order to study relative displacements of two geodesics $\Gamma(v)$ and $\Gamma(v+dv)$, we shall study the vector of their infinitesimal relative displacement

$$\eta^\alpha = \frac{\partial x^\alpha}{\partial v}\, dv\,, \tag{2}$$

As Synge had deduced, a deviation of the geodesic line $\Gamma(v+dv)$ from the geodesic line $\Gamma(v)$ can be found as the solution of his obtained equation

$$\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \eta^\gamma = 0\,, \tag{3}$$

that describes relative accelerations of two neighbour free-particles ($R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}$ is Riemann-Chrostoffel's curvature tensor). This formula is known as the geodesic lines deviation equation or the *Synge equation*.

In Weber's statement [21] the difference is that he considers two particles connected by a non-gravitational force Φ^α, a spring for instance. So their world-trajectories are non-geodesic, they are determined by the equation

$$\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu}\, U^\mu \frac{dx^\nu}{ds} = \frac{\Phi^\alpha}{m_0 c^2}\,, \tag{4}$$

which is different from the geodesic equation in that the right part in not zero here. His deduced improved equation of the world lines deviation

$$\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha\cdot\cdot\cdot}_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \eta^\gamma = \frac{1}{m_0 c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}\, dv\,, \tag{5}$$

describes relative accelerations of two particles (of the same rest-mass m_0), connected by a spring. His deviation equation is that of Synge, except of that non-gravitational force Φ^α in the right part. This formula is known as the *Synge-Weber equation*. In this case the angle between the vectors U^α and η^α does not remain unchanged along the trajectories

$$\frac{\partial}{\partial s}(U_\alpha \eta^\alpha) = \frac{1}{m_0 c^2}\, \Phi_\alpha \eta^\alpha. \tag{6}$$

Now, proceeding from this problem statement, we are going to introduce entangled states into General Relativity. At first we determine such states in the space-time of General Relativity, then we find specific physical conditions under which two particles reach a state to be entangled.

Definition Two particles A and B, located in the same spatial section* at the distance $dx^i \neq 0$ from each other, are filled in non-separable states if the observable time interval $d\tau$ between linked events in the particles† is zero $d\tau = 0$. If only $d\tau = 0$, the states become non-separated one from the other, so the particles A and B become entangled.

So we will refer to $d\tau = 0$ as the *entanglement condition* in General Relativity.

Let us consider the entanglement condition $d\tau = 0$ in connection with the world-lines deviation equations.

In General Relativity, the interval of physical observable time $d\tau$ between two events distant at dx^i one from the other is determined through components of the fundamental metric tensor as

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i, \tag{7}$$

see §84 in the well-known *The Classical Theory of Fields* by Landau and Lifshitz [19]. The mathematical apparatus of physical observable quantities (Zelmanov's theory of chronometric invariants [31, 32], see also the brief account in [30, 29]) transforms this formula to

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i, \tag{8}$$

where $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ is the gravitational potential of an acting gravitational field, and $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation.

So, following the theory of physical observable quantities, in real observations where the observer accompanies his references the space-time interval $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ is

$$ds^2 = c^2 d\tau^2 - d\sigma^2, \tag{9}$$

where $d\sigma^2 = \left(-g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}\right) dx^i dx^k$ is a three-dimensional (spatial) invariant, built on the metric three-dimensional observable tensor $h_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}$. This metric observable tensor, in real observations where the observer accompanies his references, is the same that the analogous built general covariant tensor $h_{\alpha\beta}$. So, $d\sigma^2 = h_{ik}\, dx^i dx^k$ is the spatial observable interval for any observer who accompanies his references.

As it is easy to see from (9), there are two possible cases where the entanglement condition $d\tau = 0$ occurs:

(1) $ds = 0$ and $d\sigma = 0$,

(2) $ds^2 = -d\sigma^2 \neq 0$, so $d\sigma$ becomes imaginary,

*A three-dimensional section of the four-dimensional space-time, placed in a given point in the time line. In the space-time there are infinitely many spatial sections, one of which is our three-dimensional space.

†Such linked events in the particles A and B can be radiation of a signal in one and its absorbtion in the other, for instance.

we will refer to them as the *1st kind and 2nd kind entanglement auxiliary conditions*.

Let us get back to the Synge equation and the Synge-Weber equation.

According to Zelmanov's theory of physical observable quantities [31, 32], if an observer accompanies his references the projection of a general covariant quantity on the observer's spatial section is its spatial observable projection.

Following this way, Borissova has deduced (see eqs. 7.16–7.28 in [33]) that the spatial observable projection of the Synge equation is*

$$\frac{d^2\eta^i}{d\tau^2} + 2\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\frac{d\eta^k}{d\tau} = 0\,, \qquad (10)$$

she called it the *Synge equation in chronometrically invariant form*. The Weber equation is different in its right part containing the non-gravitational force that connects the particles (of course, the force should be filled in the spatially projected form). For this reason, conclusions obtained for the Synge equation will be the same that for the Weber one.

In order to make the results of General Relativity applicable to practice, we should consider tensor quantities and equations designed in chronometrically invariant form, because in such way they contain only chronometrically invariant quantities — physical quantities and geometrical properties of space, measurable in real experiment [31, 32].

Let us look at our problem under consideration from this viewpoint.

As it easy to see, the Synge equation in its chronometrically invariant form (10) under the entanglement condition $d\tau = 0$ becomes nonsense. The Weber equation becomes nonsense as well. So, the classical problem statement becomes senseless as soon as particles reach entangled states.

At the same time, in the recent theoretical research [29] two authors of the paper (Borissova and Rabounski, 2005) have found two groups of physical conditions under which particles can be teleported in non-quantum way. They have been called the *teleportation conditions*:

(1) $d\tau = 0$ $\{ds = 0\,,\ d\sigma = 0\}$, the conditions of photon teleportation;

(2) $d\tau = 0$ $\{ds^2 = -d\sigma^2 \neq 0\}$, the conditions of substantial (mass-bearing) particles teleportation.

There also were theoretically deduced physical conditions†, which should be reached in a laboratory in order to teleport particles in the non-quantum way [29].

As it is easy to see the non-quantum teleportation condition is identical to introduce here the entanglement main condition $d\tau = 0$ in couple with the 1st kind and 2nd kind auxiliary entanglement conditions!

Taking this one into account, we transform the classical Synge and Weber problem statement into another. In our statement the world-line of a particle, being entangled to itself by definition, splits into two different world-lines under teleportation conditions. In other word, as soon as the teleportation conditions occur in a research laboratory, the world-line of a teleported particle breaks in one world-point A and immediately starts in the other world-point B (Fig. 2). Both particles A and B, being actually two different states of the same teleported particle at a remote distance one from the other, are in *entangled states*. So, in this statement, the particles A and B themselves are *entangled*.

Of course, this entanglement exists in only the moment of the teleportation when the particle exists in two different states simultaneously. As soon as the teleportation process has been finished, only one particle of them remains so the entanglement disappears.

It should be noted, it follows from the entanglement conditions, that only substantial particles can reach entangled states in the basic space-time of General Relativity — the four-dimensional pseudo-Riemannian space. Not photons. Here is why.

As it is known, the interval $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$ can not be fully degenerated in a Riemannian space‡: the condition is that the determinant of the metric fundamental tensor $g_{\alpha\beta}$ must be strictly negative $g = \det\|\,g_{\alpha\beta}\| < 0$ by definition of Riemannian spaces. In other word, in the basic space-time of General Relativity the fundamental metric tensor must be strictly non-degenerated as $g < 0$.

The observable three-dimensional (spatial) interval $d\sigma^2 = h_{ik}\,dx^i dx^k$ is positive determined [31, 32], proceeding from physical sense. It fully degenerates $d\sigma^2 = 0$ if only the space compresses into point (the senseless case) or the determinant of the metric observable tensor becomes zero $h = \det\|\,h_{ik}\| = 0$.

As it was shown by Zelmanov [31, 32], in real observations where an observer accompanies his references, the determinant of the metric observable tensor is connected with the determinant of the fundamental one by the relationship $h = -\frac{g}{g_{00}}$. From here we see, if the three-dimensional observable metric fully degenerates $h = 0$, the four-dimensional metric degenerates as well $g = 0$.

We have obtained that states of two substantial particles can be entangled, if $d\tau = 0$ $\{ds^2 = -d\sigma^2 \neq 0\}$ in the space neighbourhood. So $h > 0$ and $g < 0$ in the neighbourhood,

*In this formula, according to Zelmanov's mathematical apparatus of physical observable quantities [31, 32], $D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} = \frac{1}{2\sqrt{g_{00}}}\frac{\partial h_{ik}}{\partial t}$ is the three-dimensional symmetric tensor of the space deformation observable rate while $A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right)$ is the three-dimensional antisymmetric tensor of the space rotation observable angular velocities, which indices can be lifted/lowered by the metric observable tensor so that $D^i_k = h^{im} D_{km}$ and $A^{\cdot i}_{k\cdot} = h^{im} A_{km}$. See brief account of the Zelmanov mathematical apparatus in also [30, 33, 34, 35].

†A specific correlation between the gravitational potential w, the space rotation linear velocity v_i and the teleported particle's velocity u^i.

‡It can only be partially degenerated. For instance, a four-dimensional Riemannian space can be degenerated into a three-dimensional one.

hence the four-dimensional pseudo-Riemannian space is not degenerated.

Conclusion Substantial particles can reach entangled states in the basic space-time of General Relativity (the four-dimensional pseudo-Riemannian space) under specific conditions in the neighbourhood.

Although $ds^2 = -d\sigma^2$ in the neighbourhood ($d\sigma$ should be imaginary), the substantial particles remain in regular sub-light area, they do not become super-light tachyons. It is easy to see, from the definition of physical observable time (8), the entanglement condition $d\tau = 0$ occurs only if the specific relationship holds

$$\mathrm{w} + v_i u^i = c^2 \tag{11}$$

between the gravitational potential w, the space rotation linear velocity v_i and the particles' true velocity $u^i = dx^i/dt$ in the observer's laboratory. For this reason, in the neighbourhood the space-time metric is

$$ds^2 = -d\sigma^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k, \tag{12}$$

so the substantial particles can become entangled if the space initial signature $(+---)$ becomes inverted $(-+++)$ in the neighbourhood, while the particles' velocities u^i remain no faster than light.

Another case — massless particles (photons). States of two phonos can be entangled, only if there is in the space neighbourhood $d\tau = 0$ $\{ds = 0\,, d\sigma = 0\}$. In this case the determinant of the metric observable tensor becomes $h = 0$, so the space-time metric as well degenerates $g = -g_{00}\, h = 0$. This is not the four-dimensional pseudo-Riemannian space.

Where is that area? In the previous works (Borissova and Rabounski, 2001 [30, 29]) a generalization to the basic space-time of General Relativity was introduced — the four-dimensional space which, having General Relativity's sign-alternating label $(+---)$, permits the space-time metric to be fully degenerated so that there is $g \leqslant 0$.

As it was shown in those works, as soon as the specific condition $\mathrm{w} + v_i u^i = c^2$ occurs, the space-time metric becomes fully degenerated: there are $ds = 0$, $d\sigma = 0$, $d\tau = 0$ (it can be easy derived from the above definition for the quantities) and, hence $h = 0$ and $g = 0$. Therefore, in a space-time where the *degeneration condition* $\mathrm{w} + v_i u^i = c^2$ is permitted the determinant of the fundamental metric tensor is $g \leqslant 0$. This case includes both Riemannian geometry case $g < 0$ and non-Riemannian, fully degenerated one $g = 0$. For this reason a such space is one of Smarandache geometry spaces [22–28], because its geometry is partially Riemannian, partially not*. In the such generalized space-time the 1st kind entanglement conditions $d\tau = 0$ $\{ds = 0\,, d\sigma = 0\}$ (the entanglement conditions for photons) are permitted in that area where the space metric fully degenerates (there $h = 0$ and, hence $g = 0$).

Conclusion Massless particles (photons) can reach entangled states, only if the basic space-time fully degenerates $g = \det\|g_{\alpha\beta}\| = 0$ in the neighbourhood. It is permitted in the generalized four-dimensional space-time which metric can be fully degenerated $g \leqslant 0$ in that area where the degeneration conditions occur. The generalized space-time is attributed to Smarandache geometry spaces, because its geometry is partially Riemannian, partially not.

So, entangled states have been introduced into General Relativity for both substantial particles and photons.

3 Quantum Causality Threshold in General Relativity

This term was introduced by one of the authors two years ago (Smarandache, 2003) in our common correspondence [36] on the theme:

Definition Considering two particles A and B located in the same spatial section, *Quantum Causality Threshold* was introduced as a special state in which neither A nor B can be the cause of events located "over" the spatial section on the Minkowski diagram.

The term *Quantum* has been added to the *Causality Threshold*, because in this problem statement an interaction is considered between two infinitely far away particles (in infinitesimal vicinities of each particle) so this statement is applicable to only quantum scale interactions that occur in the scale of elementary particles.

Now, we are going to find physical conditions under which particles can reach the threshold in the space-time of General Relativity.

Because in this problem statement we look at causal relations in General Relativity's space-time from "outside", it is required to use an "outer viewpoint" — a point of view located outside the space-time.

We introduce a such point of outlook in an Euclidean flat space, which is tangential to our's in that world-point, where the observer is located. In this problem statement we have a possibility to compare the absolute cause relations in that tangential flat space with those in ours. As a matter, a tangential Euclidean flat space can be introduced at any point of the pseudo-Riemannian space.

*In foundations of geometry it is known the *S-denying* of an axiom [22–25], i. e. in the same space an "axiom is false in at least two different ways, or is false and also true. Such axiom is said to be Smarandachely denied, or S-denied for short" [26]. As a result, it is possible to introduce geometries, which have common points bearing mixed properties of Euclidean, Lobachevsky-Bolyai-Gauss, and Riemann geometry in the same time. Such geometries has been called paradoxist geometries or Smarandache geometries. For instance, Iseri in his book *Smarandache Manifolds* [26] and articles [27, 28] introduced manifolds that support particular cases of such geometries.

At the same time, according to Zelmanov [31, 32], within infinitesimal vicinities of any point located in the pseudo-Riemannian space a *locally geodesic reference frame* can be introduced. In a such reference frame, within infinitesimal vicinities of the point, components of the metric fundamental tensor (marked by tilde)

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial^2 \tilde{g}_{\alpha\beta}}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}\right)(\tilde{x}^\mu - x^\mu)(\tilde{x}^\nu - x^\nu) + \dots \qquad (13)$$

are different from those $g_{\alpha\beta}$ at the point of reflection to within only the higher order terms, which can be neglected. So, in a locally geodesic reference frame the fundamental metric tensor can be accepted constant, while its first derivatives (Christoffel's symbols) are zeroes. The fundamental metric tensor of an Euclidean space is as well a constant, so values of $\tilde{g}_{\mu\nu}$, taken in the vicinities of a point of the pseudo-Riemannian space, converge to values of $g_{\mu\nu}$ in the flat space tangential at this point. Actually, we have a system of the flat space's basic vectors $\vec{e}_{(\alpha)}$ tangential to curved coordinate lines of the pseudo-Riemannian space. Coordinate lines in Riemannian spaces are curved, inhomogeneous, and are not orthogonal to each other (the latest is true if the space rotates). Therefore the lengths of the basic vectors may be very different from the unit.

Writing the world-vector of an infinitesimal displacement as $d\vec{r} = (dx^0, dx^1, dx^2, dx^3)$, we obtain $d\vec{r} = \vec{e}_{(\alpha)} dx^\alpha$, where the components of the basic vectors $\vec{e}_{(\alpha)}$ tangential to the coordinate lines are $\vec{e}_{(0)} = \{e^0_{(0)}, 0, 0, 0\}$, $\vec{e}_{(1)} = \{0, e^1_{(1)}, 0, 0\}$, $\vec{e}_{(2)} = \{0, 0, e^2_{(2)}, 0\}$, $\vec{e}_{(3)} = \{0, 0, 0, e^2_{(3)}\}$. Scalar product of $d\vec{r}$ with itself is $d\vec{r}d\vec{r} = ds^2$ or, in another $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$, so $g_{\alpha\beta} = \vec{e}_{(\alpha)}\vec{e}_{(\beta)} = e_{(\alpha)}e_{(\beta)} \cos(x^\alpha; x^\beta)$. We obtain

$$g_{00} = e^2_{(0)}\,, \qquad g_{0i} = e_{(0)}e_{(i)} \cos(x^0; x^i)\,, \qquad (14)$$

$$g_{ik} = e_{(i)}e_{(k)} \cos(x^i; x^k)\,, \qquad i, k = 1, 2, 3\,. \qquad (15)$$

Then, substituting g_{00} and g_{0i} from formulas that determine the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ and the space rotation linear velocity $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$, we obtain

$$v_i = -c\, e_{(i)} \cos(x^0; x^i)\,, \qquad (16)$$

$$h_{ik} = e_{(i)}e_{(k)}\Big[\cos(x^0; x^i)\cos(x^0; x^k) - \cos(x^i; x^k)\Big]. \qquad (17)$$

From here we see: if the pseudo-Riemannian space is free of rotation, $\cos(x^0; x^i) = 0$ so the observer's spatial section is strictly orthogonal to time lines. As soon as the space starts to do rotation, the cosine becomes different from zero so the spatial section becomes non-orthogonal to time lines (Fig. 3). Having this process, the light hypercone inclines with the time line to the spatial section. In this inclination the light hypercone does not remain unchanged, it "compresses" because of hyperbolic transformations in pseudo-Riemannian space. The more the light hypercone inclines, the more it symmetrically "compresses" because the space-time's geometrical structure changes according to the inclination.

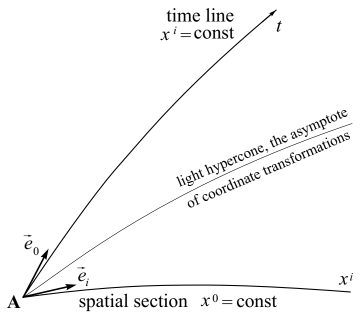

Fig. 3: The spatial section becomes non-orthogonal to time lines, as soon as the space starts rotation.

In the ultimate case, where the cosine reach the ultimate value $\cos(x^0; x^i) = 1$, time lines coincide the spatial section: time "has fallen" into the three-dimensional space. Of course, in this case the light hypercone overflows time lines and the spatial section: the light hypercone "has as well fallen" into the three-dimensional space.

As it is easy to see from formula (16), this ultimate case occurs as soon as the space rotation velocity v_i reaches the light velocity. If particles A and B are located in the space filled into this ultimate state, neither A nor B can be the cause of events located "over" the spatial section in the Minkowski diagrams we use in the pictures. So, in this ultimate case the space-time is filled into a special state called Quantum Causality Threshold.

Conclusion Particles, located in General Relativity's space-time, reach Quantum Causality Threshold as soon as the space rotation reaches the light velocity. Quantum Causality Threshold is impossible if the space does not rotate (holonomic space), or if it rotates at a sub-light speed.

So, Quantum Causality Threshold has been introduced into General Relativity.

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Bootstrap Universe from Self-Referential Noise

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We further deconstruct Heraclitean Quantum Systems giving a model for a universe using pregeometric notions in which the end-game problem is overcome by means of self-referential noise. The model displays self-organisation with the emergence of 3-space and time. The time phenomenon is richer than the present geometric modelling.

1 Heraclitean Quantum Systems

From the beginning of theoretical physics in the 6th and 5th centuries BC there has been competition between two classes of modelling of reality: one class has reality explained in terms of things, and the other has reality explained purely in terms of relationships (information).* While in conventional physics a mix of these which strongly favours the "things" approach is currently and very efficaciously used, here we address the problem of the "ultimate" modelling of reality. This we term the *end-game* problem: at higher levels in the phenomenology of reality one chooses economical and effective models — which usually have to be accompanied by meta-rules for interpretation, but at the lower levels we are confronted by the problem of the source of "things" and their rules or "laws". At one extreme we could have an infinite regress of ever different "things", another is the notion of a Platonic world where mathematical things and their rules reside [1]. In both instances we still have the fundamental problem of why the universe "ticks" — that is, why it is more than a mathematical construct; why is it experienced?

This "end-game" problem is often thought of as the unification of our most successful and deepest, but incompatible, phenomenologies: General Relativity and Quantum Theory. We believe that the failure to find a common underpinning of these models is that it is apparently often thought it would be some amalgamation of the two, and not something vastly different. Another difficulty is that the lesson from these models is often confused; for instance from the success of the geometrical modelling of space and time it is often argued that the universe "*is* a 4-dimensional manifold". However the geometrical modelling of time is actually deficient: it lacks much of the experienced nature of time — for it fails to model both the directionality of time and the phenomenon of the (local) "present moment". Indeed the geometrical model might better be thought of as a "historical model" of time, because in histories the notion of direction and present moment are absent — they must be provided by external meta-rules. General relativity then is about possible histories of the universe, and in this it is both useful and successful. Similarly quantum field theories have fields built upon a possible (historical) spacetime, and subjected to quantisation. But such quantum theories have difficulties with classicalisation and the individuality of events — as in the "measurement problem". At best the theory invokes ensemble measurement postulates as external meta-rules. So our present-day quantum theories are also historical models.

The problem of unifying general relativity and quantum theories then comes down to going beyond *historical* modelling, which in simple terms means finding a better model of time. The historical or *being* model of reality has been with us since Parmenides and Zeno, and is known as the Eleatic model. The *becoming* or *processing* model of reality dates back further to Heraclitus of Ephesus (540 – 480 BC) who argued that common sense is mistaken in thinking that the world consists of stable things; rather the world is in a state of flux. The appearances of "things" depend upon this flux for their continuity and identity. What needs to be explained, Heraclitus argued, is not change, but the appearance of stability.

Although "process" modelling can be traced through to the present time it has always been a speculative notion because it has never been implemented in a mathematical form and subjected to comparison with reality. Various proposals of a *pregeometric* nature have been considered [2, 3, 4]. Here we propose a mathematical *pregeometric process* model of reality — which in [5] was called a *Heraclitean Quantum System* (HQS). There we arrived at a HQS by deconstruction of the functional integral formulation of quantum field theories retaining only those structures which we felt would not be emergent. In this we still started with "things", namely a Grassmann algebra, and ended with the need to de-

*This is the original 1997 version of the paper which introduced the notion that reality has an *information-theoretic* intrinsic randomness. Since this pioneering paper the model of reality known as *Process Physics* has advanced enormously, and has been confirmed in numerous experiments. The book Cahill, R. T. *Process Physics: From Information Theory to Quantum Space and Matter*, Nova Science Pub. NY 2005, reviews subsequent developments. Numerous papers are available at http://www.mountainman.com.au/process_physics/ and http://www.scieng.flinders.edu.au/cpes/people/cahill_r/processphysics.html compose the mathematical structures into possible histories — each corresponding to a different possible decoherent classical sequencing. However at that level of the HQS we cannot expect anything other than the usual historical modelling of time along with its deficiencies. The problem there was that the deconstruction began with ensembled quantum field theory, and we can never recover individuality and actuality from ensembles — that has been the problem with quantum theory since its inception.

Here we carry the deconstruction one step further by exploiting the fact that functional integrals can be thought of as arising as ensemble averages of Wiener processes. These are normally associated with Brownian-type motions in which random processes are used in modelling many-body dynamical systems. We argue that random processes are a fundamental and necessary aspect of reality — that they arise in the resolution presented here to the end-game problem of modelling reality. In sect. 2 we argue that this "noise" arises as a necessary feature of the self-referential nature of the universe. In sect. 3 we discuss the nature of the self-organised space and time phenomena that arise, and argue that the time modelling is richer and more "realistic" than the geometrical model. In sect. 4 we show how the ensemble averaging of possible universe behaviour is expressible as a functional integral.

2 Self-Referential Noise

Our proposed solution to the end-game problem is to avoid the notion of things and their rules; rather to use a bootstrapped self-referential system. Put simply, this models the universe as a self-organising and self-referential information system — "information" denoting relationships as distinct from "things". In such a system there is no bottom level and we must consider the system as having an iterative character and attempt to pick up the structure by some mathematical modelling.

Chaitin [6] developed some insights into the nature of complex self-referential information systems: combining Shannon's information theory and Turing's computability theory resulted in the development of Algorithmic Information Theory (AIT). This shows that number systems contain randomness and unpredictability, and extends Gödel's discovery, which resulted from self-referenc-ing problems, of the incompleteness of such systems (see [7] for various discussions of the *physics of information*; here we are considering *information as physics*).

Hence if we are to model the universe as a closed system, and thus self-referential, then the mathematical model must necessarily contain randomness. Here we consider one very simple such model and proceed to show that it produces a dynamical 3-space and a theory for time that is richer than the historical/geometrical model.

We model the self-referencing by means of an iterative map

$$B_{ij} \to B_{ij} - (B + B^{-1})_{ij}\,\eta + w_{ij}, \quad i,j = 1,2,\ldots,M \to \infty. \tag{1}$$

We think of B_{ij} as relational information shared by two monads i and j. The monads concept was introduced by Leibniz, who espoused the *relational* mode of thinking in response to and in contrast to Newton's *absolute* space and time. Leibniz's ideas were very much in the *process* mould of thinking: in this the monad's *view* of available information and the commonality of this information is intended to lead to the emergence of space. The monad i acquires its meaning entirely by means of the information $B_{i1}, B_{i2}, \ldots$, where $B_{ij} = -B_{ji}$ to avoid self-information, and real number valued. The map in (1) has the form of a Wiener process, and the $w_{ij} = -w_{ji}$ are independent random variables for each ij and for each iteration, and with variance 2η for later convenience. The w_{ij} model the self-referential noise. The beginning of a universe is modelled by starting the iterative map with $B_{ij} \approx 0$, representing the absence of information or order. Clearly due to the B^{-1} term iterations will rapidly move the B_{ij} away from such starting conditions.

The non-noise part of the map involves B and B^{-1}. Without the non-linear inverse term the map would produce independent and trivial random walks for each B_{ij} — the inverse introduces a linking of all information. We have chosen B^{-1} because of its indirect connection with quantum field theory (see sec. 4) and because of its self-organising property. It is the conjunction of the noise and non-noise terms which leads to the emergence of self-organisation: without the noise the map converges (and this determines the signs in formula 1), in a deterministic manner to a degenerate condensate type structure, discussed in [5], corresponding to a pairing of linear combinations of monads. Hence the map models a non-local and noisy information system from which we extract spatial and time-like behaviour, but we expect residual non-local and random processes characteristic of quantum phenomena including EPR/Aspect type effects. While the map already models some time-like behaviour, it is in the nature of a bootstrap system that we start with *process*. In this system the noise corresponds to the Heraclitean flux which he also called the "cosmic fire", and from which the emergence of stable structures should be understood. To Heraclitus the flame represented one of the earliest examples of the interplay of order and disorder. The contingency and self-ordering of the process clearly suggested a model for reality.

3 Emergent Space and Time

Here we show that the HQS iterative map naturally results in dynamical 3-dimensional spatial structures. Under the mapping the noise term will produce rare large value B_{ij}.

Because the order term is generally much smaller, for small η, than the disorder term these values will persist under the mapping through more iterations than smaller valued B_{ij}. Hence the larger B_{ij} correspond to some temporary background structure which we now identify.

Consider this relational information from the point of view of one monad, call it monad i. Monad i is connected via these large B_{ij} to a number of other monads, and the whole set forms a tree-graph relationship. This is because the large links are very improbable, and a tree-graph relationship is much more probable than a similar graph with additional links. The simplest distance measure for any two nodes within a graph is the smallest number of links connecting them. Let $D_1, D_2, \ldots, D_L$ be the number of nodes of distance $1, 2, \ldots, L$ from node i (define $D_0 = 1$ for convenience), where L is the largest distance from i in a particular tree-graph, and let N be the total number of nodes in the tree. Then $\sum_{k=0}^{L} D_k = N$. See Fig.1 for an example.

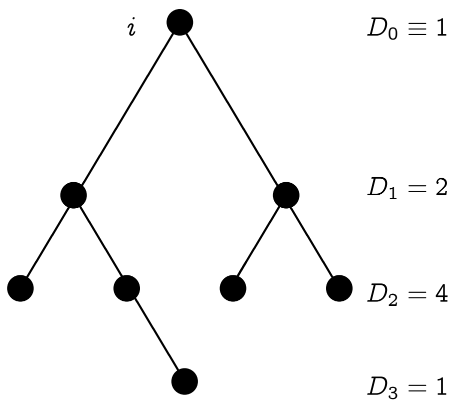

Fig. 1: An $N = 8$, $L = 3$ tree, with indicated distance distributions from monad i.

Now consider the number of different N-node trees, with the same distance distribution $\{D_k\}$, to which i can belong. By counting the different linkage patterns, together with permutations of the monads we obtain

$$\mathcal{N}(D, N) = \frac{(M-1)!\,D_1^{D_2} D_2^{D_3} \ldots D_{L-1}^{D_L}}{(M-N-2)!\,D_1!\,D_2! \ldots D_L!}\,, \qquad (2)$$

here $D_k^{D_{k+1}}$ is the number of different possible linkage patterns between levels k and $k+1$, and $(M-1)!/(M-N-2)!$ is the number of different possible choices for the monads, with i fixed. The denominator accounts for those permutations which have already been accounted for by the $D_k^{D_{k+1}}$ factors. Nagels [8] analysed $\mathcal{N}(D, N)$, and the results imply that the most likely tree-graph structure to which a monad can belong has the distance distribution

$$D_k \approx \frac{L^2 \ln L}{2\pi^2} \sin^2\left(\frac{\pi k}{L}\right) \quad k = 1, 2, \ldots, L. \qquad (3)$$

for a given arbitrary L value. The remarkable property of this most probable distribution is that the $\sin^2$ indicates that the tree-graph is embeddable in a 3-dimensional hypersphere, S^3. Most importantly, monad i "sees" its surroundings as being 3-dimensional, since $D_k \sim k^2$ for small $\pi k/L$. We call these 3-spaces *gebits* (geometrical bits). We note that the $\ln L$ factor indicates that larger gebits have a larger number density of points.

Now the monads for which the B_{ij} are large thus form disconnected gebits. These gebits however are in turn linked by smaller and more transient B_{kl}, and so on, until at some low level the remaining B_{mn} are noise only; that is they will not survive an iteration. Under iterations of the map this spatial network undergoes growth and decay at all levels, but with the higher levels (larger $\{B_{ij}\}$ gebits) showing most persistence. By a similarity transformation we can arrange the gebits into block diagonal matrices $b_1, b_2, \ldots$, within B, and embedded amongst the smaller and more common noise entries. Now each gebit matrix has $\det(b) = 0$, since a tree-graph connectivity matrix is degenerate. Hence under the mapping the B^{-1} order term has an interesting dynamical effect upon the gebits since, in the absence of the noise, B^{-1} would be singular. The outcome from the iterations is that the gebits are seen to compete and to undergo mutations, for example by adding extra monads to the gebit. Numerical studies reveal gebits competing and "consuming" noise, in a Darwinian process.

Hence in combination the order and disorder terms synthesise an evolving dynamical 3-space with hierarchical structures, possibly even being fractal. This emergent 3-space is entirely relational; it does not arise within any *a priori* geometrical background structure. By construction it is the most robust structure, — however other softer emergent modes of behaviour will be seen as attached to or embedded in this flickering 3-space. The possible fractal character could be exploited by taking a higher level view: identifying each gebit $\to I$ as a higher level monad, with appropriate informational connections $\mathcal{B}_{IJ}$, we could obtain a higher level iterative map of the form (1), with new order/disorder terms. This would serve to emphasise the notion that in self-referential systems there are no "things", but rather a complex network of iterative relations.

In the model the iterations of the map have the appearance of a cosmic time. However the analysis to reveal the internal experiential time phenomenon is non-trivial, and one would certainly hope to recover the local nature of experiential time as confirmed by special and general relativity experiments. However it is important to notice that the modelling of the time phenomenon here is much richer than that of the historical/geometric model. First the map is clearly unidirectional (there is an "arrow of time") as there is no way to even define an inverse mapping because of the role of the noise term, and this is very unlike the conventional differential equations of traditional physics. In the analysis of the gebits we noted that they show strong persistence, and in that sense the mapping shows a natural partial-memory phenomenon, but the far "future" detailed structure of even this spatial network is completely unknowable without performing the iterations. Furthermore the sequencing of the spatial and other structures is individualistic in that a rerun of the model will always produce a different outcome. Most important of all is that we also obtain a modelling of the "present moment" effect, for the outcome of the next iteration is contingent on the noise. So the system shows overall a sense of a recordable past, an unknowable future and a contingent present moment.

The HQS process model is expected to be capable of a better modelling of our experienced reality, and the key to this is the noisy processing the model requires. As well we need the "internal view", rather than the "external view" of conventional modelling in physics. Nevertheless we would expect that the internally recordable history could be indexed by the usual real-number/geometrical time coordinate.

This new self-referential process modelling requires a new mode of analysis since one cannot use externally imposed meta-rules or interpretations, rather, the internal experiential phenomena and the characterisation of the simpler ones by emergent "laws" of physics must be carefully determined. There has indeed been an ongoing study of how (unspecified) closed self-referential noisy information systems acquire self-knowledge and how the emergent hierarchical structures can "recognise" the same "individuals" [9]. These *Combinatoric Hierarchy* (CH) studies use the fact that only recursive constructions are possible in Heraclitean/Leibnizian systems. We believe that our HQS process model may provide an explicit representation for the CH studies.

4 Possible-Histories Ensemble

While the actual history of the noisy map can only be found in a particular "run", we can nevertheless show that averages over an ensemble of possible histories can be determined, and these have the form of functional integrals. The notion of an ensemble average for any function f of the B, at iteration $c = 1, 2, 3, \ldots$, is expressed by

$$< f[B] >_c = \int \mathcal{D}B f[B]\, \Phi_c[B]\,, \tag{4}$$

where $\Phi_c[B]$ is the ensemble distribution. By the usual construction for Wiener processes we obtain the Fokker-Planck equation

$$\Phi_{c+1}[B] = \Phi_c[B] - \\ -\sum_{ij} \eta \left\{ \frac{\partial}{\partial B_{ij}} \left[(B+B^{-1})_{ij}\, \Phi_c[B] \right] - \frac{\partial^2}{\partial B_{ij}^2}\, \Phi_c[B] \right\}. \tag{5}$$

For simplicity, in the quasi-stationary regime, we find

$$\Phi[B] \sim \exp\left(-S[B]\right), \tag{6}$$

where the action is

$$S[B] = \sum_{i>j} B_{ij}^2 - \mathrm{TrLn}(B)\,. \tag{7}$$

Then the ensemble average is

$$\frac{1}{Z} \int \mathcal{D}B f[B] \exp\left(-S[B]\right), \tag{8}$$

where Z ensures the correct normalisation for the averages. The connection between (1) and (7) is given by

$$(B^{-1})_{ij} = \frac{\partial}{\partial B_{ji}} \mathrm{TrLn}(B) = \frac{\partial}{\partial B_{ji}} \ln \prod_{\alpha} \lambda_\alpha[B]\,. \tag{9}$$

which probes the sensitivity of the invariant ensemble information to changes in B_{ji}, where the information is in the eigenvalues $\lambda_\alpha[B]$ of B. A further transformation is possible [5]:

$$\begin{aligned} < f[B] > &= \frac{1}{Z} \int \mathcal{D}\overline{m}\mathcal{D}m\mathcal{D}B\, f[B] \times \\ &\times \exp\left[-\sum_{i>j} B_{ij}^2 + \sum_{i,j} B_{ij}(\overline{m}_i m_j - \overline{m}_j m_i) \right] = \\ &= \frac{1}{Z} f\left[\frac{\partial}{\partial J}\right] \int \mathcal{D}\overline{m}\mathcal{D}m \exp\left[-\sum_{i>j} \overline{m}_i m_j \overline{m}_j m_i + \right. \\ &\left. + \textstyle\sum_{ij} J_{ij}(\overline{m}_i m_j - \overline{m}_j m_i) \right]. \end{aligned} \tag{10}$$

This expresses the ensemble average in terms of an anticommuting Grassmannian algebraic computation [5]. This suggests how the noisy information map may lead to fermionic modes. While functional integrals of the above forms are common in quantum field theory, it is significant that in forming the ensemble average we have lost the contingency or present-moment effect. This always happens — ensemble averages do not tell us about individuals — and then the meta-rules and "interpretations" must be supplied in order to generate some notion of what an individual might have been doing.

The Wiener iterative map can be thought of as a resolution of the functional integrals into different possible histories. However this does not imply the notion that in some sense *all* these histories must be realised, rather only *one* is required. Indeed the basic idea of the process modelling is that of individuality. Not unexpectedly we note that the modelling in (1) must be done from within that *one* closed system.

In conventional quantum theory it has been discovered that the individuality of the measurement process — the "click" of the detector — can be modelled by adding a noise term to the Schrödinger equation [10]. Then by performing an ensemble average over many individual runs of this modified Schrödinger equation one can derive the ensemble measurement postulate — namely $<A> = (\psi, A\psi)$ for the "expectation value of the operator A". This individualising of the ensemble average has been shown to also relate to the decoherence functional formalism [11]. There are a number of other proposals considering noise in spacetime modelling [12, 13].

5 Conclusion

We have addressed here the unique end-game problem which arises when we attempt to model and comprehend the universe as a closed system. The outcome is the suggestion that the peculiarities of this end-game problem are directly relevant to our everyday experience of time and space; particularly the phenomena of the contingent present moment and the three-dimensionality of space. This analysis is based upon the basic insight that a closed self-referential system is necessarily noisy. This follows from Algorithmic Information Theory. To explore the implications we have considered a simple *pregeometric non-linear noisy iterative map*. In this way we construct a process bootstrap system with minimal structure. The analysis shows that the first self-organised structure to arise is a dynamical 3-space formed from competing pieces of 3-geometry — the gebits. The analysis of experiential time is more difficult, but it will clearly be a contingent and process phenomenon which is more complex than the current geometric/historic modelling of time. To extract emergent properties of self-referential systems requires that an internal view be considered, and this itself must be a recursive process. We suggest that the non-local self-referential noise has been a major missing component of our modelling of reality. Two particular applications are an understanding of why quantum detectors "click" and of the physics of consciousness [1], since both clearly have an essential involvement with the modelling of the present-moment effect, and cannot be understood using the geometric/historic modelling of time.

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Verifying Unmatter by Experiments, More Types of Unmatter, and a Quantum Chromodynamics Formula

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As shown, experiments registered unmatter: a new kind of matter whose atoms include both nucleons and anti-nucleons, while their life span was very short, no more than 10^{-20}sec. Stable states of unmatter can be built on quarks and anti-quarks: applying the unmatter principle here it is obtained a quantum chromodynamics formula that gives many combinations of unmatter built on quarks and anti-quarks.

In the last time, before the apparition of my articles defining "matter, antimatter, and unmatter" [1, 2], and Dr. S. Chubb's pertinent comment [3] on unmatter, new development has been made to the unmatter topic.

1 Definition of Unmatter

In short, unmatter is formed by matter and antimatter that bind together [1, 2]. The building blocks (most elementary particles known today) are 6 quarks and 6 leptons; their 12 antiparticles also exist. Then *unmatter* will be formed by at least a building block and at least an antibuilding block which can bind together.

2 Exotic atom

If in an atom we substitute one or more particles by other particles of the same charge (constituents) we obtain an exotic atom whose particles are held together due to the electric charge. For example, we can substitute in an ordinary atom one or more electrons by other negative particles (say π^-, anti-ρ-meson, D^-, D_s^-- muon, τ, Ω^-, Δ^-, etc., generally clusters of quarks and antiquarks whose total charge is negative), or the positively charged nucleus replaced by other positive particle (say clusters of quarks and antiquarks whose total charge is positive, etc).

3 Unmatter atom

It is possible to define the unmatter in a more general way, using the exotic atom. The classical unmatter atoms were formed by particles like:

(a) electrons, protons, and antineutrons, or

(b) antielectrons, antiprotons, and neutrons.

In a more general definition, an unmatter atom is a system of particles as above, or such that one or more particles are replaces by other particles of the same charge. Other categories would be:

(c) a matter atom with where one or more (but not all) of the electrons and/or protons are replaced by antimatter particles of the same corresponding charges, and

(d) an antimatter atom such that one or more (but not all) of the antielectrons and/or antiprotons are replaced by matter particles of the same corresponding charges.

In a more composed system we can substitute a particle by an unmatter particle and form an unmatter atom.

Of course, not all of these combinations are stable, semi-stable, or quasi-stable, especially when their time to bind together might be longer than their lifespan.

4 Examples of unmatter

During 1970-1975 numerous pure experimental verifications were obtained proving that "atom-like" systems built on nucleons (protons and neutrons) and anti-nucleons (anti-protons and anti-neutrons) are real. Such "atoms", where nucleon and anti-nucleon are moving at the opposite sides of the same orbit around the common centre of mass, are very unstable, their life span is no more than 10^{-20} sec. Then nucleon and anti-nucleon annihilate into gamma-quanta and more light particles (pions) which can not be connected with one another, see [6, 7, 8]. The experiments were done in mainly Brookhaven National Laboratory (USA) and, partially, CERN (Switzerland), where "proton — anti-proton" and "anti-proton — neutron" atoms were observed, called them $\bar{p}p$ and $\bar{p}n$ respectively, see Fig. 1 and Fig. 2.

After the experiments were done, the life span of such "atoms" was calculated in theoretical way in Chapiro's works [9, 10, 11]. His main idea was that nuclear forces, acting between nucleon and anti-nucleon, can keep them far way from each other, hindering their annihilation. For instance, a proton and anti-proton are located at the opposite sides in the same orbit and they are moved around the orbit centre. If the diameter of their orbit is much more than the diameter of "annihilation area", they can be kept out of annihilation (see Fig. 3). But because the orbit, according to Quantum Mechanics, is an actual cloud spreading far around the average radius, at any radius between the proton and the anti-proton there is a probability that they can meet one another at the annihilation distance. Therefore "nucleon — anti-nucleon" system annihilates in any case, this system

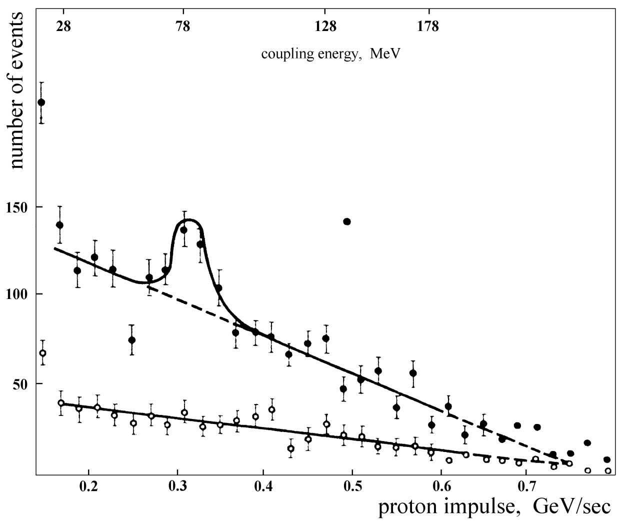

Fig. 1: Spectra of proton impulses in the reaction $\bar{p}+d \to (\bar{p}n)+p$. The upper arc — annihilation of $\bar{p}n$ into even number of pions, the lower arc — its annihilation into odd number of pions. The observed maximum points out that there is a connected system$\bar{p}n$. Abscissa axis represents the proton impulse in GeV/sec (and the connection energy of the system $\bar{p}n$). Ordinate axis — the number of events. Cited from [6].

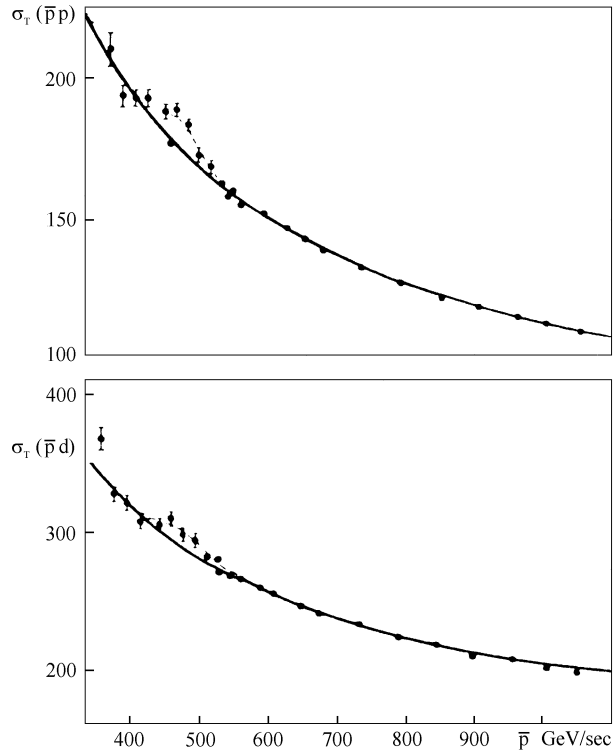

Fig. 2: Probability σ of interaction between $\bar{p}$, p and deutrons d (cited from [7]). The presence of maximum stands out the existence of the resonance state of "nucleon — anti-nucleon".

is unstable by definition having life span no more than 10^{-20} sec.

Unfortunately, the researchers limited the research to the consideration of $\bar{p}p$ and $\bar{p}n$ "atoms" only. The reason was that they, in the absence of a theory, considered $\bar{p}p$ and $\bar{p}n$ "atoms" as only a rare exception, which gives no classes of matter.

Despite Benn Tannenbaum's and Randall J. Scalise's rejections of unmatter and Scalise's personal attack on me in a true Ancient Inquisitionist style under MadSci moderator John Link's tolerance (MadSci web site, June-July 2005), the unmatter does exists, for example some messons and antimessons, through for a trifling of a second lifetime, so the pions are unmatter*, the kaon K^+ (us^), K^- (u^s), Phi (ss^), D^+ (cd^), D^0 (cu^), D_s^+ (cs^), J/Psi (cc^), B^- (bu^), B^0 (db^), B_s^0 (sb^), Upsilon (bb^), etc. are unmatter too†.

Also, the pentaquark theta-plus Θ^+, of charge $+1$, uudds^ (i. e. two quarks up, two quarks down, and one anti-strange quark), at a mass of 1.54 GeV and a narrow width of 22 MeV, is unmatter, observed in 2003 at the Jefferson Lab in Newport News, Virginia, in the experiments that involved multi-GeV photons impacting a deuterium target. Similar pentaquark evidence was obtained by Takashi Nakano of Osaka University in 2002, by researchers at the ELSA accelerator in Bonn in 1997-1998, and by researchers at ITEP in Moscow in 1986. Besides theta-plus, evidence has been found in one experiment [4] for other pentaquarks, Ξ_s^- (ddssu^) and Ξ_s^+ (uussd^).

In order for the paper to be self-contained let's recall that the *pionium* is formed by a π^+ and π^- mesons, the *positronium* is formed by an antielectron (positron) and an electron in a semi-stable arrangement, the *protonium* is formed by a proton and an antiproton also semi-stable, the *antiprotonic helium* is formed by an antiproton and electron together with the helium nucleus (semi-stable), and *muonium* is formed by a positive muon and an electron. Also, the *mesonic atom* is an ordinary atom with one or more of its electrons replaced by negative mesons. The *strange matter* is a ultra-dense matter formed by a big number of strange quarks bounded together with an electron atmosphere (this strange matter is hypothetical).

From the exotic atom, the pionium, positronium, protonium, antiprotonic helium, and muonium are unmatter. The mesonic atom is unmatter if the electron(s) are replaced by negatively-charged antimessons. Also we can define a mesonic antiatom as an ordinary antiatomic nucleous with one or more of its antielectrons replaced by positively-charged mesons. Hence, this mesonic antiatom is unmatter if the antielectron(s) are replaced by positively-charged messons. The strange matter can be unmatter if these exists at least an antiquark together with so many quarks in the nucleous. Also, we can define the strange antimatter as formed by

*Which have the composition u^d and ud^, where by u^ we mean anti-up quark, d = down quark, and analogously u = up quark and d^ = anti-down quark, while by ^ we mean "anti".

†Here c = charm quark, s = strange quark, b = bottom quark.

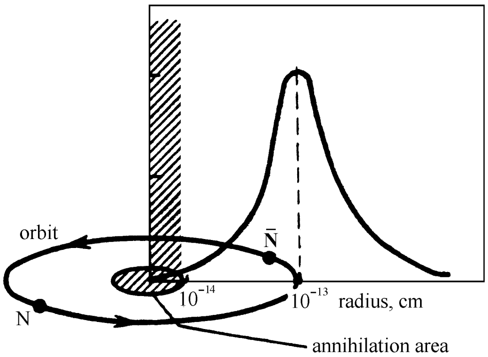

Fig. 3: Annihilation area and the probability arc in "nucleon — anti-nucleon" system (cited from [11]).

a large number of antiquarks bound together with an antielectron around them. Similarly, the strange antimatter can be unmatter if there exists at least one quark together with so many antiquarks in its nucleous.

The bosons and antibosons help in the decay of unmatter. There are 13 + 1 (Higgs boson) known bosons and 14 antibosons in present.

5 Quantum Chromodynamics formula

In order to save the colorless combinations prevailed in the Theory of Quantum Chromodynamics (QCD) of quarks and antiquarks in their combinations when binding, we devise the following formula:

$$\mathrm{Q} - \mathrm{A} \in \pm \mathrm{M3}\,, \qquad (1)$$

where M3 means multiple of three, i. e. $\pm\mathrm{M3} = \{3k \mid k \in Z\} = \{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\}$, and Q = number of quarks, A = number of antiquarks. But (1) is equivalent to

$$\mathrm{Q} \equiv \mathrm{A} \pmod 3 \qquad (2)$$

(Q is congruent to A modulo 3).

To justify this formula we mention that 3 quarks form a colorless combination, and any multiple of three (M3) combination of quarks too, i. e. 6, 9, 12, etc. quarks. In a similar way, 3 antiquarks form a colorless combination, and any multiple of three (M3) combination of antiquarks too, i. e. 6, 9, 12, etc. antiquarks. Hence, when we have hybrid combinations of quarks and antiquarks, a quark and an antiquark will annihilate their colors and, therefore, what's left should be a multiple of three number of quarks (in the case when the number of quarks is bigger, and the difference in the formula is positive), or a multiple of three number of antiquarks (in the case when the number of antiquarks is bigger, and the difference in the formula is negative).

6 Quark-antiquark combinations

Let's note by q = quark ∈ {Up, Down, Top, Bottom, Strange, Charm}, and by a = antiquark ∈ {Upˆ, Downˆ, Topˆ, Bottomˆ, Strangeˆ, Charmˆ}. Hence, for combinations of n quarks and antiquarks, $n \geqslant 2$, prevailing the colorless, we have the following possibilities:

- if n = 2, we have: qa (biquark — for example the mesons and antimessons);
- if n = 3, we have qqq, aaa (triquark — for example the baryons and antibaryons);
- if n = 4, we have qqaa (tetraquark);
- if n = 5, we have qqqqa, aaaaq (pentaquark);
- if n = 6, we have qqqaaa, qqqqqq, aaaaaa (hexaquark);
- if n = 7, we have qqqqqaa, qqaaaaa (septiquark);
- if n = 8, we have qqqqaaaa, qqqqqqaa, qqaaaaaa (octoquark);
- if n = 9, we have qqqqqqqqq, qqqqqqaaa, qqqaaaaaa, aaaaaaaaa (nonaquark);
- if n = 10, we have qqqqqaaaaa, qqqqqqqqaa, qqaaaaaaaa (decaquark); etc.

7 Unmatter combinations

From the above general case we extract the unmatter combinations:

- For combinations of 2 we have: qa (unmatter biquark), mesons and antimesons; the number of all possible unmatter combinations will be 6×6 = 36, but not all of them will bind together.

It is possible to combine an entity with its mirror opposite and still bound them, such as: uuˆ, ddˆ, ssˆ, ccˆ, bbˆ which form mesons. It is possible to combine, unmatter + unmatter = unmatter, as in udˆ + usˆ = uudˆsˆ (of course if they bind together).

- For combinations of 3 (unmatter triquark) we can not form unmatter since the colorless can not hold.
- For combinations of 4 we have: qqaa (unmatter tetraquark); the number of all possible unmatter combinations will be $6^2{\times}6^2 = 1{,}296$, but not all of them will bind together.
- For combinations of 5 we have: qqqqa, or aaaaq (unmatter pentaquarks); the number of all possible unmatter combinations will be $6^4{\times}6 + 6^4{\times}6 = 15{,}552$, but not all of them will bind together.
- For combinations of 6 we have: qqqaaa (unmatter hexaquarks); the number of all possible unmatter combinations will be $6^3{\times}6^3 = 46{,}656$, but not all of them will bind together.
- For combinations of 7 we have: qqqqqaa, qqaaaaa (unmatter septiquarks); the number of all possible unmatter combinations will be $6^5{\times}6^2 + 6^2{\times}6^5 = 559{,}872$, but not all of them will bind together.

- For combinations of 8 we have: qqqqaaaa, qqqqqqqa, qaaaaaaa (unmatter octoquarks); the number of all the unmatter combinations will be $6^4 \times 6^4 + 6^7 \times 6^1 + 6^1 \times 6^7 =$ $= 5{,}038{,}848$, but not all of them will bind together.
- For combinations of 9 we have types: qqqqqqaaa, qqqaaaaaa (unmatter nonaquarks); the number of all the unmatter combinations will be $6^6 \times 6^3 + 6^3 \times 6^6 = 2 \times 6^9$ $= 20{,}155{,}392$, but not all of them will bind together.
- For combinations of 10 we have types: qqqqqqqqaa, qqqqqaaaaa, qqaaaaaaaa (unmatter decaquarks); the number of all the unmatter combinations will be $3 \times 6^{10} = 181{,}398{,}528$, but not all of them will bind together. Etc.

I wonder if it is possible to make infinitely many combinations of quarks/antiquarks and leptons/antileptons... Unmatter can combine with matter and/or antimatter and the result may be any of these three. Some unmatter could be in the strong force, hence part of hadrons.

8 Unmatter charge

The charge of unmatter may be positive as in the pentaquark theta-plus, 0 (as in positronium), or negative as in anti-ρ-meson (u^d) (M. Jordan).

9 Containment

I think for the containment of antimatter and unmatter it would be possible to use electromagnetic fields (a container whose walls are electromagnetic fields). But its duration is unknown.

10 Further research

Let's start from neutrosophy [13], which is a generalization of dialectics, i. e. not only the opposites are combined but also the neutralities. Why? Because when an idea is launched, a category of people will accept it, others will reject it, and a third one will ignore it (don't care). But the dynamics between these three categories changes, so somebody accepting it might later reject or ignore it, or an ignorant will accept it or reject it, and so on. Similarly the dynamicity of <A>, <antiA>, <neutA>, where <neutA> means neither <A> nor <antiA>, but in between (neutral). Neutrosophy considers a kind not of di-alectics but tri-alectics (based on three components: <A>, <antiA>, <neutA>). Hence unmatter is a kind of neutrality (not referring to the charge) between matter and antimatter, i. e. neither one, nor the other.

Upon the model of unmatter we may look at ungravity, unforce, unenergy, etc.

Ungravity would be a mixture between gravity and antigravity (for example attracting and rejecting simultaneously or alternatively; or a magnet which changes the + and − poles frequently).

Unforce. We may consider positive force (in the direction we want), and negative force (repulsive, opposed to the previous). There could be a combination of both positive and negative forces in the same time, or alternating positive and negative, etc.

Unenergy would similarly be a combination between positive and negative energies (as the alternating current, a. c., which periodically reverses its direction in a circuit and whose frequency, f, is independent of the circuit's constants). Would it be possible to construct an alternating-energy generator?

To conclusion: According to the Universal Dialectic the unity is manifested in duality and the duality in unity. "Thus, Unmatter (unity) is experienced as duality (matter vs antimatter). Ungravity (unity) as duality (gravity vs antigravity). Unenergy (unity) as duality (positive energy vs negative energy) and thus also ... between duality of being (existence) vs nothingness (antiexistence) must be 'unexistence' (or pure unity)" (R. Davic).

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