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CONTENTS

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A New Method to Measure the Speed of Gravitation

Dmitri Rabounski
E-mail: rabounski@yahoo.com

According to the standard viewpoint the speed of gravitation is the speed of weak waves of the metrics. This study proposes a new approach, defining the speed as the speed of travelling waves in the field of gravitational inertial force. D'Alembert's equations of the field show that this speed is equal to the velocity of light corrected by gravitational potential. The approach leads to a new experiment to measure the speed of gravitation, which, using "detectors" such as planets and their satellites, is not linked to deviation of geodesic lines and quadrupole mass-detectors with their specific technical problems.

1 Introduction

Herein we use a pseudo-Riemannian space with the signature $(+---)$, where time is real and spatial coordinates are imaginary, because the projection of a four-dimensional impulse on the spatial section of any given observer is positive in this case. We also denote space-time indices in Greek, while spatial indices are Roman. Hence the time term in d'Alembert's operator $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ will be positive, while the spatial part (Laplace's operator) will be negative $\Delta = -g^{ik}\nabla_i\nabla_k$.

By applying the d'Alembert operator to a tensor field, we obtain the d'Alembert equations of the field. The non-zero elements are the d'Alembert equations containing the field-inducing sources. The zero elements are the equations without the sources. If there are no sources the field is free, giving a free wave. There is the time term $\frac{1}{a^2}\frac{\partial^2}{\partial t^2}$ containing the linear velocity a of the wave. For this reason, in the case of gravitational fields, the d'Alembert equations give rise to a possibility of calculating the speed of propagation of gravitational attraction (the speed of gravitation). At the same time the result may be different according to the way we define the speed as the velocity of waves of the metric, or something else.

The usual approach sets forth the speed of gravitation as follows [1, 5]. One considers the space-time metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$, composed of a Galilean metric $g^{(0)}_{\alpha\beta}$ (wherein $g^{(0)}_{00} = 1$, $g^{(0)}_{0i} = 0$, $g^{(0)}_{ik} = -\delta_{ik}$) and tiny corrections $\zeta_{\alpha\beta}$ defining a weak gravitational field. Because the $\zeta_{\alpha\beta}$ are tiny, we can raise and lower indices with the Galilean metric tensor $g^{(0)}_{\alpha\beta}$. The quantities $\zeta^{\alpha\beta}$ are defined by the main property of the fundamental metric tensor $g_{\alpha\sigma}g^{\sigma\beta} = \delta^\beta_\alpha$ as follows: $(g^{(0)}_{\alpha\sigma} + \zeta_{\alpha\sigma})\, g^{\sigma\beta} = \delta^\beta_\alpha$. Besides this approach defines $g^{\alpha\beta}$ and $g = \det\|g_{\alpha\beta}\|$ to within higher order terms withheld as $g^{\alpha\beta} = g^{(0)\alpha\beta} - \zeta^{\alpha\beta}$ and $g = g^{(0)}(1+\zeta)$, where $\zeta = \zeta^\sigma_\sigma$. Because $\zeta_{\alpha\beta}$ are tiny we can take Ricci's tensor $R_{\alpha\beta} = R^{\cdots\sigma}_{\alpha\sigma\beta}$ (the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ contracted on two indices) to within higher order terms withheld. Then the Ricci tensor for the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$ is

$$R_{\alpha\beta} = \frac{1}{2}\, g^{(0)\mu\nu}\frac{\partial^2\zeta_{\alpha\beta}}{\partial x^\mu \partial x^\nu} = \frac{1}{2}\Box\zeta_{\alpha\beta}\,,$$

which simplifies Einstein's field equations $R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = -\kappa\, T_{\alpha\beta} + \lambda g_{\alpha\beta}$, where in this case $R = g^{(0)\mu\nu}R_{\mu\nu}$. In the absence of matter and λ-fields ($T_{\alpha\beta} = 0$, $\lambda = 0$), that is, in emptiness, the Einstein equations for the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$ become

$$\Box\zeta^\beta_\alpha = 0\,.$$

Actually, these are the d'Alembert equations of the corrections $\zeta_{\alpha\beta}$ to the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$ (weak waves of the metric). Taking the flat wave travelling in the direction $x^1 = x$, we see

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\zeta^\beta_\alpha = 0\,,$$

so weak waves of the metric travel at the velocity of light in empty space.

This approach leads to an experiment, based on the principle that geodesic lines of two infinitesimally close test-particles will deviate in a field of waves of the metric. A system of two real particles connected by a spring (a quadrupole mass-detector) should react to the waves. Most of these experiments have since 1968 been linked to Weber's detector. The experiments have not been technically decisive until now, because of problems with precision of measurement and other technical problems [3] and some purely theoretical problems [4, 5].

Is the approach given above the best? Really, the resulting d'Alembert equations are derived from that form of the Ricci tensor obtained under the substantial simplifications of higher order terms withheld (i.e. to first order). Eddington [1] wrote that a source of this approximation is a specific reference frame which differs from Galilean reference frames by the tiny corrections $\zeta_{\alpha\beta}$, the origin of which could be very different from gravitation. This argument leads, as Eddington remarked, to a "vicious circle". So the standard approach has inherent drawbacks, as follows:

(1) The approach gives the Ricci tensor and hence the d'Alembert equations of the metric to within higher order terms withheld, so the velocity of waves of the metric calculated from the equations is not an exact theoretical result;

(2) A source of this approximation are the tiny corrections $\zeta_{\alpha\beta}$ to a Galilean metric, the origin of which may be very different: not only gravitation;

(3) Two bodies attract one another because of the transfer of gravitational force. A wave travelling in the field of gravitational force is not the same as a wave of the metric — these are different tensor fields. When a quadrupole mass-detector registers a signal, the detector reacts to a wave of the metric in accordance with this theory. Therefore it is concluded that quadrupole mass detectors would be the means by to discovery of waves of the metric. However, the experiment is only incidental to the measurement of the speed of gravitation.

For these reasons we are lead to consider gravitational waves as waves travelling in the field of gravitational force, which provides two important advantages:

(1) The mathematical apparatus of chronometric invariants (physical observable quantities in the General Theory of Relativity) defines gravitational inertial force F_i without the Riemann-Christoffel curvature tensor [1, 2]. Using this method, we can deduce the exact d'Alembert equations for the force field, giving an exact formula for the velocity of waves of the force;

(2) Experiments to register waves of the force field, using "detectors" such as planets or their satellites, does not involve a quadrupole mass-detector and its specific technical problems.

2 The new approach

The basis here is the mathematical apparatus of chronometric invariants, created by Zelmanov in the 1940's [1, 2]. Its essence is that if an observer accompanies his reference body, his observable quantities (chronometric invariants) are projections of four-dimensional quantities on his time line and the spatial section, made by projecting operators $b^\alpha = \frac{dx^\alpha}{ds}$ and $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$, which fully define his real reference space. Thus, chr.inv.-projections of a world-vector Q^α are $b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}}$ and $h^i_\alpha Q^\alpha = Q^i$, while chr.inv.-projections of a world-tensor of the 2nd rank $Q^{\alpha\beta}$ are $b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}$, $h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}$, $h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}$. Physical observable properties of the space are derived from the fact that the chr. inv.-differential operators $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2} v_i \frac{{}^*\partial}{\partial t}$ are non-commutative. They are the chr.inv.-vector of gravitational inertial force F_i, the chr.inv.-tensor of angular velocities of the space rotation A_{ik}, and the chr.inv.-tensor of rates of the space deformations D_{ik}, namely

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right),$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right),$$

$$v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}, \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2},$$

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}, \quad D = D^k_k = \frac{{}^*\partial \ln\sqrt{h}}{\partial t},$$

where w is gravitational potential, v_i is the linear velocity of the space rotation, $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$ is the chr.inv.-metric tensor, and also $h = \det\|h_{ik}\|$, $\sqrt{-g} = \sqrt{h}\sqrt{g_{00}}$, $g = \det\|g_{\alpha\beta}\|$. Observable non-uniformity of the space is set up by the chr.inv.-Christoffel symbols $\Delta^i_{jk} = h^{im}\Delta_{jk,m}$, which are built just like Christoffel's usual symbols $\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$, using h_{ik} instead of $g_{\alpha\beta}$.

The four-dimensional generalization of the chr.inv.-quantities F_i, A_{ik}, and D_{ik} had been obtained by Zelmanov [8] as $F_\alpha = -2c^2 b^\beta a_{\beta\alpha}$, $A_{\alpha\beta} = ch^\mu_\alpha h^\nu_\beta a_{\mu\nu}$, $D_{\alpha\beta} = ch^\mu_\alpha h^\nu_\beta d_{\mu\nu}$, where $a_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha b_\beta - \nabla_\beta b_\alpha)$, $d_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha b_\beta + \nabla_\beta b_\alpha)$.

Following the study [9], we consider a field of the gravitational inertial force $F_\alpha = -2c^2 b^\beta a_{\beta\alpha}$, the chr.inv.-spatial projection of which is F^i, so that $F_i = h_{ik}F^k$. The d'Alembert equations of the vector field $F^\alpha = -2c^2 b^\beta a_{\beta\cdot}^{\cdot\alpha}$ in the absence of sources are

$$\Box F^\alpha = 0\,.$$

Their chr.inv.-projections (referred to as the chr.inv.-d'Alembert equations) can be deduced as follows

$$b_\sigma\, g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta F^\sigma = 0\,, \qquad h^i_\sigma\, g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta F^\sigma = 0\,.$$

After some algebra we obtain the chr.inv.-d'Alembert equations for the field of the gravitational inertial force $F^\alpha = -2c^2 b^\beta a_{\beta\cdot}^{\cdot\alpha}$ in their final form. They are

$$\frac{1}{c^2}\frac{{}^*\partial}{\partial t}\left(F_k F^k\right) + \frac{1}{c^2}F_i\frac{{}^*\partial F^i}{\partial t} + D^k_m\frac{{}^*\partial F^m}{\partial x^k} + $$
$$+ h^{ik}\frac{{}^*\partial}{\partial x^i}\left[(D_{kn} + A_{kn})F^n\right] - \frac{2}{c^2}A_{ik}F^i F^k + $$
$$+ \frac{1}{c^2}F_m F^m D + \Delta^m_{kn}D^k_m F^n - $$
$$- h^{ik}\Delta^m_{ik}\left(D_{mn} + A_{mn}\right)F^n = 0\,,$$

$$\frac{1}{c^2}\frac{^*\partial^2 F^i}{\partial t^2} - h^{km}\frac{^*\partial^2 F^i}{\partial x^k \partial x^m} + \frac{1}{c^2}\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\frac{^*\partial F^k}{\partial t} + $$
$$ + \frac{1}{c^2}\frac{^*\partial}{\partial t}\left[\left(D^i_k + A^{\cdot i}_{k\cdot}\right)F^k\right] + \frac{1}{c^2}D\frac{^*\partial F^i}{\partial t} + \frac{1}{c^2}F^k\frac{^*\partial F^i}{\partial x^k} + $$
$$ + \frac{1}{c^2}\left(D^i_n + A^{\cdot i}_{n\cdot}\right)F^n D + \frac{1}{c^4}F_k F^k F^i + \frac{1}{c^2}\Delta^i_{km}F^k F^m - $$
$$ - h^{km}\left\{\frac{^*\partial}{\partial x^k}\left(\Delta^i_{mn}F^n\right) + \left(\Delta^i_{kn}\Delta^n_{mp} - \Delta^n_{km}\Delta^i_{np}\right)F^p + \right.$$
$$\left. + \Delta^i_{kn}\frac{^*\partial F^n}{\partial x^m} - \Delta^n_{km}\frac{^*\partial F^i}{\partial x^n}\right\} = 0\,.$$

Calling upon the formulae for chr.inv.-derivatives, we transform the first term in the chr.inv.-d'Alembert vector equations into the form

$$\frac{1}{c^2}\frac{^*\partial^2 F^i}{\partial t^2} = \frac{1}{c^2 g_{00}}\frac{\partial^2 F^i}{\partial t^2} + \frac{1}{c^4\sqrt{g_{00}}}\frac{^*\partial \mathrm{w}}{\partial t}\frac{^*\partial F^i}{\partial t}\,,$$

so waves of gravitational inertial force travel at a velocity u^k, the square of which is $u_k u^k = c^2 g_{00}$ and the modulus

$$u = \sqrt{u_k u^k} = c\left(1 - \frac{\mathrm{w}}{c^2}\right).$$

Because waves of the field of gravitational inertial force transfer gravitational interaction, this wave speed is the speed of gravitation as well. The speed depends on the scalar potential w of the field itself, which leads us to the following conclusions:

(1) In a weak gravitational field, the potential w of which is negligible but its gradient F_i is non-zero, the speed of gravitation equals the velocity of light;

(2) According to this formula, the speed of gravitation will be less than the velocity of light near bulky bodies like stars or planets, where gravitational potential is perceptible. On the Earth's surface slowing gravitation will be slower than light by 21 cm/sec. Gravitation near the Sun will be about 6.3×10^4 cm/sec slower than light;

(3) Under gravitational collapse ($\mathrm{w}=c^2$) the speed of gravitation becomes zero.

Let us turn now from theory to experiment. An idea as to how to measure the speed of gravitation as the speed to transfer of the attracting force between space bodies had been proposed by the mathematician Dombrowski [10] in conversation with me more than a decade ago. But in the absence of theory the idea had not developed to experiment in that time. Now we have an exact formula for the speed of waves travelling in the field of gravitational inertial force, so we can propose an experiment to measure the speed (a Weber detector reacts to weak waves of the metric, so it does not apply to this experiment).

The Moon attracts the Earth's surface, causing the flow "hump" in the ocean surface that follows the moving Moon, producing ebbs and flows. An analogous "hump" follows the Sun: its magnitude is more less. A satellite in an Earth orbit has the same ebb and flow oscillations — its orbit rises and falls a little, following the Moon and the Sun as well. A satellite in space experiences no friction, contrary of the viscous waters of the oceans. A satellite is a perfect system, which reacts instantly to the flow. If the speed of gravitation is limited, the moment of the satellite's maximum flow rise should be later than the lunar/solar upper transit by the amount of time taken by waves of the gravitational force field to travel from the Moon/Sun to the satellite.

The Earth's gravitational field is not absolutely symmetric, because of the imperfect form of the terrestrial globe. A real satellite reacts to the field defects during its orbital flight around the Earth — the height of its orbit oscillates in decimetres, giving rise to substantial noise in the experiment. For this reason a geostationary satellite would be best. Such a satellite, having an equatorial orbit, requires an angular velocity the same as that of the Earth. As a result, the height of a geostationary satellite above the Earth does not depend on non-uniformities of the Earth's gravitational field. The height could be measured with high precision by a laser rangefinder, almost without interruption, providing a possibility of registering the moment of the maximum flow rise of the satellite, perfectly.

In accordance with our formula the speed of gravitation near the Earth is 21 cm/sec less than the velocity of light. In this case the maximum of the lunar flow wave in a satellite orbit will be about 1 sec later than the lunar upper culmination. The lateness of the flow wave of the Sun will be about 500 sec after the upper transit of the Sun. The question is how precisely could the moment of the maximum flow rise of a satellite in its orbit be determined, because the real maximum can be "fuzzy" in time.

3 Effect of the curvature

If a space is homogeneous ($\Delta^i_{km}=0$) and it is free of rotation and deformation ($A_{ik}=0$, $D_{ik}=0$), then the chr.inv.-d'Alembert equations for the field of gravitational inertial force take the form

$$\frac{1}{c^2}\frac{^*\partial}{\partial t}\left(F_k F^k\right) + \frac{1}{c^2}F_i\frac{^*\partial F^i}{\partial t} = 0\,,$$

$$\frac{1}{c^2}\frac{^*\partial^2 F^i}{\partial t^2} - h^{km}\frac{^*\partial^2 F^i}{\partial x^k\partial x^m} + \frac{1}{c^2}F^k\frac{^*\partial F^i}{\partial x^k} + \frac{1}{c^4}F_k F^k F^i = 0\,,$$

so waves of gravitational inertial force are permitted even in this very simple case.

Are waves of the metric possible in this case or not?

As it is known, waves of the metric are linked to the space-time curvature derived from the Riemann-Christoffel curvature tensor. If the first derivatives of the metric (the space deformations) are zero, then its second derivatives (the curvature) are zero too. Therefore waves of the metric have no place in a non-deforming space, while waves of gravitational inertial force are possible there.

In connection with this fact, following the study [9], another question arises. By how much does the curvature affect waves of gravitational inertial force?

To answer the question let us recall that Zelmanov, following the same procedure by which the Riemann-Christoffel tensor was introduced, after considering non-commutativity of the chr.inv.-second derivatives of a vector ${}^*\nabla_i{}^*\nabla_k\,Q_l - {}^*\nabla_k{}^*\nabla_i\,Q_l = \frac{2A_{ik}}{c^2}\frac{{}^*\partial Q_l}{\partial t} + H_{lki}^{\cdot\cdot\cdot j}Q_j$, had obtained the chr. inv.-tensor $H_{lki}^{\cdot\cdot\cdot j}$ like Schouten's tensor [11]. Its generalization gives the chr.inv.-curvature tensor $C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right)$, which has all the properties of the Riemann-Christoffel tensor in the observer's spatial section. So the chr.inv.-spatial projection $Z^{iklj} = -c^2 R^{iklj}$ of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$, after contraction twice by h_{ik}, is $Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2C$, where $C = C_j^j = h^{lj}C_{lj}$ and $C_{kj} = C_{kij\cdot}^{\cdot\cdot\cdot i} = h^{im}C_{kimj}$ [1].

At the same time, as Synge's well-known book [12] shows, in a space of constant four-dimensional curvature, $K = \text{const}$, we have $R_{\alpha\beta\gamma\delta} = K\left(g_{\alpha\gamma}\,g_{\beta\delta} - g_{\alpha\delta}\,g_{\beta\gamma}\right)$, $R_{\alpha\beta} = -3K g_{\alpha\beta}$, $R = -12K$. With these formulae as a basis, after calculation of the chr.inv.-spatial projection of the Riemann-Christoffel tensor, we deduce that in a constant curvature space $Z = 6c^2K$. Equating this to the same quantity in an arbitrary curvature space, we obtain a correlation between the four-dimensional curvature K and the observable three-dimensional curvature in the constant curvature space

$$6c^2K = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2C\,.$$

If the four-dimensional curvature is zero ($K = 0$), and the space does no deformations ($D_{ik} = 0$ — its metric is stationary, $h_{ik} = \text{const}$), then no waves of the metric are possible. In such a space the observable three-dimensional curvature is

$$C = -\frac{1}{c^2}A_{ik}A^{ik},$$

which is non-zero ($C \neq 0$), only if the space rotates ($A_{ik} \neq 0$). If aside of these factors, the space does not rotate, then its observable curvature also becomes zero; $C = 0$. Even in this case the chr.inv.- d'Alembert equations show the presence of waves of gravitational inertial force.

What does this imply? As a matter of fact, gravitational attraction is an everyday reality in our world, so waves of gravitational inertial force transferring the attraction shall be incontrovertible. Therefore we adduce the alternatives:

(1) Waves of gravitational inertial force depend on a curvature of space — then the real space-time is not a space of constant curvature, or,

(2) Waves of gravitational inertial force do not depend on the curvature.

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Quantum Quasi-Paradoxes and Quantum Sorites Paradoxes

Florentin Smarandache
Dept. of Mathematics, University of New Mexico, 200 College Road, Gallup, NM 87301, USA
E-mail: fsmarandache@yahoo.com; smarand@unm.edu

There can be generated many paradoxes or quasi-paradoxes that may occur from the combination of quantum and non-quantum worlds in physics. Even the passage from the micro-cosmos to the macro-cosmos, and reciprocally, can generate unsolved questions or counter-intuitive ideas. We define a quasi-paradox as a statement which has a *prima facie* self-contradictory support or an explicit contradiction, but which is not completely proven as a paradox. We present herein four elementary quantum quasi-paradoxes and their corresponding quantum Sorites paradoxes, which form a class of quantum quasi-paradoxes.

1 Introduction

According to the Dictionary of Mathematics (Borowski and Borwein, 1991 [1]), the **paradox** is "an apparently absurd or self-contradictory statement for which there is *prima facie* support, or an explicit contradiction derived from apparently unexceptionable premises". Some paradoxes require the revision of their intuitive conception (Russell's paradox, Cantor's paradox), others depend on the inadmissibility of their description (Grelling's paradox), others show counter-intuitive features of formal theories (Material implication paradox, Skolem Paradox), others are self-contradictory — Smarandache Paradox: "All is <A> the <Non-A> too!", where <A> is an attribute and <Non-A> its opposite; for example "All is possible the impossible too!" (Weisstein, 1998 [2]).

Paradoxes are normally true and false in the same time.

The **Sorites paradoxes** are associated with Eubulides of Miletus (fourth century B. C.) and they say that there is not a clear frontier between visible and invisible matter, determinist and indeterminist principle, stable and unstable matter, long time living and short time living matter.

Generally, between <A> and <Non-A> there is no clear distinction, no exact frontier. Where does <A> really end and <Non-A> begin? One extends Zadeh's "fuzzy set" concept to the "neutrosophic set" concept.

Let's now introduce the notion of quasi-paradox:

A **quasi-paradox** is a statement which has a *prima facia* self-contradictory support or an explicit contradiction, but which is not completely proven as a paradox. A quasi-paradox is an *informal* contradictory statement, while a paradox is a *formal* contradictory statement.

Some of the below quantum quasi-paradoxes can later be proven as real quantum paradoxes.

2 Quantum Quasi-Paradoxes and Quantum Sorites Paradoxes

The below quasi-paradoxes and Sorites paradoxes are based on the antinomies: visible/invisible, determinist/indeterminist, stable/unstable, long time living/short time living, as well as on the fact that there is not a clear separation between these pairs of antinomies.

2.1.1 **Invisible Quasi-Paradox:** Our visible world is composed of a totality of invisible particles.

2.1.2 **Invisible Sorites Paradox:** There is not a clear frontier between visible matter and invisible matter.

(a) An invisible particle does not form a visible object, nor do two invisible particles, three invisible particles, etc. However, at some point, the collection of invisible particles becomes large enough to form a visible object, but there is apparently no definite point where this occurs.

(b) A similar paradox is developed in an opposite direction. It is always possible to remove a particle from an object in such a way that what is left is still a visible object. However, repeating and repeating this process, at some point, the visible object is decomposed so that the left part becomes invisible, but there is no definite point where this occurs.

2.2.1 **Uncertainty Quasi-Paradox:** Large matter, which is at some degree under the "determinist principle", is formed by a totality of elementary particles, which are under Heisenberg's "indeterminacy principle".

2.2.2 **Uncertainty Sorites Paradox:** Similarly, there is not a clear frontier between the matter under the "determinist principle" and the matter under "indeterminist principle".

2.3.1 **Unstable Quasi-Paradox:** "Stable" matter is formed by "unstable" elementary particles (elementary particles decay when free).

2.3.2 **Unstable Sorites Paradox:** Similarly, there is not a clear frontier between the "stable matter" and the "unstable matter".

2.4.1 **Short-Time-Living Quasi-Paradox:** "Long-time-living" matter is formed by very "short-time-living" elementary particles.

2.4.2 **Short-Time-Living Sorites Paradox:** Similarly, there is not a clear frontier between the "long-time-living" matter and the "short-time-living" matter.

3 Conclusion

"More such quantum quasi-paradoxes and paradoxes can be designed, all of them forming a class of Smarandache quantum quasi-paradoxes." (Dr. M. Khoshnevisan, Griffith University, Gold Coast, Queensland, Australia [3])

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A New Form of Matter — Unmatter, Composed of Particles and Anti-Particles

Florentin Smarandache
Dept. of Mathematics, University of New Mexico, 200 College Road, Gallup, NM 87301, USA
E-mail: fsmarandache@yahoo.com; smarand@unm.edu

Besides *matter* and *antimatter* there must exist *unmatter* (as a new form of matter) in accordance with the neutrosophy theory that between an entity <A> and its opposite <AntiA> there exist intermediate entities <NeutA>. Unmatter is neither matter nor antimatter, but something in between. An atom of unmatter is formed either by (1): electrons, protons, and antineutrons, or by (2): antielectrons, antiprotons, and neutrons. At CERN it will be possible to test the production of unmatter. The existence of unmatter in the universe has a similar chance to that of the antimatter, and its production also difficult for present technologies.

1 Introduction

This article is an improved version of an old manuscript [1]. This is a theoretical assumption about the possible existence of a new form of matter. Up to day the unmatter was not checked in the lab.

According to the neutrosophy theory in philosophy [2], between an entity <A> and its opposite <AntiA> there exist intermediate entities <NeutA> which are neither <A> nor <AntiA>.

Thus, between "matter" and "antimatter" there must exist something which is neither matter nor antimatter, let's call it UNMATTER.

In neutrosophy, <NonA> is what is not <A>, i. e. <NonA> = <AntiA> $\cup$ <NeutA>. Then, in physics, NONMATTER is what is not matter, i. e. nonmatter means antimatter together with unmatter.

2 Classification

A. Matter is made out of electrons, protons, and neutrons.

Each matter atom has electrons, protons, and neutrons, except the atom of ordinary hydrogen which has no neutron.

The number of electrons is equal to the number of protons, and thus the matter atom is neutral.

B. Oppositely, the **antimatter** is made out of antielectrons, antiprotons, and antineutrons.

Each antimatter atom has antielectrons (positrons), antiprotons, and antineutrons, except the antiatom of ordinary hydrogen which has no antineutron.

The number of antielectrons is equal to the number of antiprotons, and thus the antimatter atom is neutral.

C. Unmatter means neither matter nor antimatter, but in between, an entity which has common parts from both of them.

Etymologically "un-matter" comes from [ME $<$ OE, akin to Gr. *an-*, *a-*, Latin *in-*, and to the negative elements in *no*, *not*, *nor*] and [ME *matière* $<$ OFr $<$ Latin *material*] matter (see [3]), signifying no/without/off the matter.

There are two types of unmatter atoms, that we call unatoms:

u1. The first type is derived from matter; and a such unmatter atom is formed by electrons, protons, and antineutrons;

u2. The second type is derived from antimatter, and a such unmatter atom is formed by antielectrons, antiprotons, and neutrons.

One unmatter type is oppositely charged with respect to the other, so when they meet they annihilate.

The unmatter nucleus, called **unnucleus**, is formed either by protons and antineutrons in the first type, or by antiprotons and neutrons in the second type.

The charge of unmatter should be neutral, as that of matter or antimatter.

The charge of un-isotopes will also be neutral, as that of isotopes and anti-isotopes. But, if we are interested in a negative or positive charge of un-matter, we can consider an un-ion. For example an anion is negative, then its corresponding unmatter of type 1 will also be negative. While taking a cation, which is positive, its corresponding unmatter of type 1 will also be positive.

Sure, it might be the question of how much *stable* the unmatter is, as J. Murphy pointed out in a private e-mail. But Dirac also theoretically supposed the existence of antimatter in 1928 which resulted from Dirac's mathematical equation, and finally the antimatter was discovered/produced in large accelerators in 1996 when it was created the first atom of antihydrogen which lasted for 37 nanoseconds only.

There does not exist an unmatter atom of ordinary hydrogen, neither an unnucleus of ordinary hydrogen since the ordinary hydrogen has no neutron. Yet, two isotopes of the hydrogen, *deuterium* (^{2}H) which has one neutron, and artificially made *tritium* (^{3}H) which has two neutrons have corresponding unmatter atoms of both types, *un-deuterium* and *un-tritium* respectively. The isotopes of an element X differ in the number of neutrons, thus their nuclear mass is different, but their nuclear charges are the same.

For all other matter atom X, there is corresponding an antimatter atom and two unmatter atoms

The unmatter atoms are also neutral for the same reason that either the number of electrons is equal to the number of protons in the first type, or the number of antielectrons is equal to the number of antiprotons in the second type.

If antimatter exists then a higher probability would be for the unmatter to exist, and reciprocally.

Unmatter atoms of the same type stick together form an **unmatter molecule** (we call it **unmolecule**), and so on. Similarly one has two types of unmatter molecules.

The *isotopes* of an atom or element X have the same atomic number (same number of protons in the nucleus) but different atomic masses because the different number of neutrons.

Therefore, similarly the **un-isotopes of type 1** of X will be formed by electrons, protons, and antineutrons, while the **un-isotopes of type 2** of X will be formed by antielectrons, antiprotons, and neutrons.

An *ion* is an atom (or group of atoms) X which has last one or more electrons (and as a consequence carries a negative charge, called *anion*, or has gained one or more electrons (and as a consequence carries a positive charge, called *cation*).

Similarly to isotopes, the **un-ion of type 1** (also called **un-anion 1** or **un-cation 1** if resulted from a negatively or respectively positive charge ion) of X will be formed by electrons, protons, and antineutrons, while the **un-ion of type 2** of X (also called **un-anion 2** or **un-cation 2** if resulted from a negatively or respectively positive charge ion) will be formed by antielectrons, antiprotons, and neutrons.

The ion and the un-ion of type 1 have the same charges, while the ion and un-ion of type 2 have opposite charges.

D. Nonmatter means what is not matter, therefore nonmatter actually comprises antimatter and unmatter. Similarly one defines a nonnucleus.

3 Unmatter propulsion

We think (as a prediction or supposition) it could be possible at using unmatter as fuel for space rockets or for weapons platforms because, in a similar way as antimatter is presupposed to do [4, 5], its mass converted into energy will be fuel for propulsion.

It seems to be a little easier to build unmatter than antimatter because we need say antielectrons and antiprotons only (no need for antineutrons), but the resulting energy might be less than in matter-antimatter collision.

We can collide unmatter 1 with unmatter 2, or unmatter 1 with antimatter, or unmatter 2 with matter.

When two, three, or four of them (unmatter 1, unmatter 2, matter, antimatter) collide together, they annihilate and turn into energy which can materialize at high energy into new particles and antiparticles.

4 Existence of unmatter

The existence of unmatter in the universe has a similar chance to that of the antimatter, and its production also difficult for present technologies. At CERN it will be possible to test the production of unmatter.

If antimatter exists then a higher probability would be for the unmatter to exist, and reciprocally.

The 1998 Alpha Magnetic Spectrometer (AMS) flown on the International Space Station orbiting the Earth would be able to detect, besides cosmic antimatter, unmatter if any.

5 Experiments

Besides colliding electrons, or protons, would be interesting in colliding neutrons. Also, colliding a neutron with an antineutron in accelerators.

We think it might be easier to produce in an experiment an unmatter atom of deuterium (we can call it un-deuterium of type 1). The deuterium, which is an isotope of the ordinary hydrogen, has an electron, a proton, and a neutron. The idea would be to convert/transform in a deuterium atom the neutron into an antineutron, then study the properties of the resulting un-deuterium 1.

Or, similarly for un-deuterium 2, to convert/transform in a deuterium atom the electron into an antielectron, and the proton into an antiproton (we can call it un-deuterium of type 2).

Or maybe choose another chemical element for which any of the previous conversions/transformations might be possible.

6 Neutrons and antineutrons

Hadrons consist of baryons and mesons and interact via strong force.

Protons, neutrons, and many other hadrons are composed from quarks, which are a class of fermions that possess a fractional electric charge. For each type of quark there exists a corresponding antiquark. Quarks are characterized by properties such as *flavor* (up, down, charm, strange, top, or bottom) and *color* (red, blue, or green).

A neutron is made up of quarks, while an antineutron is made up of antiquarks.

A neutron (see [9]) has one Up quark (with the charge of $+\frac{2}{3}\times 1.606\times 10^{19}$ C) and two Down quarks (each with the charge of $-\frac{1}{3}\times1.606\times10^{19}$ C), while an antineutron has one anti Up quark (with the charge of $-\frac{2}{3}\times1.606\times10^{19}$ C) and two anti Down quarks (each with the charge of $+\frac{1}{3}\times1.606\times\times10^{19}$ C).

An antineutron has also a neutral charge, through it is opposite to a neutron, and they annihilate each other when meeting.

Both, the neutron and the antineutron, are neither attracted to nor repelling from charges particles.

7 Characteristics of unmatter

Unmatter should look identical to antimatter and matter, also the gravitation should similarly act on all three of them. Unmatter may have, analogously to antimatter, utility in medicine and may be stored in vacuum in traps which have the required configuration of electric and magnetic fields for several months.

8 Open Questions

8.a Can a matter atom and an unmatter atom of first type stick together to form a molecule?

8.b Can an antimatter atom and an unmatter atom of second type stick together to form a molecule?

8.c There might be not only a You and an anti-You, but some versions of an un-You in between You and anti-You. There might exist un-planets, un-stars, un-galaxies? There might be, besides our universe, an anti-universe, and more un-universes?

8.d Could this unmatter explain why we see such an imbalance between matter and antimatter in our corner of the universe? (Jeff Farinacci)

8.e If matter is thought to create gravity, is there any way that antimatter or unmatter can create antigravity or ungravity? (Mike Shafer from Cornell University)

I assume that since the magnetic field or the gravitons generate gravitation for the matter, then for antimatter and unmatter the corresponding magnetic fields or gravitons would look different since the charges of subatomic particles are different. . .

I wonder how would the universal law of attraction be for antimmater and unmatter?

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Fractality Field in the Theory of Scale Relativity

Laurent Nottale

Centre National de la Recherche Scientifique, Laboratoire "Univers et Théories", Observatoire de Paris-Meudon, F-92195 Meudon Cedex, France

E-mail: laurent.nottale@obspm.fr

In the theory of scale relativity, space-time is considered to be a continuum that is not only curved, but also non-differentiable, and, as a consequence, fractal. The equation of geodesics in such a space-time can be integrated in terms of quantum mechanical equations. We show in this paper that the quantum potential is a manifestation of such a fractality of space-time (in analogy with Newton's potential being a manifestation of curvature in the framework of general relativity).

1 Introduction

The theory of scale relativity aims at describing a non-differentiable continuous manifold by the building of new tools that implement Einstein's general relativity concepts in the new context (in particular, covariant derivative and geodesics equations). We refer the reader to Refs. [1, 2, 3, 4] for a detailed description of the construction of these tools. In the present short research note, we want to address a specific point of the theory, namely, the emergence of an additional potential energy which manifests the fractal and nondifferentiable geometry.

2 Non relativistic quantum mechanics

2.1 Quantum potential

In the scale relativity approach, one decomposes the velocity field on the geodesics bundle of a nondifferentiable space-time in terms of a classical, differentiable part, $\mathcal{V}$, and of a fractal, divergent, nondifferentiable part $\mathcal{W}$ of zero mean. Both velocity fields are complex due to a fundamental two-valuedness of the classical (differentiable) velocity issued from the nondifferentiability [1]. Then one builds a complex covariant total derivative that reads in the simplest case (spinless particle, nonrelativistic velocities and no external field) [1, 2, 3]

$$\frac{\widehat{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\nabla - i\,\mathcal{D}\Delta\,. \tag{1}$$

The constant $2\mathcal{D} = \langle d\xi^2 \rangle / dt$ ($= \hbar/m$ in standard quantum mechanics) measures the amplitude of the fractal fluctuations. Note that it is possible to have a more complete construction in which the full velocity field $\mathcal{V} + \mathcal{W}$ intervenes in the covariant derivative [6]. In the same way as in general relativity, the geodesics equation can therefore be written, using this covariant derivative, in terms of a free, inertial motion-like equation,

$$\frac{\widehat{d}\mathcal{V}}{dt} = 0\,. \tag{2}$$

Let us explicitly introduce the real and imaginary parts of the complex velocity $\mathcal{V} = V - iU$,

$$\frac{\widehat{d}\mathcal{V}}{dt} = \left(\left\{\frac{\partial}{\partial t} + V.\nabla\right\} - i\left\{U.\nabla + \mathcal{D}\Delta\right\}\right)(V - iU) = 0\,. \tag{3}$$

We see in this expression that the real part of the covariant derivative, $\widehat{d}_R/dt = \partial/\partial t + V.\nabla$, is the standard total derivative expressed in terms of partial derivatives, while the new terms are included in the imaginary part, $\widehat{d}_I/dt = -(U.\nabla + \mathcal{D}\Delta)$. The field will find its origin in the consequences of these additional terms on the imaginary part of the velocity $-U$. Indeed, by separating the real and imaginary parts, equation (3) reads:

$$\begin{aligned} &\left\{\left(\frac{\partial}{\partial t} + V.\nabla\right)V - (U.\nabla + \mathcal{D}\Delta)U\right\} - \\ &- i\left\{(U.\nabla + \mathcal{D}\Delta)V + \left(\frac{\partial}{\partial t} + V.\nabla\right)U\right\} = 0\,. \end{aligned} \tag{4}$$

Therefore the real part of this equation takes the form of an Euler-Newton equation of dynamics

$$\left(\frac{\partial}{\partial t} + V.\nabla\right)V = (U.\nabla + \mathcal{D}\Delta)\,U\,, \tag{5}$$

i. e.,

$$\frac{dV}{dt} = \frac{F}{m}\,, \tag{6}$$

where the total derivative of the velocity field V takes its standard form $dV/dt = (\partial/\partial t + V.\nabla)V$ and where the force F is given by $F = m(U.\nabla U + \mathcal{D}\Delta U)$.

Recall that, after one has introduced the wave function ψ from the complex action $\mathcal{S} = S_R + iS_I$, namely, $\psi = \exp(i\mathcal{S}/2m\mathcal{D}) = \sqrt{P}\exp(iS_R/2m\mathcal{D})$, equation (2) and its generalization including a scalar field, $m\,\widehat{d}\mathcal{V}/dt = -\nabla\phi$ can be integrated under the form of a Schrödinger equation [1]

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial\psi}{\partial t} - \frac{\phi}{2m}\psi = 0\,. \tag{7}$$

Let us now show that the additional force derives from a potential. Indeed, the imaginary part of the complex velocity field is given, in terms of the modulus of ψ, by the expression:

$$U = \mathcal{D}\nabla \ln P \,. \tag{8}$$

The force becomes

$$F = m\mathcal{D}^2 \left[(\nabla \ln P.\nabla)(\nabla \ln P) + \Delta(\nabla \ln P) \right]. \tag{9}$$

Now, by introducing $\sqrt{P}$ in this expression, one makes explicitly appear the remarkable identity that is already at the heart of the proof of the Schrödinger equation ([1], p. 151), namely,

$$F = 2m\mathcal{D}^2 \Big[2(\nabla \ln \sqrt{P}.\nabla)(\nabla \ln \sqrt{P}) + + \Delta(\nabla \ln \sqrt{P}) \Big] = 2m\mathcal{D}^2 \nabla \left(\frac{\Delta\sqrt{P}}{\sqrt{P}} \right). \tag{10}$$

Therefore the force F derives from a potential energy

$$Q = -2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}} \,, \tag{11}$$

which is nothing but the standard "quantum potential", but here established as a mere manifestation of the nondifferentiable and fractal geometry instead of being deduced from a postulated Schrödinger equation.

The real part of the motion equation finally takes the standard form of the equation of dynamics in presence of a scalar potential,

$$\frac{dV}{dt} = \left(\frac{\partial}{\partial t} + V.\nabla \right) V = -\frac{\nabla Q}{m} \,, \tag{12}$$

while the imaginary part is the equation of continuity $\partial P/\partial t + \mathrm{div}(PV) = 0$. The fact that the field equation is derived from the same remarkable identity that gives rise to the Schrödinger equation is also manifest in the similarity of its form with the free stationary Schrödinger equation, namely,

$$\mathcal{D}^2 \Delta\sqrt{P} + \frac{Q}{2m}\sqrt{P} = 0 \quad \longleftrightarrow \quad \mathcal{D}^2 \Delta\psi + \frac{E}{2m}\psi = 0 \,. \tag{13}$$

Now, the form (11) of the field equation means that the field can be known only after having solved the Schrödinger equation for the wave function. This is a situation somewhat different from that of general relativity, where, at least for test-particles, the description is reversed: given the energy-momentum tensor, one solves the Einstein field (i. e. space-time geometry) equations for the metric potentials, then one writes the geodesics equation in the space-time so determined and solve it for the motion of the particle. However, even in general relativity this case is an ideally simplified situation, since already in the two-body problem the motion of the bodies should be injected in the energy-momentum tensor, so that this is a looped system which has no exact analytical solution.

In the case of a quantum mechanical particle considered in scale relativity, the loop between the motion (geodesics) equation and the field equation is even more tight. Indeed, here the concept of test-particle loses its meaning. Even in the case of only one "particle", the space-time geometry is determined by the particle itself and by its motion, so that the field equation and the geodesics equation now participate of the same level of description. This explains why the motion/geodesics equation, in its Hamilton-Jacobi form that takes the form of the Schrödinger equation, is obtained without having first written the field equation in an explicit way. Actually, the potential Q is implicitly contained in the Schrödinger form of the equations, and it is made explicit only when coming back to a fluid-like Euler-Newton representation. In the end, the particle is described by a wave function (which is constructed, in the scale relativity theory, from the geodesics), of which only the square of the modulus P is observable. Therefore one expects the "field" to be given by a function of P, which is exactly what is found.

2.2 Invariants and energy balance

Let us now make explicit the energy balance by accounting for this additional potential energy. This question has already been discussed in [7, 8] and in [9], but we propose here a different presentation. We shall express the energy equation in terms of the various equivalent variables which we use in scale relativity, namely, the wave function ψ, the complex velocity $\mathcal{V}$ or its real and imaginary parts V and $-U$.

The first and main form of the energy equation is the Schrödinger equation itself, that we have derived as a prime integral of the geodesics equation. The Schrödinger equation is therefore the quantum equivalent of the metric form (i. e., of the equation of conservation of the energy). It may be written in the free case under the form

$$\mathcal{D}^2 \frac{\Delta\psi}{\psi} = -i\mathcal{D} \frac{\partial \ln \psi}{\partial t} \,. \tag{14}$$

In the stationary case with given energy E, it becomes:

$$E = -2m\mathcal{D}^2 \frac{\Delta\psi}{\psi} \,. \tag{15}$$

Now we can use the fundamental remarkable identity $\Delta\psi/\psi = (\nabla \ln \psi)^2 + \Delta \ln \psi$. Re-introducing the complex velocity field $\mathcal{V} = -2i\mathcal{D}\nabla \ln \psi$ in this expression we finally obtain the correspondence:

$$E = -2m\mathcal{D}^2 \frac{\Delta\psi}{\psi} = \frac{1}{2} m \left(\mathcal{V}^2 - 2i\mathcal{D}\nabla.\mathcal{V} \right). \tag{16}$$

Note that when a potential term is present, all these relations remain true by replacing E by $E - \phi$.

This is the non-relativistic equivalent of Pissondes' relation [8] in the relativistic case, $\mathcal{V}^\mu \mathcal{V}_\mu + i\lambda\partial^\mu \mathcal{V}_\mu = 1$ (see also hereafter). Therefore the form of the energy $E = (1/2)mV^2$ is not conserved: this is precisely due to the existence of the additional potential energy of geometric origin. Let us prove this statement.

From equation (16) we know that the imaginary part of $(\mathcal{V}^2 - 2i\mathcal{D}\nabla.\mathcal{V})$ is zero. By writing its real part in terms of the real velocities U and V, we find:

$$E = \frac{1}{2}\, m\left(\mathcal{V}^2 - 2i\mathcal{D}\nabla.\mathcal{V}\right) = \\ = \frac{1}{2}\, m\left(V^2 - U^2 - 2\mathcal{D}\nabla.U\right). \tag{17}$$

Now we can express the potential energy Q given in equation (11) in terms of the velocity field U:

$$Q = -\frac{1}{2}\, m\left(U^2 - 2\mathcal{D}\nabla.U\right), \tag{18}$$

so that we finally write the energy balance under the three equivalent forms:

$$E = -2m\mathcal{D}^2\,\frac{\Delta\psi}{\psi} = \\ = \frac{1}{2}\, m\left(\mathcal{V}^2 - 2i\mathcal{D}\nabla.\mathcal{V}\right) = \frac{1}{2}\, mV^2 + Q\,. \tag{19}$$

More generally, in presence of an external potential energy ϕ and in the non-stationary case, it reads:

$$-\frac{\partial S_R}{\partial t} = \frac{1}{2}\, mV^2 + Q + \phi\,, \tag{20}$$

where S_R is the real part of the complex action (i. e., $S_R/2m\mathcal{D}$ is the phase of the wave function).

3 Relativistic quantum mechanics

3.1 Quantum potential

All the above description can be directly generalized to relativistic QM and the Klein-Gordon equation [10, 2, 3]. The geodesics equation still reads in this case:

$$\frac{\widehat{d}\mathcal{V}_\alpha}{ds} = 0\,, \tag{21}$$

where the total derivative is given by [10, 3]

$$\frac{\widehat{d}}{ds} = \left(\mathcal{V}^\mu + i\,\frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu\,. \tag{22}$$

The complex velocity field $\mathcal{V}_\alpha$ reads in terms of the wave function

$$\mathcal{V}_\alpha = i\lambda\,\partial_\alpha \ln\psi\,. \tag{23}$$

The relation between the non-relativistic fractal parameter $\mathcal{D}$ and the relativistic one λ is simply $2\mathcal{D} = \lambda c$. In particular, in the standard QM case, λ is the Compton length of the particle, $\lambda = \hbar/mc$, and we recover $\mathcal{D} = \hbar/2m$.

The calculations are similar to the non-relativistic case. We decompose the complex velocity in terms of its real and imaginary parts, $\mathcal{V}_\alpha = V_\alpha - i\,U_\alpha$, so that the geodesics equation becomes

$$\left\{V^\mu - i\left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\right\}\partial_\mu\left(V_\alpha - i\,U_\alpha\right) = 0\,, \tag{24}$$

i. e.,

$$\left\{V^\mu\partial_\mu V_\alpha - \left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu U_\alpha\right\} - \\ - i\left\{\left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu V_\alpha + V^\mu\partial_\mu U_\alpha\right\} = 0\,. \tag{25}$$

The real part of this equation takes the form of a relativistic Euler-Newton equation of dynamics:

$$\frac{dV_\alpha}{ds} = V^\mu\partial_\mu V_\alpha = \left(U^\mu - \frac{\lambda}{2}\,\partial^\mu\right)\partial_\mu U_\alpha\,. \tag{26}$$

Therefore the relativistic case is similar to the non-relativistic one, since a generalized force also appears in the right-hand side of this equation. Let us now prove that it also derives from a potential. Using the expression for U_α in terms of the modulus $\sqrt{P}$ of the wave function,

$$U_\alpha = -\lambda\,\partial_\alpha \ln\sqrt{P}\,, \tag{27}$$

we may write the force under the form

$$\frac{F_\alpha}{m} = -\lambda\,\partial^\mu \ln\sqrt{P}\,\partial_\mu(-\lambda\,\partial_\alpha \ln\sqrt{P}) + \\ + \frac{\lambda^2}{2}\,\partial^\mu\partial_\mu\partial_\alpha \ln\sqrt{P} = \\ = \lambda^2\left(\partial^\mu \ln\sqrt{P}\,\partial_\mu\partial_\alpha \ln\sqrt{P} + \frac{1}{2}\partial^\mu\partial_\mu\partial_\alpha \ln\sqrt{P}\right). \tag{28}$$

Since $\partial^\mu\partial_\mu\partial_\alpha = \partial_\alpha\partial^\mu\partial_\mu$ commutes and since $\partial_\alpha(\partial^\mu \ln f\,\partial_\mu \ln f) = 2\,\partial^\mu \ln f\,\partial_\alpha\partial^\mu \ln f$, we obtain

$$\frac{F_\alpha}{m} = \frac{1}{2}\,\lambda^2\,\partial_\alpha\left(\partial^\mu \ln\sqrt{P}\,\partial_\mu \ln\sqrt{P} + \partial^\mu\partial_\mu \ln\sqrt{P}\right). \tag{29}$$

We can now make use of the remarkable identity (that generalizes to four dimensions the one which is also at the heart of the non-relativistic case)

$$\partial^\mu \ln\sqrt{P}\,\partial_\mu \ln\sqrt{P} + \partial^\mu\partial_\mu \ln\sqrt{P} = \frac{\partial^\mu\partial_\mu\sqrt{P}}{\sqrt{P}}\,, \tag{30}$$

and we finally obtain

$$\frac{dV_\alpha}{ds} = \frac{1}{2}\,\lambda^2\,\partial_\alpha\left(\frac{\partial^\mu\partial_\mu\sqrt{P}}{\sqrt{P}}\right). \tag{31}$$

Therefore, as in the non-relativistic case, the force derives from a potential energy

$$Q_R = \frac{1}{2} mc^2 \, \lambda^2 \, \frac{\partial^\mu \partial_\mu \sqrt{P}}{\sqrt{P}} \,, \tag{32}$$

that can also be expressed in terms of the velocity field U as

$$Q_R = \frac{1}{2} mc^2 \left(U^\mu U_\mu - \lambda \, \partial^\mu U_\mu \right). \tag{33}$$

At the non-relativistic limit ($c \to \infty$), the Dalembertian $\partial^\mu \partial_\mu = (\partial^2/c^2 \partial t^2 - \Delta)$ is reduced to $-\Delta$, and since $\lambda = 2\mathcal{D}/c$, we recover the nonrelativistic potential energy $Q = -2m\mathcal{D}^2 \Delta \sqrt{P}/\sqrt{P}$. Note the correction to the potential introduced by Pissondes [7] which is twice this potential and therefore cannot agree with the nonrelativistic limit.

3.2 Invariants and energy balance

As shown by Pissondes [7, 8], the four-dimensional energy equation $u^\mu u_\mu = 1$ is generalized in terms of the complex velocity under the form $\mathcal{V}^\mu \mathcal{V}_\mu + i\lambda \partial^\mu \mathcal{V}_\mu = 1$. Let us show that the additional term is a manifestation of the new scalar field Q which takes its origin in the fractal and nondifferentiable geometry. Start with the geodesics equation

$$\frac{\widehat{d}\mathcal{V}_\alpha}{ds} = \left(\mathcal{V}^\mu + i\, \frac{\lambda}{2}\, \partial^\mu \right) \partial_\mu \, \mathcal{V}_\alpha = 0 \,. \tag{34}$$

Then, after introducing the wave function by using the relation $\mathcal{V}_\alpha = i\lambda\, \partial_\alpha \ln\psi$, after calculations similar to the above ones (now on the full function ψ instead of only its modulus $\sqrt{P}$), the geodesics equation becomes:

$$\begin{aligned} \frac{\widehat{d}\mathcal{V}_\alpha}{ds} &= -\frac{\lambda^2}{2}\, \partial_\alpha \left(\partial^\mu \ln\psi \, \partial_\mu \ln\psi + \partial^\mu \partial_\mu \ln\psi \right) = \\ &= \frac{1}{2}\, \partial_\alpha \left(-\lambda^2 \, \frac{\partial^\mu \partial_\mu \psi}{\psi} \right) = 0 \,. \end{aligned} \tag{35}$$

Under its right-hand form, this equation is integrated in terms of the Klein-Gordon equation,

$$\lambda^2 \, \partial^\mu \partial_\mu \psi + \psi = 0 \,. \tag{36}$$

Under its left hand form, the integral writes

$$-\lambda^2 (\partial^\mu \ln\psi \; \partial_\mu \ln\psi + \partial^\mu \partial_\mu \ln\psi) = 1 \,. \tag{37}$$

It becomes in terms of the complex velocity [8]

$$\mathcal{V}^\mu \mathcal{V}_\mu + i\lambda \partial^\mu \mathcal{V}_\mu = 1 \,, \tag{38}$$

which is therefore but another form taken by the KG equation (as expected from the fact that the KG equation is the quantum equivalent of the Hamilton-Jacobi equation). Let us now separate the real and imaginary parts of this equation. One obtains:

$$\begin{aligned} &V^\mu V_\mu - (U^\mu U_\mu - \lambda\, \partial^\mu U_\mu) = 1 \,, \\ &2\, V^\mu U_\mu - \lambda\, \partial^\mu V_\mu = 0 \,. \end{aligned} \tag{39}$$

Then the energy balance writes, in terms of the additional potential energy Q_R

$$V^\mu V_\mu = 1 + 2\, \frac{Q_R}{mc^2} \,. \tag{40}$$

Let us show that we actually expect such a relation for the quadratic invariant in presence of an external potential ϕ. The energy relation writes in this case $(E - \phi)^2 = p^2c^2 + m^2c^4$, i. e. $E^2 - p^2c^2 = m^2c^4 + 2E\phi - \phi^2$. Introducing the rest frame energy by writing $E = mc^2 + E'$, we obtain

$$\begin{aligned} V^\mu V_\mu &= \frac{E^2 - p^2c^2}{m^2c^4} = \\ &= 1 + 2\, \frac{\phi}{mc^2} + \left[2\, \frac{E'}{mc^2}\, \frac{\phi}{mc^2} - \frac{\phi^2}{m^2c^4} \right]. \end{aligned} \tag{41}$$

This justifies the relativistic factor 2 in equation (40) and supports the interpretation of Q_R in terms of a potential, at least at the level of the leading terms.

Now, concerning the additional terms, it should remain clear that this is only an approximate description in terms of field theory of what are ultimately (in this framework) the manifestations of the fractal and nondifferentiable geometry of space-time. Therefore we expect the field theory description to be a first order approximation in the same manner as, in general relativity, the description in terms of Newtonian potential.

In particular, in the non-relativistic limit $c \to \infty$ the last two terms of equation (41) vanish and we recover the energy equation (19) which is therefore exact in this case.

4 Conclusion

Placing ourselves in the framework of the scale-relativity theory, we have shown in a detailed way that the quantum potential, whose origin remained mysterious in standard quantum mechanics, is a manifestation of the nondifferentiability and fractality of space-time in the new approach.

This result is expected to have many applications, as well in physics as in other sciences, including biology [4]. It has been used, in particular, to suggest a new solution to the problem of “dark matter” in cosmology [11, 5], based on the proposal that chaotic gravitational system can be described on long time scales (longer than their horizon of predictibility) by the scale-relativistic equations and therefore by a macroscopic Schrödinger equation [12]. In this case there would be no need for additional non baryonic dark matter, since the various observed non-Newtonian dynamical effects (that the hypothesis of dark matter wants to explain despite the check of all attempts of detection) would be readily accounted for by the new scalar field that manifests the fractality of space.

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On the Possibility of Instant Displacements in the Space-Time of General Relativity

Larissa Borissova and Dmitri Rabounski
E-mail: lborissova@yahoo.com; rabounski@yahoo.com

Employing the mathematical apparatus of chronometric invariants (physical observable quantities), this study finds a theoretical possibility for the instant displacement of particles in the space-time of the General Theory of Relativity. This is to date the sole theoretical explanation of the well-known phenomenon of photon teleportation, given by the purely geometrical methods of Einstein's theory.

As it is known, the basic space-time of the General Theory of Relativity is a four-dimensional pseudo-Riemannian space, which is, in general, inhomogeneous, curved, rotating, and deformed. There the square of the space-time interval $ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta$, being expressed in the terms of physical observable quantities — chronometric invariants [1, 2], takes the form

$$ds^2 = c^2 d\tau^2 - d\sigma^2.$$

Here the quantity

$$d\tau = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\, v_i dx^i$$

is an interval of physical observable time, $\mathrm{w}=c^2(1-\sqrt{g_{00}})$ is the gravitational potential, $v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation, $d\sigma^2 = h_{ik}\,dx^i dx^k$ is the square of a spatial observable interval, $h_{ik} = -g_{ik} + \frac{1}{c^2}\, v_i v_k$ is the metric observable tensor, g_{ik} are spatial components of the fundamental metric tensor $g_{\alpha\beta}$ (space-time indices are Greek $\alpha, \beta = 0, 1, 2, 3$, while spatial indices — Roman $i, k = 1, 2, 3$).

Following this form we consider a particle displaced by ds in the space-time. We write ds^2 as follows

$$ds^2 = c^2 d\tau^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right),$$

where $\mathrm{v}^2 = h_{ik}\,\mathrm{v}^i \mathrm{v}^k$, and $\mathrm{v}^i = \frac{dx^i}{d\tau}$ is the three-dimensional observable velocity of the particle. So ds is: (1) a substantial quantity under $\mathrm{v} < c$; (2) a zero quantity under $\mathrm{v} = c$; (3) an imaginary quantity under $\mathrm{v} > c$.

Particles of non-zero rest-masses $m_0 \neq 0$ (substance) can be moved: (1) along real world-trajectories $cd\tau > d\sigma$, having real relativistic masses $m = \frac{m_0}{\sqrt{1 - \mathrm{v}^2/c^2}}$; (2) along imaginary world-trajectories $cd\tau < d\sigma$, having imaginary relativistic masses $m = \frac{im_0}{\sqrt{\mathrm{v}^2/c^2 - 1}}$ (tachyons). World-lines of both kinds are known as *non-isotropic trajectories*.

Particles of zero rest-masses $m_0 = 0$ (massless particles), having non-zero relativistic masses $m \neq 0$, move along world-trajectories of zero four-dimensional lengths $cd\tau = d\sigma$ at the velocity of light. They are known as *isotropic trajectories*. Massless particles are related to light-like particles — quanta of electromagnetic fields (photons).

A condition under which a particle may realize an instant displacement (*teleportation*) is equality to zero of the observable time interval $d\tau = 0$ so that the *teleportation condition* is

$$\mathrm{w} + v_i u^i = c^2,$$

where $u^i = \frac{dx^i}{dt}$ is its three-dimensional coordinate velocity. From this the square of that space-time interval by which this particle is instantly displaced takes the form

$$ds^2 = -d\sigma^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\,dx^i dx^k,$$

where $1 - \frac{\mathrm{w}}{c^2} = \frac{v_i u^i}{c^2}$ in this case, because $d\tau = 0$.

Actually, the signature $(+---)$ in the space-time area of a regular observer becomes $(-+++)$ in that space-time area where particles may be teleported. So the terms "time" and "three-dimensional space" are interchanged in that area. "Time" of teleporting particles is "space" of the regular observer, and vice versa "space" of teleporting particles is "time" of the regular observer.

Let us first consider substantial particles. As it easy to see, instant displacements (teleportation) of such particles manifests along world-trajectories in which $ds^2 = -d\sigma^2 \neq 0$ is true. So the trajectories represented in the terms of observable quantities are purely spatial lines of imaginary three-dimensional lengths $d\sigma$, although being taken in ideal world-coordinates t and x^i the trajectories are four-dimensional. In a particular case, where the space is free of rotation ($v_i = 0$) or its rotation velocity v_i is orthogonal to the particle's coordinate velocity u^i (so that $v_i u^i = |v_i||u^i| \cos(v_i; u^i) = 0$), substantial particles may be teleported only if gravitational collapse occurs ($\mathrm{w} = c^2$). In this case world-trajectories of teleportation taken in ideal world-coordinates become also purely spatial $ds^2 = g_{ik}\,dx^i dx^k$.

Second, massless light-like particles (photons) may be teleported along world-trajectories located in a space of the metric

$$ds^2 = -d\sigma^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\,dx^i dx^k = 0\,,$$

because for photons $ds^2 = 0$ by definition. So the space of photon teleportation characterizes itself by the conditions $ds^2 = 0$ and $d\sigma^2 = c^2 d\tau^2 = 0$.

The obtained equation is like the "light cone" equation $c^2 d\tau^2 - d\sigma^2 = 0$ ($d\sigma \neq 0$, $d\tau \neq 0$), elements of which are world-trajectories of light-like particles. But, in contrast to the light cone equation, the obtained equation is built by ideal world-coordinates t and x^i — not this equation in the terms of observable quantities. So teleporting photons move along trajectories which are elements of the world-cone (like the light cone) in that space-time area where substantial particles may be teleported (the metric inside that area has been obtained above).

Considering the photon teleportation cone equation from the viewpoint of a regular observer, we can see that the spatial observable metric $d\sigma^2 = h_{ik}\, dx^i dx^k$ becomes degenerate, $h = \det \| h_{ik} \| = 0$, in the space-time area of that cone. Taking the relationship $g = -h\, g_{00}$ [1, 2] into account, we conclude that the four-dimensional metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ degenerates there as well $g = \det \| g_{\alpha\beta} \| = 0$. The last fact implies that signature conditions defining pseudo-Riemannian spaces are broken, so that photon teleportation manifests outside the basic space-time of the General Theory of Relativity. Such a fully degenerate space was considered in [3, 4], where it was referred to as a *zero-space* because, from viewpoint of a regular observer, all spatial intervals and time intervals are zero there.

When $d\tau = 0$ and $d\sigma = 0$ observable relativistic mass m and the frequency ω become zero. Thus, from the viewpoint of a regular observer, all particles located in zero-space (in particular, teleporting photons) having zero rest-masses $m_0 = 0$ appear as zero relativistic masses $m = 0$ and the frequencies $\omega = 0$. Therefore particles of this kind may be assumed to be the ultimate case of massless light-like particles.

We will refer to all particles located in zero-space as *zero-particles*.

In the frames of the particle-wave concept each particle is given by its own wave world-vector $K_\alpha = \dfrac{\partial \psi}{\partial x^\alpha}$, where ψ is the wave phase (eikonal). The eikonal equation $K_\alpha K^\alpha = 0$ [5], setting forth that the length of the wave vector K^α remains unchanged*, for regular massless light-like particles (regular photons), becomes a travelling wave equation

$$\frac{1}{c^2}\left(\frac{{}^*\partial \psi}{\partial t}\right)^2 + h^{ik}\,\frac{{}^*\partial \psi}{\partial x^i}\,\frac{{}^*\partial \psi}{\partial x^k} = 0\,,$$

that may be obtained after taking $K_\alpha K^\alpha = g^{\alpha\beta}\,\dfrac{\partial \psi}{\partial x^\alpha}\,\dfrac{\partial \psi}{\partial x^\beta} = 0$ in the terms of physical observable quantities [1, 2], where we formulate regular derivatives through chronometrically invariant (physical observable) derivatives $\dfrac{{}^*\partial}{\partial t} = \dfrac{1}{\sqrt{g_{00}}}\dfrac{\partial}{\partial t}$ and $\dfrac{{}^*\partial}{\partial x^i} = \dfrac{\partial}{\partial x^i} + \dfrac{1}{c^2}\, v_i \dfrac{{}^*\partial}{\partial t}$ and we use $g^{00} = \dfrac{1}{g_{00}}\left(1 - \dfrac{1}{c^2}\, v_i v^i\right)$, $v_k = h_{ik} v^i$, $v^i = -c g^{0i} \sqrt{g_{00}}$, $g^{ik} = -h^{ik}$.

The eikonal equation in zero-space takes the form

$$h^{ik}\,\frac{{}^*\partial \psi}{\partial x^i}\,\frac{{}^*\partial \psi}{\partial x^k} = 0$$

because there $\omega = \dfrac{{}^*\partial \psi}{\partial t} = 0$, putting the equation's time term into zero. It is a standing wave equation. So, from the viewpoint of a regular observer, in the frames of the particle-wave concept, all particles located in zero-space manifest as *standing light-like waves*, so that all zero-space appears filled with a system of light-like standing waves — a light-like hologram. This implies that an experiment for discovering non-quantum teleportation of photons should be linked to stationary light.

There is no problem in photon teleportation being realised along fully degenerate world-trajectories ($g = 0$) outside the basic pseudo-Riemannian space ($g < 0$), while teleportation trajectories of substantial particles are strictly non-degenerate ($g < 0$) so the lines are located in the pseudo-Riemannian space†. It presents no problem because at any point of the pseudo-Riemannian space we can place a tangential space of $g \leqslant 0$ consisting of the regular pseudo-Riemannian space ($g < 0$) and the zero-space ($g = 0$) as two different areas of the same manifold. A space of $g \leqslant 0$ is a natural generalization of the basic space-time of the General Theory of Relativity, permitting non-quantum ways for teleportation of both photons and substantial particles (previously achieved only in quantum fashion — quantum teleportation of photons in 1998 [6] and of atoms in 2004 [7, 8]).

Until now teleportation has had an explanation given only by Quantum Mechanics [9]. Now the situation changes: with our theory we can find physical conditions for the realisation of teleportation of both photons and substantial particles in a non-quantum way.

The only difference is that from the viewpoint of a regular observer the square of any parallely transported vector remains unchanged. It is also an "observable truth" for vectors in zero-space, because the observer reasons standards of his pseudo-Riemannian space anyway. The eikonal equation in zero-space, expressed in his observable world-coordinates, is $K_\alpha K^\alpha = 0$. But in ideal world-coordinates t and x^i the metric inside zero-space, $ds^2 = -\left(1 - \frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + g_{ik}\, dx^i dx^k = 0$, degenerates into a three-dimensional $d\mu^2$ which, depending

*According to Levi-Civita's rule, in a Riemannian space of n dimensions the length of any n-dimensional vector Q^α remains unchanged in parallel transport, so $Q_\alpha Q^\alpha = \text{const}$. So it is also true for the four-dimensional wave vector K^α in a four-dimensional pseudo-Riemannian space — the basic space-time of the General Theory of Relativity. Since $ds = 0$ is true along isotropic trajectories (because $c d\tau = d\sigma$), the length of any isotropic vector is zero, so that we have $K_\alpha K^\alpha = 0$.

†Any space of Riemannian geometry has the strictly non-degenerate metric feature $g < 0$ by definition. Pseudo-Riemannian spaces are a particular case of Riemannian spaces, where the metric is sign-alternating. So the four-dimensional pseudo-Riemannian space of the signature $(+---)$ or $(-+++)$ on which Einstein based the General Theory of Relativity is also a strictly non-degenerated metric ($g < 0$).

on gravitational potential w uncompensated by something else, is not invariant, $d\mu^2 = g_{ik}\,dx^i dx^k = \left(1-\frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 \neq \neq \text{inv}$. As a result, within zero-space, the square of a transported vector, a four-dimensional coordinate velocity vector U^α for instance, being degenerated into a spatial U^i, does not remain unchanged

$$U_i U^i = g_{ik}\,U^i U^k = \left(1-\frac{\mathrm{w}}{c^2}\right)^2 c^2 \neq \text{const},$$

so that although the geometry is Riemannian for a regular observer, the real geometry of zero-space within the space itself is non-Riemannian.

We conclude from this brief study that instant displacements of particles are naturally permitted in the space-time of the General Theory of Relativity. As it was shown, teleportation of substantial particles and photons realizes itself in different space-time areas. But it would be a mistake to think that teleportation requires the acceleration of a substantial particle to super-light speeds (the tachyons area), while a photon needs to be accelerated to infinite speed. No — as it is easy to see from the teleportation condition $\mathrm{w} + v_i u^i = c^2$, if gravitational potential is essential and the space rotates at a speed close to the velocity of light, substantial particles may be teleported at regular sub-light speeds. Photons can reach the teleportation condition easier, because they move at the velocity of light. From the viewpoint of a regular observer, as soon as the teleportation condition is realised in the neighbourhood of a moving particle, such particle "disappears" although it continues its motion at a sub-light coordinate velocity u^i (or at the velocity of light) in another space-time area invisible to us. Then, having its velocity reduced, or by the breaking of the teleportation condition by something else (lowering gravitational potential or the space rotation speed), it "appears" at the same observable moment at another point of our observable space at that distance and in the direction which it obtained by u^i there.

In connection with the results, it is important to remember the "Infinity Relativity Principle", introduced by Abraham Zelmanov (1913–1987), a prominent cosmologist. Proceeding from his cosmological studies [1], he concluded that "... in homogeneous isotropic cosmological models spatial infinity of the Universe depends on our choice of that reference frame from which we observe the Universe (the observer's reference frame). If the three-dimensional space of the Universe, being observed in one reference frame, is infinite, it may be finite in another reference frame. The same is as well true for the time during which the Universe evolves."

We have come to the "Finite Relativity Principle" here. As it was shown, because of a difference between physical observable world-coordinates and ideal ones, the same space-time areas may be very different, being defined in each of the frames. Thus, in observable world-coordinates, zero-space is a point ($d\tau = 0$, $d\sigma = 0$), while $d\tau = 0$ and $d\sigma = 0$ taken in ideal world-coordinates become $-\left(1-\frac{\mathrm{w}}{c^2}\right)^2 c^2 dt^2 + + g_{ik}\,dx^i dx^k = 0$, which is a four-dimensional cone equation like the light cone. Actually here is the "Finite Relativity Principle" for observed objects — an observed point is the whole space taken in ideal coordinates.

Conclusions

This research currently is the sole explanation of virtual particles and virtual interaction given by the purely geometrical methods of Einstein's theory. It is possible that this method will establish a link between Quantum Electrodynamics and the General Theory of Relativity.

Moreover, this research is currently the sole theoretical explanation of the observed phenomenon of teleportation [6, 7, 8] given by the General Theory of Relativity.

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On Dual Phase-Space Relativity, the Machian Principle and Modified Newtonian Dynamics

Carlos Castro
Center for Theor. Studies of Physical Systems, Clark Atlanta University, Atlanta, USA
E-mail: czarlosromanov@yahoo.com; castro@ctsps.cau.edu

We investigate the consequences of the Mach's principle of inertia within the context of the Dual Phase Space Relativity which is compatible with the Eddington-Dirac large numbers coincidences and may provide with a physical reason behind the observed anomalous Pioneer acceleration and a solution to the riddle of the cosmological constant problem. The cosmological implications of Non-Archimedean Geometry by assigning an upper impossible scale in Nature and the cosmological variations of the fundamental constants are also discussed. We study the corrections to Newtonian dynamics resulting from the Dual Phase Space Relativity by analyzing the behavior of a test particle in a *modified* Schwarzschild geometry (due to the the effects of the maximal acceleration) that leads in the weak-field approximation to essential *modifications* of the Newtonian dynamics and to violations of the equivalence principle. Finally we follow another avenue and find modified Newtonian dynamics induced by the Yang's Noncommutative Spacetime algebra involving a lower and upper scale in Nature.

1 Introduction

In recent years we have argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein's general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature. A scale relativistic theory involving spacetime *resolutions* was developed long ago by Nottale where the Planck scale was postulated as the minimum observer independent invariant resolution in Nature [2]. Since "points" cannot be observed physically with an ultimate resolution, they are fuzzy and smeared out into fuzzy balls of Planck radius of arbitrary dimension. For this reason one must construct a theory that includes all dimensions (and signatures) on the equal footing. Because the notion of dimension is a topological invariant, and the concept of a fixed dimension is lost due to the fuzzy nature of points, dimensions are resolution-dependent, one must also include a theory with *all* topologies as well. It turned out that Clifford algebras contained the appropriate algebro-geometric features to implement this principle of polydimensional transformations that reshuffle a five-brane history for a membrane history, for example. For an extensive review of this Extended Relativity Theory in Clifford Spaces that encompasses the unified dynamics of all p-branes, for different values of the dimensions of the extended objects, and numerous physical consequences, see [1].

A Clifford-space dynamical derivation of the stringy-minimal length uncertainty relations [11] was furnished in [45]. The dynamical consequences of the minimal-length in Newtonian dynamics have been recently reviewed by [44]. The idea of minimal length (the Planck scale L_P) can be incorporated within the context of the maximal acceleration Relativity principle [68] $a_{max} = c^2/L_P$ in Finsler Geometries [56] and [14]. A different approach than the one based on Finsler Geometries is the pseudo-complex Lorentz group description by Schuller [61] related to the effects of maximal acceleration in Born-Infeld models that also maintains Lorentz invariance, in contrast to the approaches of Double Special Relativity (DSR) [70] where the Lorentz symmetry is deformed. Quantum group deformations of the Poincaré symmetry and of Gravity have been analyzed by [69] where the deformation parameter q could be interpreted in terms of an upper and lower scale as $q = e^{L_P/R}$ such that the undeformed limit $q = 1$ can be attained when $L_P \to 0$ and/or when $R \to \infty$ [68]. For a discussions on the open problems of Double Special Relativity theories based on kappa-deformed Poincaré symmetries [63] and motivated by the anomalous Lorentz-violating dispersion relations in the ultra high energy cosmic rays [71, 73], we refer to [70].

An upper limit on the maximal acceleration of particles was proposed long ago by Caianiello [52]. This idea is a direct consequence of a suggestion made years earlier by Max Born on a Dual Relativity principle operating in Phase Spaces [49], [74] where there is an upper bound on the four-force (maximal string tension or tidal forces in strings) acting on a particle as well as an upper bound in the particle's velocity given by the speed of light. For a recent status of the geometries behind maximal-acceleration see [73]; its relation to the Double Special Relativity programs was studied by [55] and the possibility that Moyal deformations of Poincaré algebras could be related to the kappa-deformed Poincaré algebras was raised in [68]. A thorough study of Finsler geometry and Clifford algebras has been undertaken by Vacaru [81] where Clifford/spinor structures were defined with respect to Nonlinear connections associated with certain non-holonomic modifications of Riemann-Cartan gravity. The study of non-holonomic Clifford-Structures in the construction of a Noncommutative Riemann-Finsler Geometry has recently been advanced by [81].

Other implications of the maximal acceleration principle in Nature, like neutrino oscillations and other phenomena, have been studied by [54], [67], [22]. Recently, the variations of the fine structure constant α [64] with the cosmological accelerated expansion of the Universe was recast as a renormalization group-like equation governing the cosmological red shift (Universe scale) variations of α based on this maximal acceleration principle in Nature [68]. The fine structure constant was smaller in the past. Pushing the cutoff scale to the minimum Planck scale led to the intriguing result that the fine structure constant could have been extremely small (zero) in the early Universe and that all matter in the Universe could have emerged via the Unruh-Rindler-Hawking effect (creation of radiation/matter) due to the acceleration w. r. t the vacuum frame of reference. For reviews on the alleged variations of the fundamental constants in Nature see [65].

The outline of this work goes as follows. In section 2 we review the Dual Phase Space Relativity and show why the Planck areas are invariant under acceleration-boosts transformations.

In 3.1 we investigate the consequences of the Mach's principle of inertia within the context of the Dual Phase Space Relativity Principle which is compatible with the Eddington-Dirac large numbers coincidence and may provide with a very plausible physical reason behind the observed anomalous Pioneer acceleration due to the fact that the universe is in accelerated motion (a non-inertial frame of reference) w. r. t the vacuum. Our proposal shares similarities with the previous work of [6], [3]. To our knowledge, the first person who *predicted* the Pioneer anomaly in 1978 was P. LaViolette [5], from an entirely different approach based on the novel theory of sub-quantum kinetics to explain the vacuum fluctuations, two years *prior* to the Anderson et al observations [7]. The cosmological implications of Non-Archimedean Geometry [94] by assigning an upper impassible scale in Nature [2] and the cosmological variations of the fundamental constants are also discussed.

In 3.2 the crucial modifications to Newtonian dynamics resulting from the Dual Phase Space Relativity are analyzed further. In particular, the physical consequences of an upper and lower bounds in the acceleration and an upper and lower bounds in the angular velocity. We study the particular behavior of a test particle living in a *modified* Schwarzschild geometry (due to the the effects of the principle of maximal acceleration) that leads in the weak-field approximation to essential *modifications* of the Newtonian dynamics and to violations of the equivalence principle. For violations of the equivalence principle in neutrino oscillations see [42], [54].

Finally, in 4 we study another interesting avenue for the origins of modified Newtonian dynamics based on Yang's Noncommutative Spacetime algebra involving a lower and upper scale [136] that has been revisited recently by us [134] in the context of holography and area-quantization in C-spaces (Clifford spaces); in the physics of D-branes and covariant Matrix models by [137] and within the context of Lie algebra stability by [48]. A different algebra with two length scales has been studied by [43] in order to account for modifications of Newtonian dynamics (that also violates the equivalence principle).

2 Dual Phase-Space Relativity

In this section we will review in detail the Born's Dual Phase Space Relativity and the principle of Maximal-acceleration Relativity [68] from the perspective of $8D$ Phase Spaces and the role of the invariance $U(1,3)$ Group. We will focus for simplicity on a *flat* $8D$ Phase Space. A *curved* case scenario has been analyzed by Brandt [56] within the context of the Finsler geometry of the $8D$ tangent bundle of spacetime and written the generalized $8D$ gravitational equations that reduce to the ordinary Einstein-Riemannian gravitational equations in the *infinite* acceleration limit. Vacaru [81] has constructed the Riemann-Finsler geometries endowed with non-holonomic structures induced by *nonlinear* connections and developed the formalism to build a Noncommutative Riemann-Finsler Geometry by introducing suitable Clifford structures. A curved *momentum* space geometry was studied by [50]. Toller [73] has explored the different possible geometries associated with the maximal acceleration principle and the physical implications of the meaning of an "observer", "measuring device" in the cotangent space.

The $U(1,3) = SU(1,3) \otimes U(1)$ Group transformations, which leave invariant the phase-space intervals under rotations, velocity and acceleration boosts, were found by Low [74] and can be simplified drastically when the velocity/acceleration boosts are taken to lie in the z-direction, leaving the transverse directions x, y, p_x, p_y intact; i. e., the $U(1,1) = SU(1,1) \otimes U(1)$ subgroup transformations that leave invariant the phase-space interval are given by (in units of $\hbar = c = 1$)

$$(d\omega)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} = (d\tau)^2 \left[1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}\right] = (d\tau)^2 \left[1 - \frac{m^2 g^2(\tau)}{m_P^2 A_{max}^2}\right], \tag{2.1}$$

where we have factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in eq-(2.1) and the maximal proper-force is set to be $b \equiv m_P A_{max}$. Here m_P is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, may also be interpreted as the maximal string tension when L_P is the Planck scale.

The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass m in the z-direction which we take to be defined by the X coordinate. The interval $(d\omega)^2$ described by Low [74] is $U(1,3)$-invariant for the most general transformations in the $8D$ phase-space. These transformations are rather elaborate, so we refer to the references [74] for details. The appearance of the $U(1,3)$ group in $8D$ Phase Space is not too surprising since it could be seen as the "complex doubling" version of the Lorentz group $SO(1,3)$. Low discussed the irreducible unitary representations of such $U(1,3)$ group and the relevance for the strong interactions of quarks and hadrons since $U(1,3)$, with 16 generators, contains the $SU(3)$ group.

The analog of the Lorentz relativistic factor in eq-(2.1) involves the ratios of two proper *forces*. One variable force is given by $mg(\tau)$ and the maximal proper force sustained by an *elementary* particle of mass m_P (a *Planckton*) is assumed to be $F_{max} = m_{Planck} c^2/L_P$. When $m = m_P$, the ratio-squared of the forces appearing in the relativistic factor of eq-(2.1) becomes then g^2/A^2_{max}, and the phase space interval coincides with the geometric interval discussed by [61], [54], [67], [22].

The transformations laws of the coordinates in that leave invariant the interval (2.1) were given by [74]:

$$T' = T\cosh\xi + \left(\frac{\xi_v X}{c^2} + \frac{\xi_a P}{b^2}\right)\frac{\sinh\xi}{\xi}, \qquad (2.2a)$$

$$E' = E\cosh\xi + (-\xi_a X + \xi_v P)\,\frac{\sinh\xi}{\xi}, \qquad (2.2b)$$

$$X' = X\cosh\xi + \left(\xi_v T - \frac{\xi_a E}{b^2}\right)\frac{\sinh\xi}{\xi}, \qquad (2.2c)$$

$$P' = P\cosh\xi + \left(\frac{\xi_v E}{c^2} + \xi_a T\right)\frac{\sinh\xi}{\xi}. \qquad (2.2d)$$

The ξ_v is velocity-boost rapidity parameter and the ξ_a is the force/acceleration-boost rapidity parameter of the primed-reference frame. They are defined respectively:

$$\tanh\left(\frac{\xi_v}{c}\right) = \frac{v}{c}, \quad \tanh\left(\frac{\xi_a}{b}\right) = \frac{ma}{m_P A_{max}}. \qquad (2.3)$$

The *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs-(2.2a, 2.2d) is defined in terms of the velocity and acceleration boosts parameters ξ_v, ξ_a respectively as:

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}}. \qquad (2.4)$$

Our definition of the rapidity parameters are *different* than those in [74].

Straightforward algebra allows us to verify that these transformations leave the interval of eq-(2.1) in classical phase space invariant. They are are fully consistent with Born's duality Relativity symmetry principle [49] $(Q,P) \to \to (P,-Q)$. By inspection we can see that under Born duality, the transformations in eqs-(2.2a, 2.2d) are *rotated* into each other, up to numerical b factors in order to match units. When on sets $\xi_a = 0$ in (2.2a, 2.2d) one recovers automatically the standard Lorentz transformations for the X,T and E,P variables *separately*, leaving invariant the intervals $dT^2 - dX^2 = (d\tau)^2$ and $(dE^2 - dP^2)/b^2$ separately.

When one sets $\xi_v = 0$ we obtain the transformations rules of the events in Phase space, from one reference-frame into another *uniformly*-accelerated frame of reference, $a =$ const, whose acceleration-rapidity parameter is in this particular case:

$$\xi \equiv \frac{\xi_a}{b}, \quad \tanh(\xi) = \frac{ma}{m_P A_{max}}. \qquad (2.5)$$

The transformations for pure acceleration-boosts in Phase Space are:

$$T' = T\cosh\xi + \frac{P}{b}\sinh\xi, \qquad (2.6a)$$

$$E' = E\cosh\xi - bX\sinh\xi, \qquad (2.6b)$$

$$X' = X\cosh\xi - \frac{E}{b}\sinh\xi, \qquad (2.6c)$$

$$P' = P\cosh\xi + bT\sinh\xi. \qquad (2.6d)$$

It is straightforward to verify that the transformations (2.6a, 2.6c) leave invariant the fully phase space interval (2.1) but *does not* leave invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$. Only the *combination*:

$$(d\omega)^2 = (d\tau)^2\left(1 - \frac{m^2 g^2}{m_P^2 A^2_{max}}\right) \qquad (2.7a)$$

is truly left invariant under pure acceleration-boosts in Phase Space. Once again, can verify as well that these transformations satisfy Born's duality symmetry principle:

$$(T,X) \to (E,P),\ (E,P) \to (-T,-X) \qquad (2.7b)$$

and $b \to \frac{1}{b}$. The latter Born duality transformation is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i. e. namely, a small/large radius duality, a winding modes/Kaluza-Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in Noncommutative Field Theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory.

The composition of two successive pure acceleration-boosts is another pure acceleration-boost with acceleration rapidity given by $\xi'' = \xi + \xi'$. The addition of *proper* forces (accelerations) follows the usual relativistic composition rule:

$$\tanh\xi'' = \tanh(\xi+\xi') = = \frac{\tanh\xi + \tanh\xi'}{1+\tanh\xi\tanh\xi'} \Rightarrow \frac{ma''}{m_P A} = \frac{\frac{ma}{m_P A} + \frac{ma'}{m_P A}}{1+\frac{m^2 aa'}{m_P^2 A^2}} \qquad (2.8)$$

and in this fashion the upper limiting *proper* acceleration is never *surpassed* like it happens with the ordinary Special Relativistic addition of velocities.

The group properties of the full combination of velocity and acceleration boosts eqs-(2.2a, 2.2d) in Phase Space requires much more algebra [68]. A careful study reveals that the composition *rule* of two successive full transformations is given by $\xi'' = \xi + \xi'$ and the transformation laws are *preserved* if, and only if, the $\xi; \xi'; \xi'' \dots$ parameters obeyed the suitable relations:

$$\frac{\xi_a}{\xi} = \frac{\xi'_a}{\xi'} = \frac{\xi''_a}{\xi''} = \frac{\xi''_a}{\xi+\xi'}\,, \qquad (2.9a)$$

$$\frac{\xi_v}{\xi} = \frac{\xi'_v}{\xi'} = \frac{\xi''_v}{\xi''} = \frac{\xi''_v}{\xi+\xi'}\,. \qquad (2.9b)$$

Finally we arrive at the composition law for the effective, velocity and acceleration boosts parameters ξ''; ξ''_v; ξ''_a respectively:

$$\xi''_v = \xi_v + \xi'_v\,, \qquad (2.10a)$$

$$\xi''_a = \xi_a + \xi'_a\,, \qquad (2.10b)$$

$$\xi'' = \xi + \xi'. \qquad (2.10c)$$

The above relations among the parameters are required in order to prove the $U(1,1)$ *group* composition law of the transformations in order to have a truly Maximal-Acceleration Phase Space Relativity theory resulting from a Phase-Space change of coordinates in the cotangent bundle of spacetime.

2.1 Planck-scale Areas are invariant under acceleration boosts

Having displayed explicitly the Group transformations rules of the coordinates in Phase space we will show why *infinite* acceleration-boosts (which is *not* the same as infinite proper acceleration) preserve Planck-scale *Areas* [68] as a result of the fact that $b = (1/L_P^2)$ equals the *maximal* invariant force, or string tension, if the units of $\hbar = c = 1$ are used.

At Planck-scale L_P intervals/increments in one reference frame we have by definition (in units of $\hbar = c = 1$): $\Delta X = \Delta T = L_P$ and $\Delta E = \Delta P = \frac{1}{L_P}$ where $b \equiv \frac{1}{L_P^2}$ is the maximal tension. From eqs-(2.6a, 2.6d) we get for the transformation rules of the finite intervals ΔX, ΔT, ΔE, ΔP, from one reference frame into another frame, in the *infinite* acceleration-boost limit $\xi \to \infty$,

$$\Delta T' = L_P(\cosh\xi + \sinh\xi) \to \infty$$

$$\Delta E' = \frac{1}{L_P}(\cosh\xi - \sinh\xi) \to 0 \qquad (2.11b)$$

by a simple use of L'Hôpital's rule or by noticing that both $\cosh\xi; \sinh\xi$ functions approach infinity at the same rate

$$\Delta X' = L_P(\cosh\xi - \sinh\xi) \to 0\,, \qquad (2.11c)$$

$$\Delta P' = \frac{1}{L_P}(\cosh\xi + \sinh\xi) \to \infty\,, \qquad (2.11d)$$

where the discrete displacements of two events in Phase Space are defined: $\Delta X = X_2 - X_1 = L_P$, $\Delta E = E_2 - E_1 = \frac{1}{L_P}$, $\Delta T = T_2 - T_1 = L_P$ and $\Delta P = P_2 - P_1 = \frac{1}{L_P}$.

Due to the identity:

$$(\cosh\xi + \sinh\xi)(\cosh\xi - \sinh\xi) = = \cosh^2\xi - \sinh^2\xi = 1 \qquad (2.12)$$

one can see from eqs-(2.11a, 2.11d) that the Planck-scale *Areas* are truly *invariant* under *infinite* acceleration-boosts $\xi = \infty$:

$$\Delta X'\Delta P' = 0\times\infty = = \Delta X\Delta P(\cosh^2\xi - \sinh^2\xi) = \Delta X\Delta P = \frac{L_P}{L_P} = 1\,, \qquad (2.13a)$$

$$\Delta T'\Delta E' = \infty\times 0 = = \Delta T\Delta E(\cosh^2\xi - \sinh^2\xi) = \Delta T\Delta E = \frac{L_P}{L_P} = 1\,, \qquad (2.13b)$$

$$\Delta X'\Delta T' = 0\times\infty = = \Delta X\Delta T(\cosh^2\xi - \sinh^2\xi) = \Delta X\Delta T = (L_P)^2\,, \qquad (2.13c)$$

$$\Delta P'\Delta E' = \infty\times 0 = = \Delta P\Delta E(\cosh^2\xi - \sinh^2\xi) = \Delta P\Delta E = \frac{1}{L_P^2}\,. \qquad (2.13d)$$

It is important to emphasize that the invariance property of the minimal Planck-scale *Areas* (maximal Tension) is *not* an exclusive property of *infinite* acceleration boosts $\xi = \infty$, but, as a result of the identity $\cosh^2\xi - \sinh^2\xi = 1$, for all values of ξ, the minimal Planck-scale *Areas* are *always* invariant under *any* acceleration-boosts transformations. Meaning physically, in units of $\hbar = c = 1$, that the Maximal Tension (or maximal Force) $b = \frac{1}{L_P^2}$ is a true physical *invariant* universal quantity. Also we notice that the Phase-space areas, or cells, in units of $\hbar$, are also invariant! The pure-acceleration boosts transformations are "symplectic". It can be shown also that areas greater (smaller) than the Planck-area remain greater (smaller) than the invariant Planck-area under acceleration-boosts transformations.

The infinite acceleration-boosts are closely related to the infinite red-shift effects when light signals barely escape Black hole Horizons reaching an asymptotic observer with an infinite red shift factor. The important fact is that the Planck-scale *Areas* are truly maintained invariant under acceleration-boosts. This could reveal very important information about Black-holes Entropy and Holography.

3 Modified Newtonian Dynamics from Phase Space Relativity

3.1 The Machian Principle and Eddington-Dirac Large Numbers Coincidence

A natural action associated with the invariant interval in Phase-Space given by eq-(2.1) is:

$$S = m \int d\tau \sqrt{1 + \frac{m^2}{m_P^2 a^2}(d^2 x^\mu/d\tau^2)(d^2 x_\mu/d\tau^2)} \,. \quad (3.1)$$

The proper-acceleration is orthogonal to the proper-velocity and this can be easily verified by differentiating the time-like proper-velocity squared:

$$\begin{aligned} V^2 = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = V^\mu V_\mu = 1 > 0 \Rightarrow \\ \Rightarrow \frac{dV^\mu}{d\tau} V_\mu = \frac{d^2 x^\mu}{d\tau^2} V_\mu = 0 \,, \end{aligned} \quad (3.2)$$

which implies that the proper-acceleration is space-like:

$$\begin{aligned} -g^2(\tau) = \frac{d^2 x^\mu}{d\tau^2}\frac{d^2 x_\mu}{d\tau^2} < 0 \Rightarrow \\ \Rightarrow S = m \int d\tau \sqrt{1 - \frac{m^2 g^2}{m_P^2 a^2}} = m \int d\omega \,, \end{aligned} \quad (3.3)$$

where the analog of the Lorentz time-dilation factor in Phase-space is now given by

$$d\omega = d\tau \sqrt{1 - \frac{m^2 g^2(\tau)}{m_P^2 a^2}} \,, \quad (3.4a)$$

namely,

$$(d\omega)^2 = \Omega^2 d\tau^2 = \left[1 - \frac{m^2 g^2(\tau)}{m_P^2 a^2}\right] g_{\mu\nu} dx^\mu dx^\nu . \quad (3.4b)$$

The invariant proper interval is no longer the standard proper-time τ but is given by the quantity $\omega(\tau)$. The deep connection between the physics of maximal acceleration and Finsler geometry has been analyzed by [56]. The action is real-valued if, and only if, $m^2 g^2 < m_P^2 a^2$ in the same fashion that the action in Minkowski spacetime is real-valued if, and only if, $v^2 < c^2$. This is the physical reason why there is an upper bound in the proper-four force acting on a fundamental particle given by $(mg)_{bound} = m_P(c^2/L_P) = m_P^2$ in natural units of $\hbar = c = 1$.

The Eddington-Dirac large numbers coincidence (and an ultraviolet/infrared entanglement) can be easily implemented if one equates the upper bound on the proper-four force sustained by a fundamental particle, $(mg)_{bound} = m_P(c^2/L_P)$, with the proper-four force associated with the mass of the (observed) universe M_U, and whose *minimal* acceleration c^2/R is given in terms of an infrared-cutoff R (the Hubble horizon radius). Equating these proper-four forces gives

$$\frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} \Rightarrow \frac{M_U}{m_P} = \frac{R}{L_P} \sim 10^{61} \,, \quad (3.5)$$

from this equality of proper-four forces associated with a maximal/minimal acceleration one infers $M_U \sim 10^{61}\, m_{Planck} \sim 10^{61} 10^{19} m_{proton} = 10^{80} m_{proton}$ which is indeed consistent with observations and agrees with the Eddington-Dirac number 10^{80}:

$$N = 10^{80} = (10^{40})^2 \sim \left(\frac{F_e}{F_G}\right)^2 \sim \left(\frac{R}{r_e}\right)^2 , \quad (3.6)$$

where $F_e = e^2/r^2$ is the electrostatic force between an electron and a proton; $F_G = G m_e m_{proton}/r^2$ is the corresponding gravitational force and $r_e = e^2/m_e \sim 10^{-13}$cm is the classical electron radius (in units $\hbar = c = 1$).

One may notice that the above equation (3.5) is also consistent with the Machian postulate that the rest mass of a particle is determined via the gravitational potential energy due to the other masses in the universe. In particular, by equating:

$$m_i c^2 = G m_i \sum_j \frac{m_j}{|r_i - r_j|} = \frac{G m_i M_U}{R} \Rightarrow \frac{c^2}{G} = \frac{M_U}{R} . \quad (3.7)$$

Due to the negative binding energy, the composite mass m_{12} of a system of two objects of mass m_1, m_2 is not equal to the sum $m_1 + m_2 > m_{12}$. We can now arrive at the conclusion that the *minimal* acceleration c^2/R is also the same acceleration induced on a test particle of mass m by a spherical mass distribution M_U inside a radius R. The acceleration felt by a test particle of mass m sitting at the edge of the observable Universe (at the Hubble horizon radius R) is:

$$\frac{G M_U}{R^2} = a \,. \quad (3.8)$$

From the last two equations (3.7, 3.8) one gets the same expression for the *minimal* acceleration:

$$a = a_{min} = \frac{c^2}{R} \,, \quad (3.9)$$

which is of the same order of magnitude as the anomalous acceleration of the Pioneer and Galileo spacecrafts $a \sim 10^{-8}$ cm/s^2. A very plausible physical reason behind the observed anomalous Pioneer acceleration could be due to the fact that the universe is in accelerated expansion and motion (a non-inertial frame of reference) w. r. t the vacuum. Our proposal shares some similarities with the previous work of [6]. To our knowledge, the first person who *predicted* the Pioneer anomaly in 1978 was P. LaViolette [5], from an entirely different approach based on the novel theory of sub-quantum kinetics to explain the vacuum fluctuations, two years *prior* to the Anderson et al observations [7]. Nottale has invoked the Machian principle of inertia [3] adopting a local and global inertial coordinate system at the scale of the solar system in order to explain the origins of this Pioneer-Galileo anomalous constant acceleration. The Dirac-Eddington large number coincidences from vacuum fluctuations was studied by [8].

Let us examine closer the equality between the proper-four forces

$$\frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} \Rightarrow \frac{m_P}{L_P} = \frac{M_U}{R} = \frac{c^2}{G}\,. \tag{3.10}$$

The last term in eq-(3.10) is directly obtained after implementing the Machian principle in eq-(3.7). Thus, one concludes from eq-(3.10) that as the universe evolves in time one must have the conserved ratio of the quantities $M_U/R = c^2/G = m_P/L_P$. This interesting possibility, advocated by Dirac long ago, for the fundamental constants $\hbar$, c, G, ... to vary over cosmological time is a plausible idea with the provision that the above ratios satisfy the relations in eq-(3.10) at any given moment of cosmological time. If the fundamental constants do not vary over time then the ratio $M_U/R = c^2/G$ must refer then to the *asymptotic* values of the Hubble horizon radius $R = R_{asymptotic}$. A related approach to the idea of an impassible upper asymptotic length R has been advocated by Scale Relativity [2] and in Khare [94] where a Cosmology based on non-Archimedean geometry was proposed by recurring to p-adic numbers. For example, a Non-Archimedean number addition law of two masses m_1, m_2 does not follow the naive addition rule $m_1 + m_2$ but instead:

$$m_1 \bullet m_2 = \frac{m_1 + m_2}{1 + (m_1 m_2 / M_U^2)}\,,$$

which is similar to the composition law of velocities in ordinary Relativity in terms of the speed of light. When the masses m_1, m_2 are much smaller than the universe mass M_U one recovers the ordinary addition law. Similar considerations follow in the Non-Archimedean composition law of lengths such that the upper length R_{asym} is never surpassed. For further references on p-adic numbers and Physics were refer to [40]. A Mersenne prime, $M_p = 2^p - 1 = prime$, for $p = prime$, p-adic hierarchy of scales in Particle physics and Cosmology has been discussed by Pitkannen and Noyes where many of the the fundamental energy scales, masses and couplings in Physics has been obtained [41], [42]. For example, the Mersenne prime $M_{127} = 2^{127} - 1 \sim 10^{38} \sim (m_{Planck}/m_{proton})^2$. The derivation of the Standard Model parameters from first principle has obtained by Smith [43] and Beck [47].

In [68] we proposed a plausible explanation of the variable fine structure constant phenomenon based on the maximal-acceleration relativity principle in phase-space by modifying the Robertson-Friedmann-Walker metric by a similar (acceleration-dependent) conformal factor as in eqs-(3.4). It led us to the conclusion that the universe could have emerged from the vacuum as a quantum bubble (or "brane-world") of Planck mass and Planck radius that expanded (w. r. t to the vacuum) at the speed of light with a *maximal* acceleration $a = c^2/L_p$. Afterwards the acceleration began to slow down as matter was being created from the vacuum, via an Unruh-Rindler-Hawking effect, from this initial maximal value c^2/L_p to the value of $c^2/R \sim 10^{-8}$cm/s^2 (of the same order of magnitude as the Pioneer anomalous acceleration). Namely, as the universe expanded, matter was being created from the vacuum via the Unruh-Rindler-Hawking effect (which must not to be confused with Hoyle's Steady State Cosmolgy) such that the observable mass of the universe enclosed within the observed Hubble horizon radius obeys (at any time) the relation $M_U \sim R$. Such latter relationship is very similar (up to a factor of 2) to the Schwarzschild black-hole horizon-radius relation $r_s = 2M$ (in units of $\hbar = c = G = 1$). As matter is being created out of the vacuum, the Hubble horizon radius grows accordingly such that $M_U/R = c^2/G$. Note that the Hubble horizon radius is one-half the Schwarzchild horizon radius $(1/2)(2GM_U/c^2) = (1/2)R_S$.

Lemaître's idea of the Universe as a "primordial atom" (like a brane-world) of Planck size has been also analyzed by [30] from a very different perspective than Born's Dual Phase Space Relativity. These authors have argued that one can have a compatible picture of the expansion of the Universe with the Eddington-Dirac large number coincidences if one invokes a variation of the fundamental constants with the cosmological evolution time as Dirac advocated long ago.

One of the most salient features of this section is that it agrees with the findings of [4] where a *geometric mean* relationship was found from first principles among the cosmological constant ρ_{vacuum}, the Planck area λ^2 and the AdS_4 throat size squared R^2 given by $(\rho_v)^{-1} = (\lambda)^2(R^2)$. Since the throat size of de Sitter space is the same as that of Anti de Sitter space, by setting the infrared scale R equal to the Hubble radius horizon observed *today* R_H and λ equal to the Planck scale one reproduces precisely the *observed* value of the vacuum energy density! [25]: $\rho \sim L_{Planck}^{-2} R_H^{-2} = L_P^{-4}(L_{Planck}/R_H)^2 \sim 10^{-122} M_{Planck}^4$.

Nottale's proposal [2] for the resolution to the cosmological constant problem is based on taking the Hubble scale R as an upper impassible scale and implementing the Scale Relativity principle so that in order to compare the vacuum energies of the Universe at the Planck scale $\rho(L_P)$ with the vacuum energy measured at the Hubble scale $\rho(R)$ one needs to include the Scale Relativistic correction factors which account for such apparent huge discrepancy: $\rho(L_P)/\rho(R) = (R/L_P)^2 \sim 10^{122}$. In contrast, the results of this work are based on Born's Dual Phase-Space Relativity principle. In the next sections we will review the dynamical consequences of the Yang's Noncommutative spacetime algebra comprised of *two* scales, the minimal Planck scale L_p (related to a minimum distance) and an upper infrared scale R related to a minimum momentum $p = \hbar/R$. Another interesting approach to dark matter, dark energy and the cosmological constant based on a vacuum condensate has been undertaken by [25].

We finalize this subsection by pointing out that the maximal/minimal angular velocity correspond to c/L_P and c/R respectively. A maximum angular velocity has important consequences in future Thomas-precession experiments [61], [73] whereas a minimal angular velocity has important consequences in galactic rotation measurements. The role of the Machian principle in constructing quantum cosmologies, models of dark energy, etc... has been studied in [52] and its relationship to modified Newtonian dynamics and fractals by [54], [3].

3.2 Modified Newtonian Dynamics from Phase-Space Relativity

Having displayed the cosmological features behind the proper-four forces equality (3.10) that relates the maximal/minimal acceleration in terms of the minimal/large scales and which is compatible with Eddington-Dirac's large number coincidences we shall derive next the *modified* Newtonian dynamics of a test particle which emerges from the Born's Dual Phase Space Relativity principle.

The modified Schwarzschild metric is defined in terms of the non-covariant acceleration as:

$$(d\omega)^2 = \Omega^2 (d\tau)^2 = \left[1 + \frac{m^2 g_{\mu\nu}(d^2x^\mu/d\tau^2)(d^2x^\nu/d\tau^2)}{m_P^2 a^2}\right] g_{\mu\nu} dx^\mu dx^\nu \,,$$

$$-g^2(\tau) \equiv g_{\mu\nu}(d^2x^\mu/d\tau^2)(d^2x^\nu/d\tau^2) < 0\,. \qquad (3.11a)$$

A covariant acceleration in curved space-times is given by:

$$\frac{Dv^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\,.$$

A particle in free fall follows a geodesic with *zero* covariant acceleration. Hence, we shall use the non-covariant acceleration in order to compute the effects of the maximal acceleration of a test particle in Schwarzschild spacetimes.

The components of the non-covariant four-acceleration $d^2x^\mu/d\tau^2$ of a test particle of mass m moving in a Schwarzschild spacetime background can be obtained in a straightforward fashion after using the on-shell condition $g_{\mu\nu}P^\mu P^\nu = m^2$ in spherical coordinates (by solving the relativistic Hamilton-Jacobi equations). The explict components of the (space-like) proper-four acceleration can be found in [22], [36] in terms of two integration constants, the energy E and angular momentum L. The latter components yields the final expression for the conformal factor Ω^2 in the case of pure radial motion [22]:

$$\begin{aligned}\Omega^2(m,a,M,E,r) &= \\ &= 1 - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\Big\{(1-2M/r)^{-1}\left(\frac{M}{r^2}\right)^2 - \\ &- \left[4M^2(E/m)^2r^{-4}(1-2M/r)^{-3}\right]\times \\ &\times\left[(E/m)^2 - (1-2M/r)\right]\Big\}\,.\end{aligned} \qquad (3.12)$$

In the Newtonian limit, to a first order approximation, we can set $1 - 2M/r \sim 1$ in eq-(3.12) since we shall be concentrating in distances larger than the Schwarzschild radius $r > r_s = 2M$, the conformal factor Ω^2 in eq-(3.12) simplifies:

$$\Omega^2 \sim 1 - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\left\{\left(\frac{M}{r^2}\right)^2 - \left[4M^2(E/m)^2r^{-4}\right]\left[(E/m)^2-1\right]\right\}, \qquad (3.13)$$

the modified Schwarzschild metric component $g'_{00} = \Omega^2 g_{00} = \Omega^2(1-2M/r) = 1 + 2\mathcal{U}'$ yields the *modified* gravitational potential $\mathcal{U}'$ in the *weak* field approximation

$$g'_{00} = 1 + 2\mathcal{U}' \sim 1 - \frac{2M}{r} - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\left(\frac{2M}{r^2}\right)^2 F(E/m) \qquad (3.14)$$

with

$$F(E/m) = \left(\frac{E}{m}\right)^2 - \left(\frac{E}{m}\right)^4 + \frac{1}{4}\,, \qquad (3.15)$$

where $F(E/m) > 0$ in the Newtonian limit $E < m$. The modified radial acceleration which encodes the modified Newtonian dynamics and which violates the equivalence principle (since the acceleration now depends on the mass of the test particle m) is

$$a' = -\frac{\partial\mathcal{U}'}{\partial r} = -\frac{M}{r^2}\left[1 + 8F\left(\frac{E}{m}\right)\left(\frac{m}{m_P}\right)^2 \times \left(\frac{M}{m_P}\right)\frac{1}{m_P^3 r^3}\right] + O(r^{-6})\,, \qquad (3.16)$$

this result is valid for distances $r \gg r_s = 2M$. We have set the maximal acceleration $a = \frac{c^2}{L_P} = m_P$ in units of $\hbar = c = G = 1$. This explains the presence of the m_P factors in the denominators. The first term in eq-(3.16) is the standard Newtonian gravitational acceleration $-M/r^2$ and the second terms are the leading corrections of order $1/r^5$. The higher order corrections $O(r^{-6})$ appear when we do not set $1 - 2M/r \sim 1$ in the expression for the conformal factor Ω^2 and when we include the extra term in the product of Ω^2 with $g_{00} = (1 - 2M/r)$.

The conformal factor Ω^2 when $L \neq 0$ (rotational degrees of freedom are switched on) such that the test particle moves in the radial and transverse (angular) directions has been given in [22]:

$$\Omega^2 = 1 - \frac{m^2}{m_P^2 a^2} \left\{ \frac{1}{1-2M/r} \times \right. \\ \times \left. \left[-\frac{3ML^2}{m^2 r^4} + \frac{L^2}{m^2 r^3} - \frac{M}{r^2} \right]^2 \right\} + \\ + \frac{m^2}{m_P^2 a^2} \left[-\frac{4L^2}{m^2 r^4} + \frac{4E^2 M^2}{m^2 r^4 (1-2M/r)^3} \right] \times \\ \times \left[\frac{E^2}{m^2} - (1-2M/r) \left(1 + \frac{L^2}{m^2 r^2} \right) \right]. \tag{3.17}$$

Following the same weak field approximation procedure $g'_{00} = \Omega^2(E, L, m) g_{00} = 1 + 2\mathcal{U}'$ yields the modified gravitational potential $\mathcal{U}'$ and modified Newtonian dynamics $a' = -\partial_r \mathcal{U}'$ that leads once again to a violation of the equivalence principle due to the fact that the acceleration depends on the values of the masses of the test particle.

4 Modified Newtonian Dynamics resulting from Yang's Noncommutative Spacetime Algebra

We end this work with some relevant remarks about the impact of Yang's Noncommutative spacetime algebra on modified Newtonian dynamics. Such algebra involves *two* length scales, the minimal Planck scale $L_P = \lambda$ and an upper infrared cutoff scale $\mathcal{R}$.

Recently in [134] an isomorphism between Yang's Noncommutative space-time algebra (involving *two* length scales) [136] and the *holographic area coordinates* algebra of C-spaces (Clifford spaces) was constructed via an AdS_5 spacetime (embedded in $6D$) which is instrumental in explaining the origins of an extra (infrared) scale $\mathcal{R}$ in conjunction to the (ultraviolet) Planck scale λ characteristic of C-spaces. Yang's Noncommutative space-time algebra allowed Tanaka [137] to explain the origins behind the *discrete* nature of the spectrum for the *spatial* coordinates and *spatial* momenta which yields a *minimum* length-scale λ (ultraviolet cutoff) and a minimum momentum $p = \hbar/\mathcal{R}$ (maximal length $\mathcal{R}$, infrared cutoff).

Related to the issue of area-quantization, the norm-squared $\mathbf{A}^2$ of the holographic Area operator $X_{AB}X^{AB}$ in Clifford-spaces has a correspondence with the quadratic Casimir operator $\lambda^4 \Sigma_{AB} \Sigma^{AB}$ of the conformal algebra $SO(4,2)$ ($SO(5,1)$ in the Euclideanized AdS_5 case). This holographic area-Casimir relationship does not differ much from the area-spin relation in Loop Quantum Gravity $\mathbf{A}^2 \sim \lambda^4 \sum j_i(j_i+1)$ in terms of the $SU(2)$ Casimir J^2 with eigenvalues $j(j+1)$, where the sum is taken over the spin network sites [111] and the minimal Planck scale emerges from a regularization procedure.

The Yang's algebra can be written in terms of the $6D$ angular momentum operators and a $6D$ pseudo-Euclidean metric η^{MN}:

$$\hat{M}^{\mu\nu} = \hbar \Sigma^{\mu\nu}, \quad \hat{M}^{56} = \hbar \Sigma^{56}, \tag{4.1}$$

$$\lambda \Sigma^{\mu 5} = \hat{x}^\mu, \quad \frac{\hbar}{\mathcal{R}} \Sigma^{\mu 6} = \hat{p}^\mu, \tag{4.2}$$

$$\mathcal{N} = \frac{\lambda}{\mathcal{R}} \Sigma^{56}, \tag{4.3}$$

as follows:

$$[\hat{p}^\mu, \mathcal{N}] = -i\eta^{66} \frac{\hbar}{R^2}\, \hat{x}^\mu, \tag{4.4}$$

$$[\hat{x}^\mu, \mathcal{N}] = i\eta^{55} \frac{L_P^2}{\hbar}\, \hat{p}^\mu, \tag{4.5}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = -i\eta^{55} L_P^2\, \Sigma^{\mu\nu}, \tag{4.6}$$

$$[\hat{p}^\mu, \hat{p}^\nu] = -i\eta^{66} \frac{\hbar^2}{R^2}\, \Sigma^{\mu\nu}, \tag{4.7}$$

$$[\hat{x}^\mu, \hat{p}^\mu] = i\hbar\eta^{\mu\nu}\, \mathcal{N}, \tag{4.8}$$

$$[\hat{x}^\mu, \Sigma^{\nu\rho}] = \eta^{\mu\rho} x^\nu - \eta^{\mu\nu} x^\rho, \tag{4.9}$$

$$[\hat{p}^\mu, \Sigma^{\nu\rho}] = \eta^{\mu\rho} p^\nu - \eta^{\mu\nu} p^\rho, \tag{4.10}$$

The dynamical consequences of the Yang's Noncommutative spacetime algebra can be derived from the quantum/classical correspondence:

$$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \leftrightarrow \{A, B\}_{PB}, \tag{4.11}$$

i. e. commutators correspond to Poisson brackets. More precisely, to Moyal brackets in Phase Space. In the classical limit $\hbar \to 0$ Moyal brackets reduce to Poisson brackets. Since the coordinates and momenta are no longer commuting variables the classical Newtonian dynamics is going to be modified since the symplectic two-form $\omega^{\mu\nu}$ in Phase Space will have additional non-vanishing elements stemming from these non-commuting coordinates and momenta.

In particular, the modified brackets read now:

$$\{\{A(x,p), B(x,p)\}\} = \partial_\mu A \omega^{\mu\nu} \partial_\nu B = \\ = \{A(x,p), B(x,p)\}_{PB} \{x^\mu, p^\nu\} + \\ + \frac{\partial A}{\partial x^\mu} \frac{\partial B}{\partial x^\nu} \{x^\mu, x^\nu\} + \frac{\partial A}{\partial p^\mu} \frac{\partial B}{\partial p^\nu} \{p^\mu, p^\nu\}. \tag{4.12}$$

If the coordinates and momenta were commuting variables the modified bracket will reduce to the first term only:

$$\{\{A(x,p), B(x,p)\}\} = \\ = \{A(x,p), B(x,p)\}_{PB} \{x^\mu, p^\nu\} = \\ = \left[\frac{\partial A}{\partial x^\mu} \frac{\partial B}{\partial p^\nu} - \frac{\partial A}{\partial p^\mu} \frac{\partial B}{\partial x^\nu} \right] \eta^{\mu\nu} \mathcal{N}. \tag{4.13}$$

The ordinary Heisenberg (canonical) algebra is recovered when $\mathcal{N} \to 1$ in eq-(4.13).

In the nonrelativistic limit, the modified dynamical equations are:

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{\partial H}{\partial p^j}\{x^i, p^j\} + \frac{\partial H}{\partial x^j}\{x^i, x^j\}, \quad (4.14)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = -\frac{\partial H}{\partial x^j}\{x^i, p^j\} + \frac{\partial H}{\partial p^j}\{p^i, p^j\}. \quad (4.15)$$

The non-relativistic Hamiltonian for a central potential $V(r)$ is:

$$H = \frac{p_i p^i}{2m} + V(r), \quad r = \Big[\sum_i x_i x^i\Big]^{1/2}. \quad (4.16)$$

Defining the magnitude of the central force by $F = -\frac{\partial V}{\partial r}$ and using $\frac{\partial r}{\partial x^i} = \frac{x_i}{r}$ one has the modified dynamical equations of motion (4.14, 4.15):

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{p_j}{m}\delta^{ij} - F\frac{x_j}{r}L_P^2\Sigma^{ij}, \quad (4.16a)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = F\frac{x_j}{r}\delta^{ij} + \frac{p_j}{m}\frac{\Sigma^{ij}}{R^2}. \quad (4.16b)$$

The angular momentum two-vector Σ^{ij} can be written as the *dual* of a vector $\vec{J}$ as follows $\Sigma^{ij} = \epsilon^{ijk}J_k$ so that:

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{p^i}{m} - L_P^2 F\frac{x_j}{r}\epsilon^{ijk}J_k, \quad (4.17a)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = F\frac{x^i}{r} + \frac{p_j}{m}\frac{\epsilon^{ijk}J_k}{R^2}. \quad (4.17b)$$

For planar motion (central forces) the cross-product of $\vec{J}$ with $\vec{p}$ and $\vec{x}$ is not zero since $\vec{J}$ points in the perpendicular direction to the plane. Thus, one will have nontrivial corrections to the ordinary Newtonian equations of motion induced from Yang's Noncommutative spacetime algebra in the nonrelativistic limit. When $\vec{J} = 0$, pure radial motion, there are no corrections. This is not the case when we studied the modified Newtonian dynamics in the previous section of the modified Schwarzschild field due to the maximal-acceleration relativistic effects. Therefore, the two routes to obtain modifications of Newtonian dynamics are very different.

Concluding, eqs-(4.16, 4.17) determine the *modified* Newtonian dynamics of a test particle under the influence of a central potential explicitly in terms of the two L_P, R minimal/maximal scales. When $L_P \to 0$ and $R \to \infty$ one recovers the ordinary Newtonian dynamics $v^i = (p^i/m)$ and $F(x^i/r) = m(dv^i/dt)$. The unit vector in the radial direction has for components $\hat{r} = (\vec{r}/r) = (x^1/r, x^2/r, x^3/r)$.

It is warranted to study the full relativistic dynamics as well, in particular the *modified* relativistic dynamics of the de-Sitter rigid top [135] due to the effects of Yang's Noncommutative spacetime algebra with a lower and an upper scale. The de Sitter rigid Top can be generalized further to Clifford spaces since a Clifford-polyparticle has more degrees of freedom than a relativistic top in ordinary spacetimes [46] and, naturally, to study the *modified* Nambu-Poisson dynamics of p-branes [49] as well. A different physical approach to the theory of large distance physics based on certain two-dim nonlinear sigma models has been advanced by Friedan [51].

An Extended Relativity theory with both an upper and lower scale can be formulated in the Clifford extension of Phase Spaces along similar lines as [1], [68] by adding the Clifford-valued polymomentum degrees of freedom to the Clifford-valued holographic coordinates. The Planck scale L_P and the minimum momentum $(\hbar/R)$ are introduced to match the dimensions in the Clifford-Phase Space interval in D-dimensions as follows:

$$\begin{aligned} d\Sigma^2 &= <dX^\dagger dX> + \frac{1}{\mathcal{F}^2}<dP^\dagger dP> = \\ &= \left(\frac{d\sigma}{L_P^{D-1}}\right)^2 + dx_\mu dx^\mu + \frac{dx_{\mu\nu}dx^{\mu\nu}}{L_P^2} + \\ &+ \frac{dx_{\mu\nu\rho}dx^{\mu\nu\rho}}{L_P^4} + \ldots + \frac{1}{\mathcal{F}^2}\Bigg[\left(\frac{d\tilde{\sigma}}{(\hbar/R)^{D-1}}\right)^2 + \\ &+ dp_\mu dp^\mu + \frac{dp_{\mu\nu}dp^{\mu\nu}}{(\hbar/R)^2} + \frac{dp_{\mu\nu\rho}dp^{\mu\nu\rho}}{(\hbar/R)^4} + \ldots\Bigg]. \end{aligned} \quad (4.18)$$

All the terms in eq-(4.18) have dimensions of length^2 and the maximal force is:

$$\mathcal{F} = \frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} = \frac{c^4}{G}. \quad (4.19)$$

The relevance of studying this extended Relativity in a Clifford-extended Phase Space is that it is the proper arena to construct a Quantum Cosmology compatible with Non-Archimedean Geometry, Yang's Noncommutative spacetime algebra [136] and Scale Relativity [2] with an upper and lower limiting scales, simultaneously. This clearly deserves further investigation.

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The Extended Relativity Theory in Clifford Spaces

Carlos Castro* and Matej Pavšič†

*Center for Theor. Studies of Physical Systems, Clark Atlanta University, Atlanta, USA
E-mail: czarlosromanov@yahoo.com; castro@ctsps.cau.edu

†Jožef Stefan Institute, Jamova 39, SI-1000 Ljubljana, Slovenia
E-mail: Matej.Pavsic@ijs.si

An introduction to some of the most important features of the Extended Relativity theory in Clifford-spaces (C-spaces) is presented whose "point" coordinates are non-commuting Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes... degrees of freedom associated with the collective particle, string, membrane, p-brane... dynamics of p-loops (closed p-branes) in target D-dimensional spacetime backgrounds. C-space Relativity naturally incorporates the ideas of an invariant length (Planck scale), maximal acceleration, non-commuting coordinates, supersymmetry, holography, higher derivative gravity with torsion and variable dimensions/signatures. It permits to study the dynamics of all (closed) p-branes, for all values of p, on a unified footing. It resolves the ordering ambiguities in QFT, the problem of time in Cosmology and admits superluminal propagation (tachyons) without violations of causality. A discussion of the maximal-acceleration Relativity principle in phase-spaces follows and the study of the invariance group of symmetry transformations in phase-space allows to show why Planck areas are *invariant* under acceleration-boosts transformations. This invariance feature suggests that a maximal-string tension principle may be operating in Nature. We continue by pointing out how the relativity of signatures of the underlying n-dimensional spacetime results from taking different n-dimensional slices through C-space. The conformal group in spacetime emerges as a natural subgroup of the Clifford group and Relativity in C-spaces involves natural *scale* changes in the sizes of physical objects without the introduction of forces nor Weyl's gauge field of dilations. We finalize by constructing the generalization of Maxwell theory of Electrodynamics of point charges to a theory in C-spaces that involves extended charges coupled to antisymmetric tensor fields of arbitrary rank. In the concluding remarks we outline briefly the current promising research programs and their plausible connections with C-space Relativity.

Contents

1 Introduction

In recent years it was argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein's general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature [8]. A theory involving spacetime *resolutions* was developed long ago by Nottale [23] where the Planck scale was postulated as the minimum observer independent invariant resolution [23] in Nature. Since "points" cannot be observed physically with an ultimate resolution, it is reasonable to postulate that they are smeared out into fuzzy balls. In refs.[8] it was assumed that those balls have the Planck radius and arbitrary dimension. For this reason it was argued in refs. [8] that one should construct a theory which includes all dimensions (and signatures) on the equal footing. In [8] this Extended Scale Relativity principle was applied to the quantum mechanics of p-branes which led to the construction of Clifford-space (C-space) where all p-branes were taken to be on the same footing, in the sense that the transformations in C-space reshuffled a string history for a five-brane history, a membrane history for a string history, for example.

Clifford algebras contained the appropriate algebraic-geometric features to implement this principle of polydimensional transformations [14]–[17]. In [14]–[16] it was proposed that every physical quantity is in fact a *polyvector*, that is, a Clifford number or a Clifford aggregate. Also, spinors are the members of left or right minimal ideals of Clifford algebra, which may provide the framework for a deeper understanding of sypersymmetries, i. e., the transformations relating bosons and fermions. The Fock-Stueckelberg theory of a relativistic particle can be embedded in the Clifford algebra of spacetime [15, 16]. Many important aspects of Clifford algebra are described in [1], [6], [7], [3], [15, 16, 17], [5], [48]. It is our belief that this may lead to the proper formulation of string and M theory.

A geometric approach to the physics of the Standard Model in terms of Clifford algebras was advanced by [4]. It was realized in [43] that the $Cl(8)$ Clifford algebra contains the 4 fundamental nontrivial representations of $Spin(8)$ that accommodate the chiral fermions and gauge bosons of the Standard Model and which also includes gravitons via the McDowell-Mansouri-Chamseddine-West formulation of gravity, which permits to construct locally, in $D=8$, a geometric Lagrangian for the Standard Model plus Gravity. Furthermore, discrete Clifford-algebraic methods based on hyperdiamond-lattices have been instrumental in constructing E_8 lattices and deriving the values of the force-strengths (coupling constants) and masses of the Standard Model with remarkable precision by [43]. These results have recently been corroborated by [46] for Electromagnetism, and by [47], where all the Standard Model parameters were obtained from first principles, despite the contrary orthodox belief that it is senseless to "derive" the values of the fundamental constants in Nature from first principles, from pure thought alone; i. e. one must invoke the Cosmological Anthropic Principle to explain why the constants of Nature have they values they have.

Using these methods the bosonic p-brane propagator, in the quenched mini superspace approximation, was constructed in [18, 19]; the logarithmic corrections to the black hole entropy based on the geometry of Clifford space (in short C-space) were obtained in [21]; the modified nonlinear de Broglie dispersion relations, the corresponding minimal-length stringy [11] and p-brane uncertainty relations also admitted a C-space interpretation [10], [19]. A generalization of Maxwell theory of electromagnetism in C-spaces comprised of extended charges coupled to antisymmetric tensor fields of arbitrary rank was attained recently in [75]. The resolution of the ordering ambiguities of QFT in curved spaces was resolved by using polyvectors, or Clifford-algebra valued objects [26]. One of the most remarkable features of the Extended Relativity in C-spaces is that a higher derivative Gravity with Torsion in ordinary spacetime follows naturally from the analog of the Einstein-Hlbert action in *curved* C-space [20].

In this new physical theory the arena for physics is no longer the ordinary spacetime, but a more general manifold of Clifford algebra valued objects, noncommuting polyvectors. Such a manifold has been called a pan-dimensional continuum [14] or C-space [8]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also closed lines, surfaces, volumes, . . . , called 0-loops (points), 1-loops (closed strings), 2-loops (closed membranes), 3-loops, etc. It is a sort of a *dimension* category, where the role of functorial maps is played by C-space transformations which reshuffles a p-brane history for a p'-brane history or a mixture of all of them, for example. The above geometric objects may be considered as to corresponding to the well-known physical objects, namely closed p-branes. Technically those transformations in C-space that reshuffle objects of different dimensions are generalizations of the ordinary Lorentz transformations to C-space.

C-space Relativity involves a generalization of Lorentz invariance (and not a deformation of such symmetry) involving superpositions of p-branes (p-loops) of all possible dimensions. The Planck scale is introduced as a natural parameter that allows us to bridge extended objects of different dimensionalities. Like the speed of light was need in Einstein Relativity to fuse space and time together in the Minkowski spacetime interval. Another important point is that the Conformal Group of four-dimensional spacetime is a consequence of the Clifford algebra in *four-dimensions* [25] and it emphasizes the fact why the natural dilations/contractions of objects in C-space is *not* the same physical phenomenon than what occurs in Weyl's geometry which requires introducing, by hand, a gauge field of dilations. Objects move dilationally, in the absence of forces, for a different physical reasoning than in Weyl's geometry: they move dilationally because of inertia. This was discussed long ago in refs. [27, 28].

This review is organized as follows: section 2 is dedicated to extending ordinary Special Relativity theory, from Minkowski spacetime to C-spaces, where the introduction of the invariant Planck scale is required to bridge objects, p-branes, of different dimensionality.

The generalized dynamics of particles, fields and branes in C-space is studied in section 3. This formalism allows us to construct for the first time, to our knowledge, a *unified* action which comprises the dynamics of *all* p-branes in C-spaces, for all values of p, in one single footing (see also [15]). In particular, the polyparticle dynamics in C-space, when reduced to 4-dimensional spacetime leads to the Stuckelberg formalism and the solution to the problem of time in Cosmology [15].

In section 4 we begin by discussing the geometric Clifford calculus that allows us to reproduce all the standard results in differential and projective geometry [41]. The resolution of the ordering ambiguities of QFT in curved spaces follows next when we review how it can be resolved by using polyvectors, or Clifford-algebra valued objects [26]. Afterwards we construct the Generalized Gravitational Theories in Curved C-spaces, in particular it is shown how Higher derivative Gravity with Torsion in ordinary spacetime follows naturaly from the Geometry of C-space [20].

In section 5 we discuss the Quantization program in C-spaces, and write the C-space Klein-Gordon and Dirac equations [15]. The coresponding bosonic/fermionic p-brane loop-wave equations were studied by [12], [13] without employing Clifford algebra and the concept of C-space.

In section 6 we review the Maximal-Acceleration Relativity in Phase-Spaces [127], starting with the construction of the submaximally-accelerated particle action of [53] using Clifford algebras in phase-spaces; the $U(1,3)$ invariance transformations [74] associated with an 8-dimensional phase space, and show why the minimal Planck-Scale areas are invariant under pure acceleration boosts which suggests that there could be a principle of maximal-tension (maximal acceleration) operating in string theory [68].

In section 7 we discuss the important point that the notion of spacetime signature is relative to a chosen n-dimensional subspace of 2^n-dimensional Clifford space. Different subspaces V_n — different sections through C-space — have in general different signature [15] We show afterwards how the Conformal algebra of spacetime emerges from the Clifford algebra [25] and emphasize the physical *differences* between our model and the one based on Weyl geometry. At the end we show how Clifford algebraic methods permits one to generalize Maxwell theory of Electrodynamics (associated with ordinary point-charges) to a generalized Maxwell theory in Clifford spaces involving *extended* charges and p-forms of arbitrary rank [75].

In the concluding remarks, we briefly discuss the possible avenues of future research in the construction of QFT in C-spaces, Quantum Gravity, Noncommutative Geometry, and other lines of current promising research in the literature.

2 Extending Relativity from Minkowski spacetime to C-space

We embark into the construction of the extended relativity theory in C-spaces by a natural generalization of the notion of a spacetime interval in Minkowski space to C-space [8, 14, 16, 15, 17]:

$$dX^2 = d\sigma^2 + dx_\mu dx^\mu + dx_{\mu\nu} dx^{\mu\nu} + \dots\,, \tag{1}$$

where $\mu_1 < \mu_2 < \dots$. The Clifford valued polyvector:*

$$X = X^M E_M = \sigma \underline{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \dots + \\ + x^{\mu_1 \mu_2 \dots \mu_D} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \dots \wedge \gamma_{\mu_D} \tag{2}$$

denotes the position of a point in a manifold, called Clifford space or C-space. The series of terms in (2) terminates at a *finite* grade depending on the dimension D. A Clifford algebra $Cl(r,q)$ with $r+q=D$ has 2^D basis elements. For simplicity, the gammas γ^μ correspond to a Clifford algebra associated with a flat spacetime:

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}\,, \tag{3}$$

but in general one could extend this formulation to curved spacetimes with metric $g^{\mu\nu}$ (see section 4).

The connection to strings and p-branes can be seen as follows. In the case of a closed string (a 1-loop) embedded in a target flat spacetime background of D-dimensions, one represents the projections of the closed string (1-loop) onto the embedding spacetime coordinate-planes by the variables $x^{\mu\nu}$. These variables represent the respective *areas* enclosed by the projections of the closed string (1-loop) onto the corresponding embedding spacetime planes. Similary, one can embed a closed membrane (a 2-loop) onto a D-dim flat spacetime, where the projections given by the antisymmetric variables $x^{\mu\nu\rho}$ represent the corresponding *volumes* enclosed by the projections of the 2-loop along the hyperplanes of the flat target spacetime background.

This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p=0,1,2,3,\dots$. The $p=0$ value represents the center of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}, \dots$ have been *coined* in the string-brane literature [24]. as the *holographic* areas, volumes, ... projections of the nested family of p-loops (closed p-branes) onto the embedding spacetime coordinate planes/hyperplanes. In ref. [17]

*If we do not restrict indices according to $\mu_1 < \mu_2 < \mu_3 < \dots$, then the factors $1/2!$, $1/3!$, respectively, have to be included in front of every term in the expansion (1).

they were interpreted as the generalized centre of mass coordinates of an extended object. Extended objects were thus modeled in C-space.

The scalar coordinate σ entering a polyvector X is a measure associated with the p-brane's world manifold V_{p+1} (e. g., the string's 2-dimensional worldsheet V_2): it is proportional to the $(p+1)$-dimensional area/volume of V_{p+1}. In other words, σ is proportional to the areal-time parameter of the Eguchi-Schild formulation of string dynamics [126, 37, 24].

We see in this generalized scheme the objects as observed in spacetime (which is a section through C-space) need not be infinitely extended along time-like directions. They need not be infinitely long world lines, world tubes. They can be finite world lines, world tubes. The σ coordinate measures how long are world lines, world tubes. During evolution they can becomes longer and longer or shorter and shorter.

If we take the differential dX of X and compute the scalar product among two polyvectors $<dX^\dagger dX>_0 \equiv \mathrm{d}X^\dagger * \mathrm{d}X \equiv |\mathrm{d}X|^2$ we obtain the C-space extension of the particles proper time in Minkowski space. The symbol $X^\dagger$ denotes the *reversion* operation and involves reversing the order of all the basis γ^μ elements in the expansion of X. It is the analog of the transpose (Hermitian) conjugation. The C-space proper time associated with a polyparticle motion is then the expression (1) which can be written more explicitly as:

$$\begin{aligned} |\mathrm{d}X|^2 &= G_{MN}\,\mathrm{d}X^M\mathrm{d}X^N = \mathrm{d}S^2 = \\ &= \mathrm{d}\sigma^2 + L^{-2}dx_\mu dx^\mu + L^{-4}dx_{\mu\nu}dx^{\mu\nu} + \ldots + \\ &+ L^{-2D}\mathrm{d}x_{\mu_1\ldots\mu_D}\,\mathrm{d}x^{\mu_1\cdots\mu_D}, \end{aligned} \tag{4}$$

where $G_{MN} = E_M^\dagger * E_N$ is the C-space metric.

Here we have introduced the Planck scale L since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops, ..., p-loops. Einstein introduced the speed of light as a universal absolute invariant in order to "unite" space with time (to match units) in the Minkowski space interval:

$$ds^2 = c^2dt^2 + dx_i dx^i.$$

A similar unification is needed here to "unite" objects of different dimensions, such as x^μ, $x^{\mu\nu}$, etc... . The Planck scale then emerges as another universal invariant in constructing an extended relativity theory in C-spaces [8].

Since the D-dimensional Planck scale is given explicitly in terms of the Newton constant: $L_D = (G_N)^{1/(D-2)}$, in natural units of $\hbar = c = 1$, one can see that when $D = \infty$ the value of L_D is then $L_\infty = G^0 = 1$ (assuming a finite value of G). Hence in $D = \infty$ the Planck scale has the natural value of unity. However, if one wishes to avoid any serious algebraic divergence problems in the series of terms appearing in the expansion of the analog of proper time in C-spaces, in the extreme case when $D = \infty$, from now on we shall focus solely on a *finite* value of D. In this fashion we avoid any serious algebraic convergence problems. We shall not be concerned in this work with the representations of Clifford algebras in different dimensions and with different signatures.

The line element $\mathrm{d}S$ as defined in (4) is *dimensionless*. Alternatively, one can define [8, 9] the line element whose dimension is that of the D-volume so that:

$$\begin{aligned} \mathrm{d}\Sigma^2 &= L^{2D}\mathrm{d}\sigma^2 + L^{2D-2}\mathrm{d}x_\mu \mathrm{d}^\mu + \\ &+ L^{2D-4}\mathrm{d}x_{\mu\nu}\mathrm{d}x^{\mu\nu} + \ldots + \mathrm{d}x_{\mu_1\ldots\mu_D}\mathrm{d}x^{\mu_1\cdots\mu_D}. \end{aligned} \tag{5}$$

Let us use the relation

$$\gamma_{\mu_1} \wedge \ldots \wedge \gamma_{\mu_D} = \gamma\epsilon_{\mu_1\ldots\mu_D} \tag{6}$$

and write the *volume element* as

$$\mathrm{d}x^{\mu_1\cdots\mu_D}\gamma_{\mu_1} \wedge \ldots \wedge \gamma_{\mu_D} \equiv \gamma\mathrm{d}\tilde{\sigma}\,, \tag{7}$$

where

$$\mathrm{d}\tilde{\sigma} \equiv \mathrm{d}x^{\mu_1\cdots\mu_D}\epsilon_{\mu_1\ldots\mu_D}\,. \tag{8}$$

In all expressions we assume the ordering prescription $\mu_1 < \mu_2 < \ldots < \mu_r$, $r = 1, 2, \ldots, D$. The line element can then be written in the form

$$\begin{aligned} \mathrm{d}\Sigma^2 &= L^{2D}\mathrm{d}\sigma^2 + L^{2D-2}\mathrm{d}x_\mu \mathrm{d}x^\mu + \\ &+ L^{2D-4}\mathrm{d}x_{\mu\nu}\mathrm{d}x^{\mu\nu} + \ldots + |\gamma|^2\,\mathrm{d}\tilde{\sigma}^2, \end{aligned} \tag{9}$$

where

$$|\gamma|^2 \equiv \gamma^\dagger * \gamma\,. \tag{10}$$

Here γ is the pseudoscalar basis element and can be written as $\gamma_0 \wedge \gamma_1 \wedge \ldots \gamma_{D-1}$. In flat spacetime M_D we have that $|\gamma|^2 = +1$ or -1, depending on dimension and signature. In M_4 with signature $(+---)$ we have $\gamma^\dagger * \gamma = \gamma^\dagger\gamma = \gamma^2 = -1$ $(\gamma \equiv \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3)$, whilst in M_5 with signature $(+----)$ it is $\gamma^\dagger\gamma = 1$.

The analog of Lorentz transformations in C-spaces which transform a polyvector X into another poly-vector X' is given by

$$X' = RXR^{-1} \tag{11}$$

with

$$R = e^{\theta^A E_A} = \exp\left[(\theta I + \theta^\mu\gamma_\mu + \theta^{\mu_1\mu_2}\gamma_{\mu_1}\wedge\gamma_{\mu_2}\ldots)\right] \tag{12}$$

and also

$$R^{-1} = e^{-\theta^A E_A} = \exp[-(\theta I + \theta^\nu\gamma_\nu + \theta^{\nu_1\nu_2}\gamma_{\nu_1}\wedge\gamma_{\nu_2}\ldots)] \tag{13}$$

where the theta parameters in (12), (13) are the components of the Clifford-value parameter $\Theta = \theta^M E_M$:

$$\theta; \theta^\mu; \theta^{\mu\nu}; \ldots \tag{14}$$

they are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation, one can write the norm of a poly-vectors in terms of the trace operation as: $||X||^2 = \mathit{Trace} X^2$. Hence under C-space Lorentz transformation the norms of poly-vectors behave like follows:

$$\mathit{Trace}\, X'^2 = \mathit{Trace}\,[RX^2R^{-1}] = \\ = \mathit{Trace}\,[RR^{-1}X^2] = \mathit{Trace}\, X^2. \tag{15}$$

These norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and $RR^{-1} = 1$. If one writes the invariant norm in terms of the *reversal* operation $<X^\dagger X>_s$ this will constrain the explicit form of the terms in the exponential which define the rotor R so the rotor R obeys the analog condition of an orthogonal rotation matrix $R^\dagger = R^{-1}$. Hence the appropriate poly-rotations of poly-vectors which preserve the norm must be:

$$||(X')^2|| = <X'^\dagger X'>_s = \\ = <(R^{-1})^\dagger X^\dagger R^\dagger R X R^{-1}>_s = \\ = <RX^\dagger X R^{-1}>_s = <X^\dagger X>_s = ||X^2||\,, \tag{16}$$

where once again, we made use of the analog of the cyclic property of the trace, $<RX^\dagger XR^{-1}>_s = <X^\dagger X>_s$.

This way of rewriting the inner product of poly-vectors by means of the reversal operation that reverses the order of the Clifford basis generators: $(\gamma^\mu \wedge \gamma^\nu)^\dagger = \gamma^\nu \wedge \gamma^\mu$, etc... has some subtleties. The analog of an orthogonal matrix in Clifford spaces is $R^\dagger = R^{-1}$ such that

$$<X'^\dagger X'>_s = <(R^{-1})^\dagger X^\dagger R^\dagger R X R^{-1}>_s = \\ = <RX^\dagger X R^{-1}>_s = <X^\dagger X>_s = \text{invariant}.$$

This condition $R^\dagger = R^{-1}$, of course, will *restrict* the type of terms allowed inside the exponential defining the rotor R because the *reversal* of a p-vector obeys

$$(\gamma_{\mu_1} \wedge \gamma_{\mu_2} \ldots \wedge \gamma_{\mu_p})^\dagger = \gamma_{\mu_p} \wedge \gamma_{\mu_{p-1}} \ldots \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1} = \\ = (-1)^{p(p-1)/2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \ldots \wedge \gamma_{\mu_p}\,.$$

Hence only those terms that change sign (under the reversal operation) are permitted in the exponential defining $R = \exp[\theta^A E_A]$.

Another possibility is to *complexify* the C-space polyvector valued coordinates $Z = Z^A E_A = X^A E_A + i Y^A E_A$ and the boosts/rotation parameters θ allowing the unitary condition $\bar{U}^\dagger = U^{-1}$ to hold in the generalized Clifford unitary transformations $Z' = UZU^\dagger$ associated with the complexified polyvector $Z = Z^A E_A$ such that the interval

$$<d\bar{Z}^\dagger dZ>_s = d\bar{\Omega} d\Omega + d\bar{z}^\mu dz_\mu + d\bar{z}^{\mu\nu} dz_{\mu\nu} + d\bar{z}^{\mu\nu\rho} dz_{\mu\nu\rho} + \ldots$$

remains invariant (upon setting the Planck scale $\Lambda = 1$).

The unitary condition $\bar{U}^\dagger = U^{-1}$ under the *combined* reversal and complex-conjugate operation will constrain the form of the complexified boosts/rotation parameters θ^A appearing in the rotor: $U = \exp\left[\theta^A E_A\right]$. The theta parameters θ^A are either purely real or purely imaginary depending if the reversal ${E_A}^\dagger = \pm E_A$, to ensure that an overall *change* of sign occurs in the terms $\theta^A E_A$ inside the exponential defining U so that $\bar{U}^\dagger = U^{-1}$ holds and the norm $<\bar{Z}^\dagger Z>_s$ remains invariant under the analog of unitary transformations in *complexified* C-spaces. These techniques are not very different from Penrose Twistor spaces. As far as we know a Clifford-Twistor space construction of C-spaces has not been performed so far.

Another alternative is to define the polyrotations by $R = \exp(\Theta^{AB}[E_A, E_B])$ where the commutator $[E_A, E_B] = F_{ABC} E_C$ is the C-space analog of the $i\,[\gamma_\mu, \gamma_\nu]$ commutator which is the generator of the Lorentz algebra, and the theta parameters Θ^{AB} are the C-space analogs of the rotation/boots parameters $\theta^{\mu\nu}$. The diverse parameters Θ^{AB} are purely real or purely imaginary depending whether the reversal $[E_A, E_B]^\dagger = \pm\,[E_A, E_B]$ to ensure that $R^\dagger = R^{-1}$ so that the scalar part $<X^\dagger X>_s$ remains invariant under the transformations $X' = RXR^{-1}$. This last alternative seems to be more physical because a poly-rotation should map the E_A direction into the E_B direction in C-spaces, hence the meaning of the generator $[E_A, E_B]$ which extends the notion of the $[\gamma_\mu, \gamma_\nu]$ Lorentz generator.

The above transformations are *active transformations* since the transformed Clifford number X' (polyvector) is different from the "original" Clifford number X. Considering the transformations of components we have

$$X' = X'^M E_M = L^M{}_N\, X^N E_M\,. \tag{17}$$

If we compare (17) with (11) we find

$$L^M{}_N E_N = R E_N R^{-1} \tag{18}$$

from which it follows that

$$L^M{}_N = \langle E^M R E_N R^{-1} \rangle_0 \equiv E^M * (R E_N R^{-1}) = E^M * E'_N, \tag{19}$$

where we have labelled E'_N as new basis element since in the active interpretation one may perform either a change of the polyvector components or a change of the basis elements. The $\langle\quad\rangle_0$ means the scalar part of the expression and "$*$" the scalar product. Eq-(19) has been obtained after multiplying (18) from the left by E^J, taking into account that $\langle E^J E_N \rangle_0 \equiv E^J * E_N = \delta^J{}_N$, and renamiming the index J into M.

3 Generalized dynamics of particles, fields and branes in C-space

An immediate application of this theory is that one may consider "strings" and "branes" in C-spaces as a unifying description of *all* branes of different dimensionality. As we have already indicated, since spinors are in left/right ideals of a Clifford algebra, a supersymmetry is then naturally incorporated into this approach as well. In particular, one can have world manifold and target space supersymmetry *simultaneously* [15]. We hope that the C-space "strings" and "branes" may lead us towards discovering the physical foundations of string and M-theory. For other alternatives to supersymmetry see the work by [50]. In particular, Z_3 generalizations of supersymmetry based on ternary algebras and Clifford algebras have been proposed by Kerner [128] in what has been called Hypersymmetry.

3.1 The Polyparticle Dynamics in C-space

We will now review the theory [15, 17] in which an extended object is modeled by the components $\sigma,\ x^{\mu}, x^{\mu\nu}, \ldots$ of the Clifford valued polyvector (2). By assumption the extended objects, as observed from Minkowski spacetime, can in general be localized not only along space-like, but also along time-like directions [15, 17]. In particular, they can be "instantonic" p-loops with either space-like or time-like orientation. Or they may be long, but finite, tube-like objects. The theory that we consider here goes beyond the ordinary relativity in Minkowski spacetime, therefore such localized objects in Minkowski spacetime pose no problems. They are postulated to satisfy the dynamical principle which is formulated in C-space. All conservation laws hold in C-space where we have infinitely long world "lines" or Clifford lines. In Minkowski spacetime M_4 — which is a subspace of C-space — we observe the intersections of Clifford lines with M_4. And those intersections appear as localized extended objects, p-loops, described above.

Let the motion of such an extended object be determined by the action principle

$$I = \kappa \int \mathrm{d}\tau\, (\dot{X}^{\dagger} * \dot{X})^{1/2} = \kappa \int \mathrm{d}\tau\, (\dot{X}^{A}\dot{X}_{A})^{1/2}\,, \qquad (20)$$

where κ is a constant, playing the role of "mass" in C-space, and τ is an arbitrary parameter. The C-space velocities $\dot{X}^{A} = \mathrm{d}X^{A}/\mathrm{d}\tau = (\dot{\sigma}, \dot{x}^{\mu}, \dot{x}^{\mu\ nu}, \ldots)$ are also called "holographic" velocities.

The equation of motion resulting from (20) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{X}^{A}}{\sqrt{\dot{X}^{B}\dot{X}_{B}}}\right) = 0\,. \qquad (21)$$

Taking $\dot{X}^{B}\dot{X}_{B} = \text{constant} \neq 0$ we have that $\ddot{X}^{A} = 0$, so that $x^{A}(\tau)$ is a straight worldline in C-space. The components x^{A} then change linearly with the parameter τ. This means that the extended object position x^{μ}, effective area $x^{\mu\nu}$, 3-volume $x^{\mu\nu\alpha}$, 4-volume $x^{\mu\nu\alpha\beta}$, etc., they all change with time. That is, such object experiences a sort of generalized dilational motion [17].

We shall now review the procedure exposed in ref. [17] according to which in such a generalized dynamics an object may be accelerated to faster than light speeds as viewed from a 4-dimensional Minkowski space, which is a subspace of C-space. For a different explanation of superluminal propagation based on the modified nonlinear de Broglie dispersion relations see [68].

The canonical momentum belonging to the action (20) is

$$P_{A} = \frac{\kappa \dot{X}_{A}}{(\dot{X}^{B}\dot{X}_{B})^{1/2}}\,. \qquad (22)$$

When the denominator in eq.-(22) is zero the momentum becomes infinite. We shall now calculate the speed at which this happens. This will be the *maximum speed* that an object accelerating in C-space can reach. Although an initially slow object cannot accelerate beyond that speed limit, this does not automatically exclude the possibility that fast objects traveling at a speed above that limit may exist. Such objects are C-space analog of tachyons [31, 32]. All the well known objections against tachyons should be reconsidered for the case of C-space before we could say for sure that C-space tachyons do not exist as freely propagating objects. We will leave aside this interesting possibility, and assume as a working hypothesis that there is no tachyons in C-space.

Vanishing of $\dot{X}^{B}\dot{X}_{B}$ is equivalent to vanishing of the C-space line element

$$\begin{aligned} \mathrm{d}X^{A}\mathrm{d}X_{A} = \mathrm{d}\sigma^{2} + \left(\frac{\mathrm{d}x^{0}}{L}\right)^{2} - \left(\frac{\mathrm{d}x^{1}}{L}\right)^{2} - \left(\frac{\mathrm{d}x^{01}}{L^{2}}\right)^{2} \ldots \\ \ldots + \left(\frac{\mathrm{d}x^{12}}{L^{2}}\right)^{2} - \left(\frac{\mathrm{d}x^{123}}{L^{3}}\right)^{2} - \left(\frac{\mathrm{d}x^{0123}}{L^{4}}\right)^{2} + \ldots = 0\,, \end{aligned} \qquad (23)$$

where by "..." we mean the terms with the remaining components such as x^{2}, x^{01}, x^{23}, ..., x^{012}, etc. The C-space line element is associated with a particular *choice* of C-space metric, namely $G_{MN} = E^{\dagger}_{M} * E_{N}$. If the basis E_{M}, $M = 1, 2, \ldots, 2^{D}$ is generated by the flat space γ^{μ} satisfying (3), then the C-space has the diagonal metric of eq.-(23) with $+, -$ signa. In general this is not necessarily so and the C-space metric is a more complicated expression. We take now dimension of spacetime being 4, so that x^{0123} is the highest grade coordinate. In eq.-(23) we introduce a length parameter L. This is necessary, since $x^{0} = ct$ has dimension of length, x^{12} of length square, x^{123} of length to the third power, and x^{0123} of length to the forth power. It is natural to assume that L is the *Planck length*, that is $L = 1.6{\times}10^{-35}$m.

Let us assume that the coordinate time $t = x^{0}/c$ is the parameter with respect to which we define the speed V in C-space.

So we have

$$\begin{aligned} V^{2} = -\left(L\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^{2} + \left(\frac{\mathrm{d}x^{1}}{\mathrm{d}t}\right)^{2} + \left(\frac{\mathrm{d}x^{01}}{L^{2}}\right)^{2} \ldots \\ \ldots - \left(\frac{1}{L}\frac{\mathrm{d}x^{12}}{\mathrm{d}t}\right)^{2} + \left(\frac{1}{L^{2}}\frac{\mathrm{d}x^{123}}{\mathrm{d}t}\right)^{2} + \left(\frac{1}{L^{3}}\frac{\mathrm{d}x^{0123}}{\mathrm{d}t}\right)^{2} - \ldots \end{aligned} \qquad (24)$$

From eqs.-(23), (24) we find that the maximum speed is the maximum speed is given by

$$V^2 = c^2 \,. \tag{25}$$

First, we see, the maximum speed squared V^2 contains not only the components of the 1-vector velocity $\mathrm{d}x^1/\mathrm{d}t$, as it is the case in the ordinary relativity, but also the multivector components such as $\mathrm{d}x^{12}/\mathrm{d}t$, $\mathrm{d}x^{123}/\mathrm{d}t$, etc.

The following special cases when only certain components of the velocity in C-space are different from zero, are of particular interest:

(i) Maximum 1-vector speed

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = c = 3.0{\times}10^8 \mathrm{m/s}\,;$$

(ii) Maximum 3-vector speed

$$\frac{\mathrm{d}x^{123}}{\mathrm{d}t} = L^2 c = 7.7{\times}10^{-62} \mathrm{m^3/s}\,;$$

$$\frac{\mathrm{d}\sqrt[3]{x^{123}}}{\mathrm{d}t} = 4.3{\times}10^{-21} \mathrm{m/s} \quad \text{(diameter speed)}\,;$$

(iii) Maximum 4-vector speed

$$\frac{\mathrm{d}x^{0123}}{\mathrm{d}t} = L^3 c = 1.2{\times}10^{-96} \mathrm{m^4/s}$$

$$\frac{\mathrm{d}\sqrt[4]{x^{0123}}}{\mathrm{d}t} = 1.05{\times}10^{-24} \mathrm{m/s} \quad \text{(diameter speed)}\,.$$

Above we have also calculated the corresponding diameter speeds for the illustration of how fast the object expands or contracts.

We see that the maximum multivector speeds are very small. The diameters of objects change very slowly. Therefore we normally do not observe the dilatational motion.

Because of the positive sign in front of the σ and x^{12}, x^{012}, etc., terms in the quadratic form (23) there are no limits to corresponding 0-vector, 2-vector and 3-vector speeds. But if we calculate, for instance, the energy necessary to excite 2-vector motion we find that it is very high. Or equivalently, to the relatively modest energies (available at the surface of the Earth), the corresponding 2-vector speed is very small. This can be seen by calculating the energy

$$p^0 = \frac{\kappa c^2}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{26}$$

(a) for the case of pure 1-vector motion by taking $V = \mathrm{d}x^1/\mathrm{d}t$, and

(b) for the case of pure 2-vector motion by taking $V = \mathrm{d}x^{12}/(L\mathrm{d}t)$.

By equating the energies belonging to the cases (a) and (b) we have

$$p^0 = \frac{\kappa c^2}{\sqrt{1 - \left(\frac{1}{c}\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2}} = \frac{\kappa c^2}{\sqrt{1 - \left(\frac{1}{Lc}\frac{\mathrm{d}x^{12}}{\mathrm{d}t}\right)^2}}\,, \tag{27}$$

which gives

$$\frac{1}{c}\frac{\mathrm{d}x^1}{\mathrm{d}t} = \frac{1}{Lc}\frac{\mathrm{d}x^{12}}{\mathrm{d}t} = \sqrt{1 - \left(\frac{\kappa c^2}{p_0}\right)^2}\,. \tag{28}$$

Thus to the energy of an object moving translationally at $\mathrm{d}x^1/\mathrm{d}t = 1\,\mathrm{m/s}$, there corresponds the 2-vector speed $\mathrm{d}x^{12}/\mathrm{d}t = L\,\mathrm{d}x^1/\mathrm{d}t = 1.6{\times}10^{-35}$ m^2/s (diameter speed 4×10^{-18} m/s). This would be a typical 2-vector speed of a macroscopic object. For a microscopic object, such as the electron, which can be accelerated close to the speed of light, the corresponding 2-vector speed could be of the order of 10^{-26} m^2/s (diameter speed 10^{-13} m/s). In the examples above we have provided rough estimations of possible 2-vector speeds. Exact calculations should treat concrete situations of collisions of two or more objects, assume that not only 1-vector, but also 2-vector, 3-vector and 4-vector motions are possible, and take into account the conservation of the polyvector momentum P_A.

Maximum 1-vector speed, i. e., the usual speed, can exceed the speed of light when the holographic components such as $\mathrm{d}\sigma/\mathrm{d}t$, $\mathrm{d}x^{12}/\mathrm{d}t$, $\mathrm{d}x^{012}/\mathrm{d}t$, *etc., are different from zero* [17]. This can be immediately verified from eqs.-(23), (24). The speed of light is no longer such a strict barrier as it appears in the ordinary theory of relativity in M_4. In C-space a particle has extra degrees of freedom, besides the translational degrees of freedom. The scalar, σ, the bivector, x^{12} (in general, x^{rs}, $r, s = 1, 2, 3$) and the three vector, x^{012} (in general, x^{0rs}, $r, s = 1, 2, 3$), contributions to the C-space quadratic form (23) have positive sign, which is just opposite to the contributions of other components, such as x^r, x^{0r}, x^{rst}, $x^{\mu\nu\rho\sigma}$. Because some terms in the quadratic form have $+$ and some $-$ sign, the absolute value of the 3-velocity $\mathrm{d}x^r/\mathrm{d}x^0$ can be greater than c.

It is known that when tachyons can induce a breakdown of causality. The simplest way to see why causality is violated when tachyons are used to exchange signals is by writing the temporal displacements $\delta t = t^B - t^A$ between two events (in Minkowski space-time) in two different frames of reference:

$$\begin{gathered}(\delta t)' = (\delta t)\cosh(\xi) + \frac{\delta x}{c}\sinh(\xi) = (\delta t)\Big[\cosh(\xi) + \\ + \left(\frac{1}{c}\frac{\delta x}{\delta t}\right)\sinh(\xi)\Big] = (\delta t)[\cosh(\xi) + (\beta_{tach.})\sinh(\xi)]\end{gathered} \tag{29}$$

the boost parameter ξ is defined in terms of the velocity as $\beta_{frame} = v_{frame}/c = \tanh(\xi)$, where v_{frame} is is the relative velocity (in the x-direction) of the two reference frames and can be written in terms of the Lorentz-boost rapidity parameter ξ by using hyperbolic functions. The Lorentz dilation factor is $\cosh(\xi) = (1-\beta^2_{frame})^{-1/2}$; whereas

$\beta_{tachyon} = v_{tachyon}/c$ is the beta parameter associated with the tachyon velocity $\delta x/\delta t$. By emitting a tachyon along the *negative* x -direction one has $\beta_{tachyon} < 0$ and such that its velocity exceeds the speed of light $|\beta_{tachyon}| > 1$.

A reversal in the sign of $(\delta t)' < 0$ in the above boost transformations occurs when the tachyon velocity $|\beta_{tachyon}| > 1$ and the relative velocity of the reference frames $|\beta_{frame}| < 1$ obey the inequality condition:

$$(\delta t)' = (\delta t)[\cosh(\xi) - |\beta_{tachyon}| \sinh(\xi)] < 0 \Rightarrow \Rightarrow 1 < \frac{1}{\tanh(\xi)} = \frac{1}{\beta_{frame}} < |\beta_{tachyon}| \quad (30)$$

thereby resulting in a causality violation in the primed reference frame since the effect (event B) occurs *before* the cause (event A) in the *primed* reference frame.

In the case of subluminal propagation $|\beta_{particle}| < 1$ there is no causality violation since one would have:

$$(\delta t)' = (\delta t)[\cosh(\xi) - |\beta_{particle}| \sinh(\xi)] > 0 \quad (31)$$

due to the hyperbolic trigonometric relation:

$$\cosh^2(\xi) - \sinh^2(\xi) = 1 \Rightarrow \cosh(\xi) - \sinh(\xi) \geqslant 0\,. \quad (32)$$

In the theory considered here, there are no tachyons in C-space, because physical signals in C-space are constrained to live *inside* the C-space-light cone, defined by eq.-(23). However, certain worldlines in C-space, when projected onto the subspace M_4, can appear as worldlines of ordinary tachyons outside the light-cone in M_4. The physical analog of photons in C-space corresponds to tensionless p-loops, i. e., *tensionless* closed branes, since the analog of mass m in C-space is the maximal p-loop tension. By "maximal p-loop" we mean the loop with the maximum value of p associated with the hierarchy of p-loops (closed p-branes): $p = 0, 1, 2, \ldots$ living in the embedding target spacetime. One must not confuse the Stueckelberg parameter σ with the C-space Proper-time Σ eq.-(5); so one could have a world line in C-space such that

$\mathrm{d}\Sigma = 0 \leftrightarrow C$-space photon $\leftrightarrow$ Tensionless branes with a monotonically increasing Stueckelberg parameter σ.

In C-space the dynamics refers to a larger space. Minkowski space is just a subspace of C-space. "Wordlines" now live in C-space that can be projected onto the Minkowski subspace M_4. Concerning tachyons and causality within the framework of the C-space relativity, the authors of this review propose two different explanations, described below.

According to one author (C. C.) one has to take into account the fact that one is enlarging the ordinary Lorentz group to a larger group of C-space Lorentz transformations which involve poly-rotations and generalizations of boosts transformations. In particular, the C-space generalization of the ordinary boost transformations associated with the boost rapidity parameter ξ such that $\tanh(\xi) = \beta_{frame}$ will involve now the family of C-space boost rapidity parameters θ^{t1}, θ^{t12}, θ^{t123}, $\ldots$ $\theta^{t123\cdots}$, $\ldots$ since boosts are just (poly) rotations along directions involving the *time* coordinate. Thus, one is replacing the ordinary boost transformations in Minkowski spacetime for the more general C-space boost transformations as we go from one frame of reference to another frame of reference.

Due to the linkage among the C-space coordinates (polydimensional covariance) when we envision an ordinary boost along the x^1-direction, we must not forget that it is also? interconnected to the area-boosts in the x^{12}-direction as well, and, which in turn, is also linked to the x^2 direction. Because the latter direction is *transverse* to the original tachyonic? x^1-motion? the latter x^2-boosts? won't affect things and we may concentrate? on the area-boosts along the x^{12} direction involving the θ^{t12} parameter that will appear in the C-space boosts and which contribute to a crucial extra term in the transformations such that no sign-change in $\delta t'$? will occur.

More precisely, let us set *all* the values of the theta parameters to zero *except* the parameters θ^{t1} and θ^{t12} related to the ordinary boosts in the x^1 direction and area-boosts in the x^{12} directions of C-space. This requires, for example, that one has at least one spatial-area component, and one temporal coordinate, which implies that the dimensions must be at least $D = 2 + 1 = 3$. Thus, we have in this case:

$$X' = RXR^{-1} = e^{\theta^{t1}\gamma_t\wedge\gamma_1 + \theta^{t12}\gamma_t\wedge\gamma_1\wedge\gamma_2} \times \times X^M E_M e^{-\theta^{t1}\gamma_t\wedge\gamma_1 - \theta^{t12}\gamma_t\wedge\gamma_1\wedge\gamma_2} \Rightarrow X'^N = L^N_M X^M, \quad (33)$$

where as we shown previously $L^N_M = \langle E^N R E_M R^{-1} \rangle_0$. When one concentrates on the transformations of the time coordinate, we have now that the C-space boosts do *not* coincide with ordinary boosts in the x^1 direction:

$$t' = L^t_M X^M = \langle E^t R E_M R^{-1} \rangle_0 X^M \neq (L^t_t) t + (L^t_1) x^1, \quad (34)$$

because of the extra non-vanishing θ parameter θ^{t12}.

This is because the rotor R includes the extra generator $\theta^{t12}\gamma_t \wedge \gamma_1 \wedge \gamma_2$ which will bring extra terms into the transformations; i. e. it will rotate the $E_{[12]}$ bivector-basis, that couples to the holographic coordinates x^{12}, into the E_t direction which is being contracted with the E^t element in the definition of L^t_M. There are extra terms in the C-space boosts because the poly-particle dynamics is taking place in C-space and all coordinates X^M which contain the t, x^1, x^{12} directions will contribute to the C-space boosts in $D = 3$, since one is projecting down the dynamics from C-space onto the (t, x^1) plane when one studies the motion of the tachyon in M_4.

Concluding, in the case when one sets all the θ parameters to zero, except the θ^{t1} and θ^{t12}, the $X' = RX^M E_M R^{-1}$

transformations will be:

$$(\delta t)' = L^t_M(\theta^{t1};\theta^{t12})(\delta X^M) \neq L^t_t(\delta t) + L^t_1(\delta x^1)\,, \quad (35)$$

due to the presence of the *extra* term $L^t_{12}(\delta X^{12})$ in the transformations. In the more general case, when there are more non-vanishing θ parameters, the indices M of the X^M coordinates must be *restricted* to those directions in C-space which involve the t, x^1, x^{12}, $x^{123}\dots$ directions as required by the C-space poly-particle dynamics. The generalized C-space boosts involve now the ordinary tachyon velocity component of the poly-particle as well as the generalized holographic areas, volumes, hyper-volumes... velocities $V^M = (\delta X^M/\delta t)$ associated with the poly-vector components of the Clifford-valued C-space velocity.

Hence, at the expense of *enlarging* the ordinary Lorentz boosts to the C-space Lorentz boosts, and the degrees of freedom of a point particle into an extended poly-particle by including the holographic coordinates, in C-space one can still have ordinary point-particle tachyons without changing the sign of δt, and without violating causality, due to the presence of the *extra* terms in the C-space boosts transformations which ensure us that the sign of $\delta t > 0$ is maintained as we go from one frame of reference to another one. Naturally, if one were to *freeze* all the θ parameters to zero except one θ^{t1} one would end up with the standard Lorentz boosts along the x^1-direction and a violation of causality would occur for tachyons as a result of the sign-change in $\delta t'$.

In future work we shall analyze in more detail if the condition $\delta t' = L^t_M(\delta X^M) > 0$ is satisfied for *any* physical values of the θ C-space boosts parameters and for *any* physical values of the holographic velocities consistent with the condition that the C-space velocity $V_M V^M \geqslant 0$. What one cannot have is a C-space tachyon; i. e. the physical signals in C-space must be constrained to live *inside* the C-space light-cone. The analog of "photons" in C-space are *tensionless* branes. The corresponding analog of C-space tachyons involve branes with imaginary tensions, not unlike ordinary tachyons $m^2 < 0$ of imaginary mass.

To sum up: Relativity in C-space demands *enlarging* the ordinary Lorentz group (boosts) to a larger symmetry group of C-space Lorentz group and enlarging the degrees of freedom by including Clifford-valued coordinates $X = X^M E_M$. This is the only way one can have a point-particle tachyonic speed in a Minkowski subspace without violating causality in C-space. Ordinary Lorentz boosts are incompatible with tachyons if one wishes to preserve causality. In C-space one requires to have, at least, two theta parameters θ^{t1} and θ^{t12} with the inclusion, at least, of the t, x^1, x^{12} coordinates in a C-space boost, to be able to enforce the condition $\delta t' > 0$ under (combined) boosts along the x^1 direction accompanied by an *area*-boost along the x^{12} direction of C-space. It is beyond the scope of this review to analyze all the further details of the full-fledged C-boosts transformations in order to check that the condition $\delta t' > 0$ is obeyed for *any* physical values of the θ parameters and holographic velocities.

According to the other author (M. P.), the problem of causality could be explained as follows. In the usual theory of relativity the existence of tachyons is problematic because one can arrange for situations such that tachyons are sent into the past. A tachyon T_1 is emitted from an apparatus worldline $\mathcal{C}$ at x^0_1 and a second tachyon T_2 can arrive to the same worldline $\mathcal{C}$ at an earlier time $x'^0 < x^0_1$ and trigger destruction of the apparatus. The spacetime event E' at which the apparatus is destroyed coincides with the event E at which the apparatus by initial assumption kept on functioning normally and later emitted T_1. So there is a paradox from the ordinary (constrained) relativistic particle dynamics.

There is no paradox if one invokes the unconstrained Stueckelberg description of superluminal propagation in M_4. It can be described as follows. A C-space worldline can be described in terms of five functions $x^\mu(\tau)$, $\sigma(\tau)$ (all other C-space coordinates being kept constant). In C-space we have the *constrained action* (20), whilst in Minkowski space we have a reduced, *unconstrained* action. A reduction of variables can be done by choosing a gauge in which $\sigma(\tau) = \tau$. It was shown in ref. [16, 15, 17] that the latter unconstrained action is equivalent to the well known Stueckelberg action [33, 34]. In other words, the Stueckelberg relativistic dynamics is embedded in C-space. In Stueckelberg theory all four spacetime coordinates x^μ are independent dynamical degrees of freedom that evolve in terms of an extra parameter σ which is invariant under Lorentz transformations in M_4.

From the C-space point of view, the evolution parameter σ is just one of the C-space coordintes X^M. By assumption, σ is monotonically increasing along particles' worldlines. Certain C-space worldlines may appear tachyonic from the point of view of M_4. If we now repeat the above experiment with the emission of the first and absorption of the second tachyon we find out that the second tachyon T_2 cannot reach the aparatus worldline earlier than it was emmitted from. Namely, T_2 can arrive at a C-space event E' with $x'^0 < x^0_1$, but the latter event does not coincide with the event E on the aparatus worldline, since although having the same coordinates $x'^\mu = x^\mu$, the events E and E' have different extra coordinates $\sigma' \neq \sigma$. In other words, E and E' are different points in C-space. Therefore T_2 cannot destroy the apparatus and there is no paradox.

If nature indeed obeys the dynamics in Clifford space, then a particle, as observed from the 4-dimensional Minkowski space, can be accelerated beyond the speed of light [17], provided that its extra degrees of freedom $x^{\mu\nu}$, $x^{\mu\nu\alpha}, \dots$, are changing simultaneously with the ordinary position x^μ. But such a particle, although moving faster than light in the subspace M_4, is moving slower than light in C-space, since its speed V, defined in eq.-(24), is smaller than *c*. *In* *this respect, our particle is not tachyon at all!* In C-space we thus retain all the nice features of relativity, but in the subspace M_4 we have, as a particular case, the unconstrained Stueckelberg theory in which faster-than-light propagation is not paradoxical and is consistent with the quantum field theory as well [15]. This is so, because the unconstrained Stueckelberg theory is quite different from the ordinary (constrained) theory of relativity in M_4, and faster than light motion in the former theory is of totally different nature from the faster that light motion in the latter theory. The tachyonic "world lines" in M_4 are just projections of trajectories in C-space onto Minkowski space, however, the true world lines of M_4 must be interpreted always as being embedded onto a larger C-space, such that they cannot take part in the paradoxical arrangement in which future could influence the past. The well known objections against tachyons are not valid for our particle which moves according to the relativity in C-space.

We have described how one can obtain faster than light motion in M_4 from the theory of relativity in C-space. There are other possible ways to achieve superluminal propagation. One such approach is described in refs. [84]

An alternative procedure In ref. [9] an alternative factorization of the C-space line element has been undertaken. Starting from the line element $\mathrm{d}\Sigma$ of eq.-(5), instead of factoring out the $(dx^0)^2$ element, one may factor out the $(\mathrm{d}\Omega)^2 \equiv \equiv L^{2D}\mathrm{d}\sigma^2$ element, giving rise to the generalized "holographic" velocities measured w. r. t the Ω parameter, for example the areal-time parameter in the Eguchi-Schild formulation of string dynamics [126], [37], [24], instead of the x^0 parameter (coordinate clock). One then obtains

$$\mathrm{d}\Sigma^2 = \mathrm{d}\Omega^2 \left[1 + L^{2D-2} \frac{\mathrm{d}x_\mu}{\mathrm{d}\Omega} \frac{\mathrm{d}x^\mu}{\mathrm{d}\Omega} + + L^{2D-4} \frac{\mathrm{d}x_{\mu\nu}}{\mathrm{d}\Omega} \frac{\mathrm{d}x^{\mu\nu}}{\mathrm{d}\Omega} + \ldots + |\gamma|^2 \left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\Omega} \right)^2 \right]. \tag{36}$$

The idea of ref. [9] was to restrict the line element (36) to the non tachyonic values which imposes un upper limit on the holographic velocities. The motivation was to find a lower bound of length scale. This upper holographic-velocity bound does not necessarily translate into a lower bound on the values of lengths, areas, volumes... without the introduction of quantum mechanical considerations. One possibility could be that the upper limiting speed of light and the upper bound of the momentum $m_p c$ of a Planck-mass elementary particle (the so-called *Planckton* in the literature) generalizes now to an upper-bound in the p-loop holographic velocities and the p-loop holographic momenta associated with elementary closed p-branes whose tensions are given by powers of the Planck mass. And the latter upper bounds on the holographic p-loop momenta implies a lower-bound on the holographic areas, volumes, ..., resulting from the string/brane uncertainty relations [11], [10], [19]. Thus, Quantum Mechanics is required to implement the postulated principle of minimal lengths, areas, volumes... and which cannot be derived from the classical geometry alone. The emergence of minimal Planck areas occurs also in the Loop Quantum Gravity program [111] where the expectation values of the Area operator are given by multiples of Planck area.

Recently in [134] an isomorphism between Yang's Noncommutative space-time algebra (involving *two* length scales) [136] and the *holographic area coordinates* algebra of C-spaces (Clifford spaces) was constructed via an AdS_5 space-time which is instrumental in explaining the origins of an extra (infrared) scale R in conjunction to the (ultraviolet) Planck scale λ characteristic of C-spaces. Yang's Noncommutative space-time algebra allowed Tanaka [137] to explain the origins behind the *discrete* nature of the spectrum for the *spatial* coordinates and *spatial* momenta which yields a *minimum* length-scale λ (ultraviolet cutoff) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff). In particular, the norm-squared $\mathbf{A}^2$ of the holographic Area operator $X_{AB}X^{AB}$ has a correspondence with the quadratic Casimir operator $\Sigma_{AB}\Sigma^{AB}$ of the conformal algebra $SO(4,2)$ ($SO(5,1)$ in the Euclideanized AdS_5 case). This holographic area-Casimir relationship does not differ much from the area-spin relation in Loop Quantum Gravity $\mathbf{A}^2 \sim \sim \lambda^4 \sum j_i\,(j_i+1)$ in terms of the $SU(2)$ Casimir J^2 with eigenvalues $j\,(j+1)$ and where the sum is taken over the spin network sites.

3.2 A unified theory of all p-Branes in C-spaces

The generalization to C-spaces of string and p-brane actions as embeddings of world-manifolds onto target spacetime backgrounds involves the embeddings of polyvector-valued world-manifolds (of dimensions 2^d) onto polyvector-valued target spaces (of dimensions 2^D), given by the Clifford-valued maps $X = X(\Sigma)$ (see [15]). These are maps from the Clifford-valued world-manifold, parametrized by the polyvector-valued variables Σ, onto the Clifford-valued target space parametrized by the polyvector-valued coordinates X. Physically one envisions these maps as taking an n-dimensional simplicial cell (n-loop) of the world-manifold onto an m-dimensional simplicial cell (m-loop) of the target C-space manifold; i. e. maps from n-dim objects onto m-dim objects generalizing the old maps of taking points onto points. One is basically dealing with a dimension-category of objects. The size of the simplicial cells (p-loops), upon quantization of a generalized harmonic oscillator, for example, are given by multiples of the Planck scale, in area, volume, hypervolume units or Clifford-bits.

In compact multi-index notation $X = X^M\Gamma_M$ one denotes for each one of the components of the target space polyvector X:

$$X^M \equiv X^{\mu_1\mu_2\ldots\mu_r},\ \mu_1 < \mu_2 < \ldots < \mu_r \tag{37}$$

and for the world-manifold polyvector $\Sigma = \Sigma^A E_A$:

$$\Sigma^A \equiv \xi^{a_1 a_2 \ldots a_s}, a_1 < a_2 < \ldots < a_s \,, \tag{38}$$

where $\Gamma_M = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu}, \ldots)$ and $E_A = (\underline{1}, e_a, e_{ab}, \ldots)$ form the basis of the target manifold and world manifold Clifford algebra, respectively. It is very important to order the indices within each multi-index M and A as shown above. The above Clifford-valued coordinates X^M, Σ^A correspond to antisymmetric tensors of ranks r, s in the target spacetime background and in the world-manifold, respectively.

There are many different ways to construct C-space brane actions which are on-shell equivalent to the analogs of the Dirac-Nambu-Goto action for extended objects and that are given by the world-volume spanned by the branes in their motion through the target spacetime background.

One of these actions is the Polyakov-Howe-Tucker one:

$$I = \frac{T}{2} \int [D\Sigma] \sqrt{|H|} \left[H^{AB} \partial_A X^M \partial_B X^N G_{MN} + (2 - 2^d) \right] \tag{39}$$

with the 2^d-dim world-manifold measure:

$$[D\Sigma] = (d\xi)(d\xi^a)(d\xi^{a_1 a_2})(d\xi^{a_1 a_2 a_3}) \ldots \tag{40}$$

Upon the algebraic elimination of the auxiliary world-manifold metric H^{AB} from the action (39), via the equations of motion, yields for its on-shell solution the pullback of the target C-space metric onto the C-space world-manifold:

$$H_{AB}(on - shell) = G_{AB} = \partial_A X^M \partial_B X^N G_{MN} \tag{41}$$

upon inserting back the on-shell solutions (41) into (39) gives the Dirac-Nambu-Goto action for the C-space branes directly in terms of the C-space determinant, or measure, of the induced C-space world-manifold metric G_{AB}, as a result of the embedding:

$$I = T \int [\mathrm{D}\Sigma] \sqrt{\mathrm{Det}(\partial_A X^M \partial_B X^N G_{MN})} \,. \tag{42}$$

However in C-space, the Polyakov-Howe-Tucker action admits an even further generalization that is comprised of two terms $S_1 + S_2$. The first term is [15]:

$$S_1 = \int [\mathrm{D}\Sigma] |E| E^A E^B \partial_A X^M \partial_B X^N \Gamma_M \Gamma_N \,. \tag{43}$$

Notice that this is a generalized action which is written in terms of the C-space coordinates $X^M(\Sigma)$ and the C-space analog of the target-spacetime vielbein/frame one-forms $e^m = e^m{}_\mu dx^\mu$ given by the Γ^M variables. The auxiliary world-manifold vielbein variables e^a, are given now by the Clifford-valued frame E^A variables.

In the conventional Polyakov-Howe-Tucker action, the auxiliary world-manifold metric h^{ab} associated with the standard p-brane actions is given by the usual scalar product of the frame vectors $e^a, e^b = e^a_\mu e^b_\nu g^{\mu\nu} = h^{ab}$. Hence, the C-space world-manifold metric H^{AB} appearing in (41) is given by scalar product $\langle (E^A)^\dagger E^B \rangle_0 = H^{AB}$, where $(E^A)^\dagger$ denotes the reversal operation of E^A which requires reversing the ordering of the vectors present in the Clifford aggregate E^A.

Notice, however, that the form of the action (43) is far more general than the action in (39). In particular, the S_1 itself can be decomposed further into two additional pieces by rewriting the Clifford product of two basis elements into a symmetric plus an antisymmetric piece, respectively:

$$E^A E^B = \frac{1}{2}\{E^A, E^B\} + \frac{1}{2}[E^A, E^B] \,, \tag{44}$$

$$\Gamma_M \Gamma_N = \frac{1}{2}\{\Gamma_M, \Gamma_N\} + \frac{1}{2}[\Gamma_M, \Gamma_N] . \tag{45}$$

In this fashion, the S_1 component has *two* kinds of terms. The first term containing the symmetric combination is just the analog of the standard non-linear sigma model action, and the second term is a Wess-Zumino-like term, containing the antisymmetric combination. To extract the non-linear sigma model part of the generalized action above, we may simply take the scalar product of the vielbein-variables as follows:

$$(S_1)_{sigma} = \\ = \frac{T}{2} \int [\mathrm{D}\Sigma] |E| \langle (E^A \partial_A X^M \Gamma_M)^\dagger (E^B \partial_B X^N \Gamma_N) \rangle_0 \tag{46}$$

where once again we have made use of the reversal operation (the analog of the hermitian adjoint) before contracting multi-indices. In this fashion we recover again the Clifford-scalar valued action given by [15].

Actions like the ones presented here in terms of derivatives with respect to quantities with multi-indices can be mapped to actions involving *higher* derivatives, in the same fashion that the C-space scalar curvature, the analog of the Einstein-Hilbert action, could be recast as a higher derivative gravity with torsion (reviewed in sec. 4). Higher derivatives actions are also related to theories of Higher spin fields [117] and W-geometry, W-algebras [116], [122]. For the role of Clifford algerbras to higher spin theories see [51].

The S_2 (scalar) component of the C-space brane action is the usual cosmological constant term given by the C-space determinant $|E| = \det(H^{AB})$ based on the scalar part of the geometric product $\langle (E^A)^\dagger E^B \rangle_0 = H^{AB}$

$$S_2 = \frac{T}{2} \int [\mathrm{D}\Sigma] |E| \,, (2 - 2^d) \,, \tag{47}$$

where the C-space determinant $|E| = \sqrt{|\det(H^{AB})|}$ of the $2^d \times 2^d$ generalized world-manifold metric H^{AB} is given by:

$$\det(H^{AB}) = \frac{1}{(2^d)!} \epsilon_{A_1 A_2 \ldots A_{2^d}} \epsilon_{B_1 B_2 \ldots B_{2^d}} \times \\ \times H^{A_1 B_1} H^{A_2 B_2} \ldots H^{A_{2^d} B_{2^d}} . \tag{48}$$

The $\epsilon_{A_1 A_2 \ldots A_{2d}}$ is the totally antisymmetric tensor density in C-space.

There are many different forms of p-brane actions, with and without a cosmological constant [123], and based on a new integration measure by recurring to auxiliary scalar fields [115], that one could have used to construct their C-space generalizations. Since all of them are on-shell equivalent to the Dirac-Nambu-Goto p-brane actions, we decided to focus solely on those actions having the Polyakov-Howe-Tucker form.

4 Generalized gravitational theories in curved C-spaces: higher derivative gravity and torsion from the geometry of C-space

4.1 Ordinary space

4.1.1 Clifford algebra based geometric calculus in curved space(time)

Clifford algebra is a very useful tool for description of geometry, especially of curved space V_n. Let us first review how it works in curved space(time). Later we will discuss a generalization to curved Clifford space [20].

We would like to make those techniques accessible to a wide audience of physicists who are not so familiar with the rigorous underlying mathematics, and demonstrate how Clifford algebra can be straightforwardly employed in the theory of gravity and its generalization. So we will leave aside the sophisticated mathematical approach, and rather follow as simple line of thought as possible, a praxis that is normally pursued by physicists. For instance, physicists in their works on general relativity employ a mathematical formulation and notation which is much simpler from that of purely mathematical or mathematically oriented works. For rigorous mathematical treatment the reader is advised to study, refs. [1, 76, 77, 78, 79].

Let the vector fields γ_μ, $\mu = 1, 2, \ldots, n$ be a coordinate basis in V_n satisfying the Clifford algebra relation

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}\left(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu\right) = g_{\mu\nu}\,, \tag{49}$$

where $g_{\mu\nu}$ is the metric of V_n. In curved space γ_μ and $g_{\mu\nu}$ cannot be constant but necessarily depend on position x^μ. An arbitrary vector is a linear superposition [1]

$$a = a^\mu \gamma_\mu\,, \tag{50}$$

where the components a^μ are *scalars* from the geometric point of view, whilst γ_μ are *vectors*.

Besides the basis $\{\gamma_\mu\}$ we can introduce the reciprocal basis* $\{\gamma^\mu\}$ satisfying

$$\gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) = g^{\mu\nu}\,, \tag{51}$$

*In Appendix A of the Hesteness book [1] the frame $\{\gamma^\mu\}$ is called *dual* frame because the duality operation is used in constructing it.

where $g^{\mu\nu}$ is the covariant metric tensor such that $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$, $\gamma^\mu \gamma_\nu + \gamma_\nu \gamma^\mu = 2\delta^\mu{}_\nu$ and $\gamma^\mu = g^{\mu\nu}\gamma_\nu$.

Following ref. [1] (see also [15]) we consider the *vector derivative* or *gradient* defined according to

$$\partial \equiv \gamma^\mu \partial_\mu\,, \tag{52}$$

where ∂_μ is an operator whose action depends on the quantity it acts on [26].

Applying the vector derivative ∂ on a *scalar* field ϕ we have

$$\partial\phi = \gamma^\mu \partial_\mu \phi\,, \tag{53}$$

where $\partial_\mu \phi \equiv (\partial/\partial x^\mu)\phi$ coincides with the partial derivative of ϕ.

But if we apply it on a *vector* field a we have

$$\partial a = \gamma^\mu \partial_\mu (a^\nu \gamma_\nu) = \gamma^\mu (\partial_\mu a^\nu \gamma_\nu + a^\nu \partial_\mu \gamma_\nu)\,. \tag{54}$$

In general γ_ν is not constant; it satisfies the relation to works [1, 15]

$$\partial_\mu \gamma_\nu = \Gamma^\alpha_{\mu\nu} \gamma_\alpha\,, \tag{55}$$

where $\Gamma^\alpha_{\mu\nu}$ is the *connection*. Similarly, for $\gamma^\nu = g^{\nu\alpha}\gamma_\alpha$ we have

$$\partial_\mu \gamma^\nu = -\Gamma^\nu_{\mu\alpha} \gamma^\alpha\,. \tag{56}$$

The *non commuting* operator ∂_μ so defined determines *the parallel transport* of a basis vector γ^ν. Instead of the symbol ∂_μ Hestenes uses $\square_\mu$, whilst Wheeler et. al. [36] use ∇_μ and call it "covariant derivative". In modern, mathematically oriented literature more explicit notation such as D_{γ_μ} or ∇_{γ_μ} is used. However, such a notation, although mathematically very relevant, would not be very practical in long computations. We find it very convenient to keep the symbol ∂_μ for components of the geometric operator $\partial = \gamma^\mu \partial_\mu$. When acting on a scalar field the derivative ∂_μ happens to be commuting and thus behaves as the ordinary partial derivative. When acting on a vector field, ∂_μ is a *non commuting operator*. In this respect, there can be no confusion with partial derivative, because the latter normally acts on *scalar fields*, and in such a case partial derivative and ∂_μ are one and the same thing. However, when acting on a vector field, the derivative ∂_μ is non commuting. Our operator ∂_μ when acting on γ_μ or γ^μ should be distinguished from the ordinary — *commuting* — partial derivative, let be denoted $\gamma^\nu{}_{,\mu}$, usually used in the literature on the Dirac equation in curved spacetime. The latter derivative is not used in the present paper, so there should be no confusion.

Using (55), eq.-(54) becomes

$$\partial a = \gamma^\mu \gamma_\nu (\partial_\mu a^\nu + \Gamma^\nu_{\mu\alpha} a^\alpha) \equiv \gamma^\mu \gamma_\nu \mathrm{D}_\mu a^\nu = \gamma^\mu \gamma^\nu \mathrm{D}_\mu a_\nu \tag{57}$$

where D_μ is the covariant derivative of tensor analysis.

Decomposing the Clifford product $\gamma^\mu \gamma^\nu$ into its symmetric and antisymmetric part [1]

$$\gamma^\mu \gamma^\nu = \gamma^\mu \cdot \gamma^\nu + \gamma^\mu \wedge \gamma^\nu\,, \tag{58}$$

where

$$\gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu} \tag{59}$$

is the *inner product* and

$$\gamma^\mu \wedge \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{60}$$

the *outer product*, we can write eq.-(57) as

$$\begin{aligned} \partial a &= g^{\mu\nu}\,\mathrm{D}_\mu a_\nu + \gamma^\mu \wedge \gamma^\nu \mathrm{D}_\mu a_\nu = \\ &= \mathrm{D}_\mu a^\mu + \frac{1}{2}\gamma^\mu \wedge \gamma^\nu (\mathrm{D}_\mu a_\nu - \mathrm{D}_\nu a_\mu)\,. \end{aligned} \tag{61}$$

Without employing the expansion in terms of γ_μ we have simply

$$\partial a = \partial \cdot a + \partial \wedge a\,. \tag{62}$$

Acting twice on a vector by the operator ∂ we have*

$$\begin{aligned} \partial\partial a &= \gamma^\mu \partial_\mu (\gamma^\nu \partial_\nu)(a^\alpha \gamma_\alpha) = \gamma^\mu \gamma^\nu \gamma_\alpha \mathrm{D}_\mu \mathrm{D}_\nu a^\alpha = \\ &= \gamma_\alpha \mathrm{D}_\mu \mathrm{D}^\mu a^\alpha + \frac{1}{2}(\gamma^\mu \wedge \gamma^\nu)\gamma_\alpha [\mathrm{D}_\mu, \mathrm{D}_\nu] a^\alpha = \\ &= \gamma_\alpha \mathrm{D}_\mu \mathrm{D}^\mu a^\alpha + \gamma^\mu (R_{\mu\rho} a^\rho + K_{\mu\alpha}{}^\rho \mathrm{D}_\rho a^\alpha) + \\ &+ \frac{1}{2}(\gamma^\mu \wedge \gamma^\nu \wedge \gamma_\alpha)(R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha)\,. \end{aligned} \tag{63}$$

We have used

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] a^\alpha = R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha\,, \tag{64}$$

where

$$K_{\mu\nu}{}^\rho = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \tag{65}$$

is *torsion* and $R_{\mu\nu\rho}{}^\alpha$ the *curvature tensor*. Using eq.-(55) we find

$$[\partial_\alpha, \partial_\beta]\gamma_\mu = R_{\alpha\beta\mu}{}^\nu \gamma_\nu\,, \tag{66}$$

from which we have

$$R_{\alpha\beta\mu}{}^\nu = ([[\partial_\alpha, \partial_\beta]\gamma_\mu) \cdot \gamma^\nu\,. \tag{67}$$

Thus in general the commutator of derivatives ∂_μ acting on a vector does not give zero, but is given by the curvature tensor.

In general, for an r-vector $A = a^{\alpha_1 \ldots \alpha_r}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}$ we have

$$\begin{aligned} &\partial\partial \ldots \partial A = (\gamma^{\mu_1}\partial_{\mu_1})(\gamma^{\mu_2}\partial_{\mu_2})\ldots(\gamma^{\mu_k}\partial_{\mu_k}) \times \\ &\times (a^{\alpha_1\ldots\alpha_r}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}) = \gamma^{\mu_1}\gamma^{\mu_2}\ldots \\ &\ldots \gamma^{\mu_k}\gamma_{\alpha_1}\gamma_{\alpha_2}\ldots\gamma_{\alpha_r}\mathrm{D}_{\mu_1}\mathrm{D}_{\mu_2}\ldots\mathrm{D}_{\mu_k} a^{\alpha_1\ldots\alpha_r}\,. \end{aligned} \tag{68}$$

*We use $(a \wedge b)\,c = (a \wedge b) \cdot c + a \wedge b \wedge c$ [1] and also $(a \wedge b) \cdot c = (b \cdot c)\,a - (a \cdot c)\,b$.

4.1.2 Clifford algebra based geometric calculus and resolution of the ordering ambiguity for the product of momentum operators

Clifford algebra is a very useful tool for description of geometry of curved space. Moreover, as shown in ref. [26] it provides a resolution of the long standing problem of the ordering ambiguity of quantum mechanics in curved space. Namely, eq.-(52) for the vector derivative suggests that the momentum operator is given by

$$p = -i\,\partial = -i\,\gamma^\mu \partial_\mu\,. \tag{69}$$

One can consider three distinct models:

(i) *The non relativistic particle* moving in $n$dimensional curved space. Then, $\mu = 1, 2, \ldots, n$, and signature is $(+ + + + \ldots)$;

(ii) *The relativistic particle* in curved spacetime, described by the *Schild action* [37]. Then, $\mu = 0, 1, 2, \ldots, n-1$ and signature is $(+ - - - \ldots)$;

(iii) *The Stueckelberg unconstrained particle* [33, 34, 35, 29].

In all three cases the classical action has the form

$$I[X^\mu] = \frac{1}{2\Lambda}\int \mathrm{d}\tau\, g_{\mu\nu}(x)\dot{X}^\mu \dot{X}^\nu \tag{70}$$

and the corresponding Hamiltonian is

$$H = \frac{\Lambda}{2}\, g^{\mu\nu}(x) p_\mu p_\nu = \frac{\Lambda}{2}\, p^2\,. \tag{71}$$

If, upon quantization we take for the momentum operator $p_\mu = -i\,\partial_\mu$, then the ambiguity arises of how to write the quantum Hamilton operator. The problem occurs because the expressions $g^{\mu\nu}p_\mu p_\nu$, $p_\mu g^{\mu\nu} p_\nu$ and $p_\mu p_\nu g^{\mu\nu}$ are not equivalent.

But, if we rewrite H as

$$H = \frac{\Lambda}{2}\, p^2\,, \tag{72}$$

where $p = \gamma^\mu p_\mu$ is the *momentum vector* which upon quantization becomes the momentum vector operator (69), we find that there is no ambiguity in writing the square p^2. When acting with H on a *scalar* wave function ϕ we obtain the unambiguous expression

$$H\phi = \frac{\Lambda}{2}p^2\phi = \frac{\Lambda}{2}(-i)^2(\gamma^\mu\partial_\mu)(\gamma^\nu\partial_\nu)\phi = -\frac{\Lambda}{2}\mathrm{D}_\mu \mathrm{D}^\mu \phi \tag{73}$$

in which there is no curvature term R. We expect that a term with R will arise upon acting with H on a *spinor* field ψ.

4.2 C-space

Let us now consider C-space and review the procedure of ref. [20]. A basis in C-space is given by

$$E_A = \{\gamma, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, \ldots\}\,, \tag{74}$$

where in an r-vector $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r}$ we take the indices so that $\mu_1 < \mu_2 < \ldots < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X = X^A E_A = s\,\gamma + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \ldots \,. \tag{75}$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, etc., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops, etc., onto the coordinate planes. By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string.

In order to avoid using the powers of the Planck scale length parameter L in the expansion of the polyvector X we use the dilatationally invariant units [15] in which L is set to 1. The dilation invariant physics was discussed from a different perspective also in refs. [23, 21].

In a flat C-space the basis vectors E^A are constants. In a curved C-space this is no longer true. Each E_A is a function of the C-space coordinates

$$X^A = \{s, x^\mu, x^{\mu\nu}, \ldots\} \tag{76}$$

which include scalar, vector, bivector, . . . , r-vector, . . . , coordinates.

Now we define the connection $\tilde{\Gamma}^C_{AB}$ in C-space according to

$$\partial_A E_B = \tilde{\Gamma}^C_{AB} E_C \,, \tag{77}$$

where $\partial_A \equiv \partial/\partial X^A$ is the derivative in C-space. This definition is analogous to the one in ordinary space. Let us therefore define the C-space curvature as

$$\mathcal{R}_{ABC}{}^D = ([\partial_A, \partial_B] E_C) * E^D \,, \tag{78}$$

which is a straightforward generalization of the relation (67). The "star" means the *scalar product* between two polyvectors A and B, defined as

$$A * B = \langle A\,B \rangle_S \,, \tag{79}$$

where "S" means "the scalar part" of the geometric product AB.

In the following we shall explore the above relation for curvature and see how it is related to the curvature of the ordinary space. Before doing that we shall demonstrate that the derivative with respect to the bivector coordinate $x^{\mu\nu}$ is equal to the commutator of the derivatives with respect to the vector coordinates x^μ.

Returning now to eq.-(77), the differential of a C-space basis vector is given by

$$\mathrm{d}E_A = \frac{\partial E_A}{\partial X^B}\,\mathrm{d}X^B = \Gamma^C_{AB}\,E_C\,\mathrm{d}X^B \,. \tag{80}$$

In particular, for $A = \mu$ and $E_A = \gamma_\mu$ we have

$$\begin{aligned}
\mathrm{d}\gamma_\mu &= \frac{\partial \gamma_\mu}{\partial X^\nu}\mathrm{d}x^\nu + \frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}}\mathrm{d}x^{\alpha\beta} + \ldots = \\
&= \tilde{\Gamma}^A_{\nu\mu} E_A \mathrm{d}x^\nu + \tilde{\Gamma}^A_{[\alpha\beta]\mu} E_A \mathrm{d}x^{\alpha\beta} + \ldots = \\
&= (\tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu}\gamma_\rho \wedge \gamma_\sigma + \ldots)\,\mathrm{d}x^\nu + \\
&+ (\tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu}\gamma_\rho \wedge \gamma_\sigma + \ldots)\,\mathrm{d}x^{\alpha\beta} + \ldots \,.
\end{aligned} \tag{81}$$

We see that the differential $\mathrm{d}\gamma_\mu$ is in general a polyvector, i. e., a Clifford aggregate. In eq.-(81) we have used

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu}\gamma_\rho \wedge \gamma_\sigma + \ldots \,, \tag{82}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu}\gamma_\rho \wedge \gamma_\sigma + \ldots \,. \tag{83}$$

Let us now consider a *restricted* space in which the derivatives of γ_μ with respect to x^ν and $x^{\alpha\beta}$ do not contain higher rank multivectors. Then eqs.-(82), (83) become

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha \,, \tag{84}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho \,. \tag{85}$$

Further we assume that:

(i) The components $\tilde{\Gamma}^\alpha_{\nu\mu}$ of the C-space connection $\tilde{\Gamma}^C_{AB}$ coincide with the connection $\Gamma^\alpha_{\nu\mu}$ of an ordinary space;

(ii) The components $\tilde{\Gamma}^\rho_{[\alpha\beta]\mu}$ of the C-space connection coincide with the curvature tensor $R_{\alpha\beta\mu}{}^\rho$ of an ordinary space.

Hence, eqs.-(84), (85) read

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \Gamma^\alpha_{\nu\mu}\gamma_\alpha \,, \tag{86}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = R_{\alpha\beta\mu}{}^\rho\gamma_\rho \,, \tag{87}$$

and the differential (81) becomes

$$\mathrm{d}\gamma_\mu = \left(\Gamma^\rho_{\alpha\mu}\mathrm{d}x^\alpha + \frac{1}{2}R_{\alpha\beta\mu}{}^\rho\mathrm{d}x^{\alpha\beta}\right)\gamma_\rho \,. \tag{88}$$

The same relation was obtained by Pezzaglia [14] by using a different method, namely by considering how polyvectors change with position. The above relation demonstrates that a geodesic in C-space is not a geodesic in ordinary spacetime. Namely, in ordinary spacetime we obtain Papapetrou's equation. This was previously pointed out by Pezzaglia [14].

Although a C-space connection does not transform like a C-space tensor, some of its components, i. e., those of eq.-(85), may have the transformation properties of a tensor in an ordinary space.

Under a general coordinate transformation in C-space

$$X^A \to X'^A = X'^A(X^B) \tag{89}$$

the connection transforms according to*

$$\tilde{\Gamma}'^C_{AB} = \frac{\partial X'^C}{\partial X^E}\frac{\partial X^J}{\partial X'^A}\frac{\partial X^K}{\partial X'^B}\tilde{\Gamma}^E_{JK} + \frac{\partial X'^C}{\partial X^J}\frac{\partial^2 X^J}{\partial X'^A \partial X'^B}. \tag{90}$$

In particular, the components which contain the bivector index $A = [\alpha\beta]$ transform as

$$\tilde{\Gamma}'^\rho_{[\alpha\beta]\mu} = \frac{\partial X'^\rho}{\partial X^E}\frac{\partial X^J}{\partial \sigma'^{\alpha\beta}}\frac{\partial X^K}{\partial x'^\mu}\tilde{\Gamma}^E_{JK} + \frac{\partial x'^\rho}{\partial X^J}\frac{\partial^2 X^J}{\partial \sigma'^{\alpha\beta}\partial x'^\mu}. \tag{91}$$

Let us now consider a particular class of coordinate transformations in C-space such that

$$\frac{\partial x'^\rho}{\partial x^{\mu\nu}} = 0\,, \qquad \frac{\partial x^{\mu\nu}}{\partial x'^\alpha} = 0\,. \tag{92}$$

Then the second term in eq.-(91) vanishes and the transformation becomes

$$\tilde{\Gamma}'^\rho_{[\alpha\beta]\mu} = \frac{\partial X'^\rho}{\partial x^\epsilon}\frac{\partial x^{\rho\sigma}}{\partial \sigma'^{\alpha\beta}}\frac{\partial x^\gamma}{\partial x'^\mu}\tilde{\Gamma}^\epsilon_{[\rho\sigma]\gamma}\,. \tag{93}$$

Now, for the bivector whose components are $\mathrm{d}x^{\alpha\beta}$ we have

$$\mathrm{d}\sigma'^{\alpha\beta}\gamma'_\alpha \wedge \gamma'_\beta = \mathrm{d}x^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta\,. \tag{94}$$

Taking into account that in our particular case (92) γ_α transforms as a basis vector in an ordinary space

$$\gamma'_\alpha = \frac{\partial x^\mu}{\partial x'^\alpha}\gamma_\mu\,, \tag{95}$$

we find that (94) and (95) imply

$$\mathrm{d}\sigma'^{\alpha\beta}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} = \mathrm{d}x^{\mu\nu}\,, \tag{96}$$

which means that

$$\frac{\partial x^{\mu\nu}}{\partial \sigma'^{\alpha\beta}} = \frac{1}{2}\left(\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} - \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x^\mu}{\partial x'^\beta}\right) \equiv \frac{\partial x^{[\mu}}{\partial x'^\alpha}\frac{\partial x^{\nu]}}{\partial x'^\beta}. \tag{97}$$

The transformation of the bivector coordinate $x^{\mu\nu}$ is thus determined by the transformation of the vector coordinates x^μ. This is so because the basis bivectors are the wedge products of basis vectors γ_μ.

From (93) and (97) we see that $\tilde{\Gamma}^\epsilon_{[\rho\sigma]\gamma}$ transforms like a 4th-rank tensor in an ordinary space.

Comparing eq.-(87) with the relation (66) we find

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = [\partial_\alpha, \partial_\beta]\gamma_\mu\,. \tag{98}$$

The derivative of a basis vector with respect to the bivector coordinates $x^{\alpha\beta}$ is equal to the commutator of the derivatives with respect to the vector coordinates x^α.

The above relation (98) holds for the basis vectors γ_μ. For an arbitrary polyvector

$$A = A^A E_A = s\gamma + a^\alpha\gamma_\alpha + a^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta + \ldots \tag{99}$$

we will assume the validity of the following relation

$$\frac{\mathrm{D}A^A}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]A^A\,, \tag{100}$$

where $\mathrm{D}/\mathrm{D}x^{\mu\nu}$ is the covariant derivative, defined in analogous way as in eqs. (57):

$$\frac{\mathrm{D}A^A}{\mathrm{D}X^B} = \frac{\partial A^A}{\partial X^B} + \tilde{\Gamma}^A_{BC}A^C\,. \tag{101}$$

From eq.-(100) we obtain

$$\frac{\mathrm{D}s}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]s = K_{\mu\nu}{}^\rho\partial_\rho s\,, \tag{102}$$

$$\frac{\mathrm{D}a^\alpha}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]a^\alpha = R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha\,. \tag{103}$$

Using (101) we have that

$$\frac{\mathrm{D}s}{\mathrm{D}x^{\mu\nu}} = \frac{\partial s}{\partial x^{\mu\nu}} \tag{104}$$

and also follows

$$\frac{\mathrm{D}a^\alpha}{\mathrm{D}x^{\mu\nu}} = \frac{\partial a^\alpha}{\partial x^{\mu\nu}} + \tilde{\Gamma}^\alpha_{[\mu\nu]\rho}a^\rho = \frac{\partial a^\alpha}{\partial x^{\mu\nu}} + R_{\mu\nu\rho}{}^\alpha a^\rho\,, \tag{105}$$

where, according to (ii), $\tilde{\Gamma}^\alpha_{[\mu\nu]\rho}$ has been identified with curvature. So we obtain, after inserting (104), (105) into (102), (103) that:

(a) The partial derivatives of the coefficients s and a^α, which are Clifford scalars†, with respect to $x^{\mu\nu}$ are related to *torsion*:

$$\frac{\partial s}{\partial x^{\mu\nu}} = K_{\mu\nu}{}^\rho\partial_\rho s\,, \tag{106}$$

$$\frac{\partial a^\alpha}{\partial x^{\mu\nu}} = K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha\,; \tag{107}$$

(b) Whilst the derivative of the basis vectors with respect to $x^{\mu\nu}$ are related to *curvature*:

$$\frac{\partial \gamma_\alpha}{\partial x^{\mu\nu}} = R_{\mu\nu\alpha}{}^\beta\gamma_\beta\,. \tag{108}$$

In other words, the dependence of coefficients s and a^α on $x^{\mu\nu}$ indicates the presence of torsion. On the contrary, when basis vectors γ_α depend on $x^{\mu\nu}$ this indicates that the corresponding vector space has non vanishing curvature.

*This can be derived from the relation $\mathrm{d}E'_A = \dfrac{\partial E'_A}{\partial X'^B}\,\mathrm{d}X'^B$, where $E'_A = \dfrac{\partial X^D}{\partial X'^A}E_D$ and $\mathrm{d}X'^B = \dfrac{\partial X'^B}{\partial X^C}\mathrm{d}X^C$.

†In the geometric calculus based on Clifford algebra, the coefficients such as s, a^α, $a^{\alpha\beta}$,..., are called *scalars* (although in tensor calculus they are called scalars, vectors and tensors, respectively), whilst the objects γ_α, $\gamma_\alpha \wedge \gamma_\beta, \ldots$, are called *vectors, bivectors*, etc.

4.3 On the relation between the curvature of C-space and the curvature of an ordinary space

Let us now consider the C-space curvature defined in eq.-(78). The indices A,B, can be of vector, bivector, etc., type. It is instructive to consider a particular example.

$A = [\mu\nu]$, $B = [\alpha\beta]$, $C = \gamma$, $D = \delta$

$$\left(\left[\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\alpha\beta}}\right]\gamma_\gamma\right)\cdot\gamma^\delta = \mathcal{R}_{[\mu\nu][\alpha\beta]\gamma}{}^\delta . \tag{109}$$

Using (87) we have

$$\frac{\partial}{\partial x^{\mu\nu}}\frac{\partial}{\partial x^{\alpha\beta}}\gamma_\gamma = \frac{\partial}{\partial x^{\mu\nu}}(R_{\alpha\beta\gamma}{}^\rho\gamma_\rho) = R_{\alpha\beta\gamma}{}^\rho R_{\mu\nu\rho}{}^\sigma\gamma_\sigma \tag{110}$$

where we have taken

$$\frac{\partial}{\partial x^{\mu\nu}}R_{\alpha\beta\gamma}{}^\rho = 0 , \tag{111}$$

which is true in the case of vanishing torsion (see also an explanation that follows after the next paragraph). Inserting (110) into (109) we find

$$\mathcal{R}_{[\mu\nu][\alpha\beta]\gamma}{}^\delta = R_{\mu\nu\gamma}{}^\rho R_{\alpha\beta\rho}{}^\delta - R_{\alpha\beta\gamma}{}^\rho R_{\mu\nu\rho}{}^\delta , \tag{112}$$

which is the product of two usual curvature tensors. We can proceed in analogous way to calculate the other components of $\mathcal{R}_{ABC}{}^D$ such as $\mathcal{R}_{[\alpha\beta\gamma\delta][\rho\sigma]\epsilon}{}^\mu$, $\mathcal{R}_{[\alpha\beta\gamma\delta][\rho\sigma\tau\kappa]\epsilon}{}^{[\mu\nu]}$, etc. These contain higher powers of the curvature in an ordinary space. All this is true in our restricted C-space given by eqs.-(84), (85) and the assumptions (i), (ii) bellow those equations. By releasing those restrictions we would have arrived at an even more involved situation which is beyond the scope of the present paper.

After performing the contractions of (112) and the corresponding higher order relations we obtain the expansion of the form

$$\mathcal{R} = R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \dots . \tag{113}$$

So we have shown that the C-space curvature can be expressed as the sum of the products of the ordinary spacetime curvature. This bears a resemblance to the string effective action in curved spacetimes given by sums of powers of the curvature tensors based on the quantization of non-linear sigma models [118].

If one sets aside the algebraic convergence problems when working with Clifford algebras in infinite dimensions, one can consider the possibility of studying Quantum Gravity in a very large number of dimensions which has been revisited recently [83] in connection to a perturbative renormalizable quantum theory of gravity in infinite dimensions. Another interesting possibility is that an infinite series expansion of the powers of the scalar curvature could yield the recently proposed modified Lagrangians $R + 1/R$ of gravity to accommodate the cosmological accelerated expansion of the Universe [131], after a judicious choice of the algebraic coefficients is taken. One may notice also that having a vanishing cosmological constant in C-space, $\mathcal{R} = \Lambda = 0$ does not necessarily imply that one has a vanishing cosmological constant in ordinary spacetime. For example, in the very special case of homogeneous symmetric spacetimes, like spheres and hyperboloids, where all the curvature tensors are proportional to suitable combinations of the metric tensor times the scalar curvature, it is possible to envision that the net combination of the sum of all the powers of the curvature tensors may cancel-out giving an overall zero value $\mathcal{R} = 0$. This possibility deserves investigation.

Let us now show that for vanishing torsion the curvature is independent of the bivector coordinates $x^{\mu\nu}$, as it was taken in eq.-(111). Consider the basic relation

$$\gamma_\mu\cdot\gamma_\nu = g_{\mu\nu} . \tag{114}$$

Differentiating with respect to $x^{\alpha\beta}$ we have

$$\frac{\partial}{\partial x^{\alpha\beta}}(\gamma_\mu\cdot\gamma_\nu) = \frac{\partial\gamma_\mu}{\partial x^{\alpha\beta}}\cdot\gamma_\nu + \gamma_\mu\cdot\frac{\partial\gamma_\nu}{\partial x^{\alpha\beta}} = = R_{\alpha\beta\mu\nu} + R_{\alpha\beta\nu\mu} = 0 . \tag{115}$$

This implies that

$$\frac{\partial g_{\mu\nu}}{\partial\sigma_{\alpha\beta}} = [\partial_\alpha, \partial_\beta]g_{\mu\nu} = 0 . \tag{116}$$

Hence the metric, in this particular case, is independent of the holographic (bivector) coordinates. Since the curvature tensor — when torsion is zero — can be written in terms of the metric tensor and its derivatives, we conclude that not only the metric, but also the curvature is independent of $x^{\mu\nu}$. In general, when the metric has a dependence on the holographic coordinates one expects further corrections to eq.-(112) that would include torsion.

5 On the quantization in C-spaces

5.1 The momentum constraint in C-space

A detailed discussion of the physical properties of all the components of the polymomentum P in four dimensions and the emergence of the physical mass in Minkowski spacetime has been provided in the book [15]. The polymomentum in $D = 4$, canonically conjugate to the position polyvector

$$X = \sigma + x^\mu\gamma_\mu + \gamma^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + \xi^\mu\gamma_5\gamma_\mu + s\gamma_5 \tag{117}$$

can be written as:

$$P = \mu + p^\mu\gamma_\mu + S^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + \pi^\mu\gamma_5\gamma_\mu + m\gamma_5 , \tag{118}$$

where besides the vector components p^μ we have the scalar component μ, the 2-vector components $S^{\mu\nu}$, that are connected to the spin as shown by [14]; the pseudovector components π^μ and the pseudoscalar component m.

The most salient feature of the polyparticle dynamics in C-spaces [15] is that one can start with a *constrained* action in C-space and arrive, nevertheless, at an *unconstrained* Stuckelberg action in Minkowski space (a subspace of C-space) in which $p_\mu p^\mu$ is a constant of motion. The true constraint in C-space is:

$$P_A P^A = \mu^2 + p_\mu p^\mu - 2S^{\mu\nu} S_{\mu\nu} + \pi_\mu \pi^\mu - m^2 = M^2\,, \quad (119)$$

where M is a *fixed* constant, the mass in C-space. The pseudoscalar component m is a variable, like μ, p_μ, $S^{\mu\nu}$, and π^μ, which altogether are constrained according to eq.-(119). It becomes the physical mass in Minkowski spacetime in the special case when other extra components vanish, i. e., when $\mu = 0$, $S^{\mu\nu} = 0$ and $\pi^\mu = 0$. This justifies using the notation m for mass. This is basically the distinction between the mass in Minkowski space which is a constant of motion $p_\mu p^\mu$ and the fixed mass M in C-space. The variable m is canonically conjugate to s which acquires the role of the Stuckelberg evolution parameter s that allowed ref. [29, 15] to propose a natural solution of the problem of time in quantum gravity. The polyparticle dynamics in C-space is a generalization of the relativistic Regge top construction which has recently been studied in de Sitter spaces by [135].

A derivation of a charge, mass, and spin relationship of a polyparticle can be obtained from the above polymomentum constraint in C-space if one relates the norm of the axial-momentum component π^μ of the polymomentum P to the charge [80]. It agrees exactly with the recent charge-mass-spin relationship obtained by [44] based on the Kerr-Newman black hole metric solutions of the Einstein-Maxwell equations. The naked singularity Kerr-Newman solutions have been interpreted by [45] as Dirac particles. Further investigation is needed to understand better these relationships, in particular, the deep reasons behind the *charge* assignment to the norm of the axial-vector π^μ component of the polymomentum which suggests that mass has a gravitational, electromagnetic and rotational aspects to it. In a Kaluza-Klein reduction from $D = 5$ to $D = 4$ it is well known that the electric charge is related to the p_5 component of the momentum. Hence, charge bears a connection to an internal momentum.

5.2 C-space Klein-Gordon and Dirac wave equations

The ordinary Klein-Gordon equation can be easily obtained by implementing the on-shell constraint $p^2 - m^2 = 0$ as an operator constraint on the physical states after replacing p_μ for $-i\partial/\partial x^\mu$ (we use units in which $\hbar = 1, c = 1$):

$$\left(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + m^2 \right) \phi = 0. \quad (120)$$

The C-space generalization follows from the $P^2 - M^2 = 0$ condition by replacing

$$P_A \to -i \frac{\partial}{\partial X^A} = -i \left(\frac{\partial}{\partial \sigma}, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^{\mu\nu}}, \dots \right), \quad (121)$$

$$\left(\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial x^\mu \partial x_\mu} + \frac{\partial^2}{\partial x^{\mu\nu} \partial x_{\mu\nu}} + \ldots + M^2 \right) \Phi = 0, \quad (122)$$

where we have set $L = \hbar = c = 1$ for convenience purposes and the C-space scalar field $\Phi\,(\sigma, x^\mu, x^{\mu\nu}, \dots)$ is a polyvector-valued *scalar* function of *all* the C-space variables. This is the Klein-Gordon equation associated with a free scalar polyparticle in C-space.

A wave equation for a generalized C-space harmonic oscillator requires to introduce the potential of the form $V = \kappa X^2$ that admits straightforward solutions in terms of Gaussians and Hermite polynomials similar to the ordinary point-particle oscillator. There are now collective excitations of the Clifford-oscillator in terms of the number of Clifford-bits and which represent the quanta of areas, volumes, hyper-volumes, ..., associated with the p-loops oscillations in Planck scale units. The logarithm of the degeneracy of the first collective state of the C-space oscillator, as a function of the number of bits, bears the same functional form as the Bekenstein-Hawking black hole entropy, with the upshot that one recovers, in a natural way, the logarithmic corrections to the black-hole entropy as well, if one identifies the number of Clifford-bits with the number of area-quanta of the black hole horizon. For further details about this derivation and the emergence of the Schwarzschild horizon radius relation, the Hawking temperature, the maximal Planck temperature condition, etc., we refer to [21]. Perhaps the most important consequence of this latter view of black hole entropy is the possibility that there is a ground state of quantum spacetime, resulting from of a Bose-Einstein condensate of the C-space harmonic oscillator.

A C-space version of the Dirac Equation, representing the dynamics of spinning-polyparticles (theories of extended-spin, extended charges) is obtained via the square-root procedure of the Klein-Gordon equation:

$$-i \left(\frac{\partial}{\partial \sigma} + \gamma^\mu \frac{\partial}{\partial x_\mu} + \gamma^\mu \wedge \gamma^\nu \frac{\partial}{\partial x_{\mu\nu}} + \ldots \right) \Psi = M \Psi\,, \quad (123)$$

where $\Psi(\sigma, x^\mu, x^{\mu\nu}, \dots)$ is a polyvector-valued function, a Clifford-number, $\Psi = \Psi^A E_A$ of *all* the C-space variables. For simplicity we consider here a *flat* C-space in which the metric $G_{AB} = E_A^\dagger * E_B = \eta_{AB}$ is diagonal, η_{AB} being the C-space analog of Minkowski tensor. In curved C-space the equation (123) should be properly generalized. This goes beyond the scope of the present paper.

Ordinary spinors are nothing but elements of the left/right ideals of a Clifford algebra. So they are automatically contained in the polyvector valued wave function Ψ. The ordinary Dirac equation can be obtained when Ψ is independent of the extra variables associated with a polyvector-valued coordinates X (i. e., of $x^{\mu\nu}$, $x^{\mu\nu\rho}, \dots$). For details see [15].

Thus far we have written ordinary wave equations in C-space, that is, we considered the wave equations for a "point particle" in C-space. From the perspective of the 4-dimensional Minkowski spacetime the latter "point particle" has, of course, a much richer structure then a mere point: it is an extended object, modeled by coordinates x^{μ}, $x^{\mu\nu}, \dots$ But such modeling does not embrace all the details of an extended object. In order to provide a description with more details, one can considere not the "point particles" in C-space, but *branes* in C-space. They are described by the embeddings $X = X(\Sigma)$, that is $X^M = X^M(\Sigma^A)$, considered in sec. 3.2. Quantization of such branes can employ wave functional equation, or other methods, including the second quantization formalism. For a more detailed study detailed study of the second quantization of extended objects using the tools of Clifford algebra see [15].

Without emplying Clifford algebra a lot of illuminating work has been done in relation to description of branes in terms of p-loop coordinates [132]. A bosonic/fermionic p-brane wave-functional equation was presented in [12], generalizing the closed-string (loop) results in [13] and the the quantum bosonic p-brane propagator, in the quenched-reduced minisuperspace approximation, was attained by [18]. In the latter work branes are described in terms of the collective coordinates which are just the highest grade components in the expansion of a poplyvector X given in eq.-(2). This work thus paved the way for the next logical step, that is, to consider other multivector components of X in a unified description of all branes.

Notice that the approach based on eqs.-(122), (123) is different from that by Hestenes [1] who proposed an equation which is known as the Dirac-Hestenes equation. Dirac's equation using quaternions (related to Clifford algebras) was first derived by Lanczos [91]. Later on the Dirac-Lanczos equation was rediscovered by many people, in particular by Hestenes and Gursey [92] in what became known as the Dirac-Hestenes equation. The former Dirac-Lanczos equation is Lorentz *covariant* despite the fact that it singles out an arbitrary but unique direction in ordinary space: the *spin* quantization axis. Lanczos, without knowing, had anticipated the existence of isospin as well. The Dirac-Hestenes equation $\partial\Psi e_{21} = m\Psi e_0$ is *covariant* under a change of frame [133], [93]. $e'_{\mu} = U e_{\mu} U^{-1}$ and $\Psi' = \Psi U^{-1}$ with U an element of the $Spin_+(1,3)$ yielding $\partial\Psi' e'_{21} = m\Psi' e'_0$. As Lanczos had anticipated, in a new frame of reference, the spin quantization axis is also rotated appropriately, thus there is no breakdown of covariance by introducing bivectors in the Dirac-Hestenes equation.

However, subtleties still remain. In the Dirac-Hestenes equation instead of the imaginary unit i there occurs the bivector $\gamma_1\gamma - 2$. Its square is -1 and commutes with all the elements of the Dirac algebra which is just a desired property. But on the other hand, the introduction of a bivector into an equation implies a selection of a preferred orientation in spacetime; i. e. the choice of the spin quantization axis in the original Dirac-Lanczos quaternionic equation. How is such preferred orientation (spin quantization axis) determined? Is there some dynamical symmetry which determines the preferred orientation (spin quantization axis)? is there an action which encodes a hidden dynamical principle that selects *dynamically* a preferred spacetime orientation (spin quantization axis)?

Many subtleties of the Dirac-Hesteness equation and its relation to the ordinary Dirac equation and the Seiberg-Witten equation are investigated from the rigorous mathematical point of view in refs. [93]. The approach in refs. [16, 15, 17, 8], reviewed here, is different. We start from the usual formulation of quantum theory and extend it to C-space. We retain the imaginary unit i. Next step is to give a geometric interpretation to i. Instead of trying to find a geometric origin of i in *spacetime* we adopt the interpretation proposed in [15] according to which the i is the bivector of the 2-dimensional *phase space* (whose direct product with the n-dimensional configuration space gives the $2n$-dimensional phase space)*. This appears to be a natural assumption due to the fact that complex valued quantum mechanical wave functions involve momenta p_{μ} and coordinates x^{μ} (e. g., a plane wave is given by $\exp[ip_{\mu}x^{\mu}]$, and arbitrary wave packet is a superposition of plane waves).

6 Maximal-acceleration Relativity in phase-spaces

In this section we shall discuss the maximal acceleration Relativity principle [68] based on Finsler geometry which does not destroy, nor deform, Lorentz invariance. Our discussion differs from the pseudo-complex Lorentz group description by Schuller [61] related to the effects of maximal acceleration in Born-Infeld models that also maintains Lorentz invariance, in contrast to the approaches of Double Special Relativity (DSR). In addition one does not need to modify the energy-momentum addition (conservation) laws in the scattering of particles which break translational invariance. For a discussions on the open problems of Double Special Relativity theories based on kappa-deformed Poincaré symmetries [63] and motivated by the anomalous Lorentz-violating dispersion relations in the ultra high energy cosmic rays [71, 72, 73], we refer to [70].

Related to the minimal Planck scale, an upper limit on the maximal acceleration principle in Nature was proposed by long ago Cainello [52]. This idea is a direct consequence of a suggestion made years earlier by Max Born on a Dual Relativity principle operating in phase spaces [49], [74] wherethere

*Yet another interpretation of the imaginary unit i present in the Heisenberg uncertainty relations has been undertaken by Finkelstein and collaborators [96].

is an upper bound on the four-force (maximal string tension or tidal forces in the string case) acting on a particle as well as an upper bound in the particle velocity. One can combine the maximum speed of light with a minimum Planck scale into a maximal proper-acceleration $a = c^2/L$ within the framework of Finsler geometry [56]. For a recent status of the geometries behind maximal-acceleration see [73]; its relation to the Double Special Relativity programs was studied by [55] and the possibility that Moyal deformations of Poincaré algebras could be related to the kappa-deformed Poincaré algebras was raised in [68]. A thorough study of Finsler geometry and Clifford algebras has been undertaken by Vacaru [81] where Clifford/spinor structures were defined with respect to Nonlinear connections associated with certain nonholonomic modifications of Riemann-Cartan gravity.

Other several new physical implications of the maximal acceleration principle in Nature, like neutrino oscillations and other phenomena, have been studied by [54], [67], [42]. Recently, the variations of the fine structure constant α [64], with the cosmological accelerated expansion of the Universe, was recast as a renormalization group-like equation governing the cosmological red shift (Universe scale) variations of α based on this maximal acceleration principle in Nature [68]. The fine structure constant was smaller in the past. Pushing the cutoff scale to the minimum Planck scale led to the intriguing result that the fine structure constant could have been extremely small (zero) in the early Universe and that all matter in the Universe could have emerged via the Unruh-Rindler-Hawking effect (creation of radiation/matter) due to the acceleration w. r. t the vacuum frame of reference. For reviews on the alleged variations of the fundamental constants in Nature see [65] and for more astonishing variations of αdriven by quintessence see [66].

6.1 Clifford algebras in phase space

We shall employ the procedure described in [15] to construct the Phase Space Clifford algebra that allowed [127] to reproduce the sub-maximally accelerated particle action of [53].

For simplicity we will focus on a two-dim phase space. Let e_p, e_q be the Clifford-algebra basis elements in a two-dim phase space obeying the following relations [15]:

$$e_p \cdot e_q \equiv \frac{1}{2}(e_q e_p + e_p e_q) = 0 \tag{124}$$

and $e_p e_p = e_q e_q = 1$.

The Clifford product of e_p, e_q is by definition the sum of the scalar and the wedge product:

$$e_p e_q = e_p \cdot e_q + e_p \wedge e_q = 0 + e_p \wedge e_q = i\,, \tag{125}$$

such that $i^2 = e_p e_q e_p e_q = -1$. Hence, the imaginary unit i, $i^2 = -1$ admits a very natural interpretation in terms of Clifford algebras, i. e., it is represented by the wedge product $i = e_p \wedge e_q$, a phase-space area element. Such imaginary unit allows us to express vectors in a C-phase space in the form:

$$\begin{aligned} &Q = q e_q + p e_q\,, \\ &Q \cdot e_q = q + p e_p \cdot e_q = q + ip = z\,, \\ &e_q \cdot Q = q + p e_q \cdot e_p = q - ip = z^*\,, \end{aligned} \tag{126}$$

which reminds us of the creation/annihilation operators used in the harmonic oscillator.

We shall now review the steps in [127] to reproduce the sub-maximally accelerated particle action [53]. The phase-space analog of the spacetime action is:

$$dQdQ = (dq)^2 + (dp)^2 \Rightarrow S = m\int\sqrt{(dq)^2 + (dp)^2}\,. \tag{127}$$

Introducing the appropriate length/mass scale parameters in order to have consistent units yields:

$$S = m\int\sqrt{(dq)^2 + \left(\frac{L}{m}\right)^2 (dp)^2}\,, \tag{128}$$

where we have introduced the Planck scale L and have chosen the natural units $\hbar = c = 1$. A detailed physical discussion of the dilational invariant system of units $\hbar = c = G = 4\pi\epsilon_0 = 1$ was presented in ref. [15]. G is the Newton constant and ϵ_0 is the permittivity of the vacuum.

Extending this two-dim result to a $2n$-dim phase space result requires to have for Clifford basis the elements e_{p_μ}, e_{q_μ}, where $\mu = 1, 2, 3, \ldots n$. The action in the $2n$-dim phase space is:

$$\begin{aligned} S &= m\int\sqrt{(dq^\mu dq_\mu) + \left(\frac{L}{m}\right)^2 (dp^\mu dp_\mu)} = \\ &= m\int d\tau\sqrt{1 + \left(\frac{L}{m}\right)^2 (dp^\mu/d\tau)(dp_\mu/d\tau)}\,, \end{aligned} \tag{129}$$

where we have factored-out of the square-root the infinitesimal proper-time displacement $(d\tau)^2 = dq^\mu dq_\mu$.

One can recognize the action (129), up to a numerical factor of m/a, where a is the proper acceleration, as the same action for a sub-maximally accelerated particle given by Nesterenko [53] by rewriting $(dp^\mu/d\tau) = m(d^2x^\mu/d\tau^2)$:

$$S = m\int d\tau\sqrt{1 + L^2(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}\,. \tag{130}$$

Postulating that the maximal proper-acceleration is given in terms of the speed of light and the minimal Planck scale by $a = c^2/L = 1/L$, the action above gives the Nesterenko action, up to a numerical m/a factor:

$$S = m\int d\tau\sqrt{1 + a^{-2}(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}\,. \tag{131}$$

The proper-acceleration is orthogonal to the proper-velocity and this can be easily verified by differentiating the time-like proper-velocity squared:

$$V^2 = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = V^\mu V_\mu = 1 > 0 \Rightarrow \\ \Rightarrow \frac{dV^\mu}{d\tau}V_\mu = \frac{d^2x^\mu}{d\tau^2}V_\mu = 0\,, \tag{132}$$

which implies that the proper-acceleration is space-like:

$$g^2(\tau) = -\frac{d^2x^\mu}{d\tau^2}\frac{d^2x_\mu}{d\tau^2} > 0 \Rightarrow \\ \Rightarrow S = m\int d\tau\sqrt{1-\frac{g^2}{a^2}} = m\int d\omega\,, \tag{133}$$

where the analog of the Lorentz time-dilation factor for a sub-maximally accelerated particle is given by

$$d\omega = d\tau\sqrt{1-\frac{g^2(\tau)}{a^2}}\,. \tag{134}$$

Therefore the dynamics of a sub-maximally accelerated particle can be reinterpreted as that of a particle moving in the spacetime tangent bundle whose Finsler-like metric is

$$(d\omega)^2 = g_{\mu\nu}(x^\mu, dx^\mu)dx^\mu dx^\nu = (d\tau)^2\left(1-\frac{g^2(\tau)}{a^2}\right). \tag{135}$$

The invariant time now is no longer the standard proper-time τ but is given by the quantity $\omega(\tau)$. The deep connection between the physics of maximal acceleration and Finsler geometry has been analyzed by [56]. This sort of actions involving second derivatives have also been studied in the construction of actions associated with rigid particles (strings) [57], [58], [59], [60] among others.

The action is real-valued if, and only if, $g^2 < a^2$ in the same fashion that the action in Minkowski spacetime is real-valued if, and only if, $v^2 < c^2$. This is the physical reason why there is an upper bound in the proper-acceleration. In the special case of uniformly-accelerated motion $g(\tau) = g_0 =$ = constant, the trajectory of the particle in Minkowski spacetime is a hyperbola.

Most recently, an Extended Relativity Theory in Born-Clifford-Phase spaces with an *upper* and *lower* length scales (infrared/ultraviolet cutoff) has been constructed [138]. The invariance symmetry associated with an $8D$ Phase Space leads naturally to the real Clifford algebra $Cl(2,6,R)$ and complexified Clifford $Cl_C(4)$ algebra related to Twistors. The consequences of Mach's principle of inertia within the context of Born's Dual Phase Space Relativity Principle were also studied in [138] and they were compatible with the Eddington-Dirac large numbers coincidence and with the observed values of the anomalous Galileo-Pioneer acceleration. The modified Newtonian dynamics due to the upper/lower scales and modified Schwarzschild dynamics due the maximal acceleration were also provided.

6.2 Invariance under the $U(1,3)$ Group

In this section we will review in detail the principle of Maximal-acceleration Relativity [68] from the perspective of $8D$ Phase Spaces and the $U(1,3)$ Group. The $U(1,3) =$ $= SU(1,3)\otimes U(1)$ Group transformations, which leave invariant the phase-space intervals under rotations, velocity and acceleration boosts, were found by Low [74] and can be simplified drastically when the velocity/acceleration boosts are taken to lie in the z-direction, leaving the transverse directions x, y, p_x, p_y intact; i. e., the $U(1,1) = SU(1,1)\otimes U(1)$ subgroup transformations that leave invariant the phase-space interval are given by (in units of $\hbar = c = 1$)

$$(d\sigma)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2-(dP)^2}{b^2} = \\ = (d\tau)^2\left[1+\frac{(dE/d\tau)^2-(dP/d\tau)^2}{b^2}\right] = \\ = (d\tau)^2\left[1-\frac{m^2g^2(\tau)}{m_P^2A_{max}^2}\right], \tag{136}$$

where we have factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in eq.-(136) and the maximal proper-force is set to be $b\equiv m_P A_{max}$. m_P is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, may also be interpreted as the maximal string tension when L_P is the Planck scale.

The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass m in the z-direction which we take to be X. Notice that the invariant interval $(d\sigma)^2$ in eq.-(136) is not strictly the *same* as the interval $(d\omega)^2$ of the Nesterenko action eq.-(131), which was invariant under a pseudo-complexification of the Lorentz group [61]. Only when $m = m_P$, the two intervals agree. The interval $(d\sigma)^2$ described by Low [74] is $U(1,3)$-invariant for the most general transformations in the $8D$ phase-space. These transformations are rather elaborate, so we refer to the references [74] for details. The analog of the Lorentz relativistic factor in eq.-(136) involves the ratios of two proper *forces*. One variable force is given by ma and the maximal proper force sustained by an *elementary* particle of mass m_P (a *Planckton*) is assumed to be $F_{max} = m_{Planck}c^2/L_P$. When $m = m_P$, the ratio-squared of the forces appearing in the relativistic factor of eq.-(136) becomes then g^2/A^2_{max}, and the phase space interval (136) coincides with the geometric interval of (131).

The transformations laws of the coordinates in that leave invariant the interval (136) are [74]:

$$T' = T\cosh\xi + \left(\frac{\xi_v X}{c^2} + \frac{\xi_a P}{b^2}\right)\frac{\sinh\xi}{\xi}\,, \tag{137}$$

$$E' = E\cosh\xi + (-\xi_a X + \xi_v P)\frac{\sinh\xi}{\xi}\,, \tag{138}$$

$$X' = X\cosh\xi + \left(\xi_v T - \frac{\xi_a E}{b^2}\right)\frac{\sinh\xi}{\xi}\,, \tag{139}$$

$$P' = P\cosh\xi + \left(\frac{\xi_v E}{c^2} + \xi_a T\right)\frac{\sinh\xi}{\xi}\,. \tag{140}$$

The ξ_v is velocity-boost rapidity parameter and the ξ_a is the force/acceleration-boost rapidity parameter of the primed-reference frame. They are defined respectively (in the special case when $m = m_P$):

$$\tanh\left(\frac{\xi_v}{c}\right) = \frac{v}{c}\,,$$

$$\tanh\frac{\xi_a}{b} = \frac{ma}{m_P A_{max}}\,. \tag{141}$$

The *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs.-(137)–(140) is defined in terms of the velocity and acceleration boosts parameters ξ_v, ξ_a respectively as:

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}}\,. \tag{142}$$

Our definition of the rapidity parameters are *different* than those in [74].

Straightforward algebra allows us to verify that these transformations leave the interval of eq.-(136) in classical phase space invariant. They are are fully consistent with Born's duality Relativity symmetry principle [49] $(Q,P)\rightarrow \rightarrow (P,-Q)$. By inspection we can see that under Born duality, the transformations in eqs.-(137)–(140) are *rotated* into each other, up to numerical b factors in order to match units. When on sets $\xi_a = 0$ in (137)–(140) one recovers automatically the standard Lorentz transformations for the X, T and E, P variables *separately*, leaving invariant the intervals $dT^2 - dX^2 = (d\tau)^2$ and $(dE^2 - dP^2)/b^2$ separately.

When one sets $\xi_v = 0$ we obtain the transformations rules of the events in Phase space, from one reference-frame into another *uniformly*-accelerated frame of reference, $a = \text{const}$, whose acceleration-rapidity parameter is in this particular case:

$$\xi \equiv \frac{\xi_a}{b}\,, \qquad \tanh\xi = \frac{ma}{m_P A_{max}}\,. \tag{143}$$

The transformations for pure acceleration-boosts in are:

$$T' = T\cosh\xi + \frac{P}{b}\sinh\xi\,, \tag{144}$$

$$E' = E\cosh\xi - bX\sinh\xi\,, \tag{145}$$

$$X' = X\cosh\xi - \frac{E}{b}\sinh\xi\,, \tag{146}$$

$$P' = P\cosh\xi + bT\sinh\xi\,. \tag{147}$$

It is straightforward to verify that the transformations (144)–(146) leave invariant the fully phase space interval (136) but *does not* leave invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$. Only the *combination*:

$$(d\sigma)^2 = (d\tau)^2\left(1 - \frac{m^2 g^2}{m_P^2 A_{max}^2}\right) \tag{148}$$

is truly left invariant under pure acceleration-boosts (144)–(146). One can verify as well that these transformations satisfy Born's duality symmetry principle:

$$(T,X)\rightarrow(E,P)\,, \qquad (E,P)\rightarrow(-T,-X) \tag{149}$$

and $b \rightarrow \frac{1}{b}$. The latter Born duality transformation is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i. e. namely, a small/large radius duality, a winding modes/Kaluza-Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in Noncommutative Field Theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory.

The composition of two successive pure acceleration-boosts is another pure acceleration-boost with acceleration rapidity given by $\xi'' = \xi + \xi'$. The addition of *proper* four-forces (accelerations) follows the usual relativistic composition rule:

$$\tanh\xi'' = \tanh(\xi+\xi') = \frac{\tanh\xi + \tanh\xi'}{1 + \tanh\xi\tanh\xi'} \Rightarrow$$
$$\Rightarrow \frac{ma''}{m_P A} = \frac{\frac{ma}{m_P A} + \frac{ma'}{m_P A}}{1 + \frac{m^2 a a'}{m_P^2 A^2}}\,, \tag{150}$$

and in this fashion the upper limiting *proper* acceleration is never *surpassed* like it happens with the ordinary Special Relativistic addition of velocities.

The group properties of the full combination of velocity and acceleration boosts (137)–(140) requires much more algebra [68]. A careful study reveals that the composition *rule* of two succesive full transformations is given by $\xi'' = = \xi + \xi'$ and the transformation laws are *preserved* if, and only if, the ξ ; ξ' ; ξ'' ... parameters obeyed the suitable relations:

$$\frac{\xi_a}{\xi} = \frac{\xi'_a}{\xi'} = \frac{\xi''_a}{\xi''} = \frac{\xi''_a}{\xi+\xi'}\,, \tag{151}$$

$$\frac{\xi_v}{\xi} = \frac{\xi'_v}{\xi'} = \frac{\xi''_v}{\xi''} = \frac{\xi''_v}{\xi+\xi'}\,. \tag{152}$$

Finally we arrive at the composition law for the effective, velocity and acceleration boosts parameters ξ''; ξ''_v; ξ''_a respectively:

$$\xi''_v = \xi_v + \xi'_v\,, \tag{153}$$

$$\xi''_a = \xi_a + \xi'_a\,, \tag{154}$$

$$\xi'' = \xi + \xi'\,. \tag{155}$$

The relations (151, 152, 153, 154, 155) are required in order to prove the *group* composition law of the transformations of (137)–(140) and, consequently, in order to have a truly Maximal-Acceleration Phase Space Relativity theory resulting from a phase-space change of coordinates in the cotangent bundle of spacetime.

6.3 Planck-Scale Areas are invariant under acceleration boosts

Having displayed explicitly the Group transformations rules of the coordinates in Phase space we will show why *infinite* acceleration-boosts (which is *not* the same as infinite proper acceleration) preserve Planck-Scale *Areas* [68] as a result of the fact that $b = (1/L_P^2)$ equals the *maximal* invariant force, or string tension, if the units of $\hbar = c = 1$ are used.

At Planck-scale L_P intervals/increments in one reference frame we have by definition (in units of $\hbar = c = 1$): $\Delta X = \Delta T = L_P$ and $\Delta E = \Delta P = \frac{1}{L_P}$ where $b \equiv \frac{1}{L_P^2}$ is the maximal tension. From eqs.-(137)–(140) we get for the transformation rules of the finite intervals ΔX, ΔT, ΔE, ΔP, from one reference frame into another frame, in the *infinite* acceleration-boost limit $\xi \to \infty$,

$$\Delta T' = L_P(\cosh\xi + \sinh\xi) \to \infty\,, \tag{156}$$

$$\Delta E' = \frac{1}{L_P}(\cosh\xi - \sinh\xi) \to 0 \tag{157}$$

by a simple use of L'Hôpital's rule or by noticing that both $\cosh\xi$; $\sinh\xi$ functions approach infinity at the same rate

$$\Delta X' = L_P(\cosh\xi - \sinh\xi) \to 0\,, \tag{158}$$

$$\Delta P' = \frac{1}{L_P}(\cosh\xi + \sinh\xi) \to \infty\,, \tag{159}$$

where the discrete displacements of two events in Phase Space are defined: $\Delta X = X_2 - X_1 = L_P$, $\Delta E = E_2 - E_1 = \frac{1}{L_P}$, $\Delta T = T_2 - T_1 = L_P$ and $\Delta P = P_2 - P_1 = \frac{1}{L_P}$.

Due to the identity:

$$(\cosh\xi + \sinh\xi)(\cosh\xi - \sinh\xi) = \cosh^2\xi - \sinh^2\xi = 1 \tag{160}$$

one can see from eqs.-(156)–(159) that the Planck-scale *Areas* are truly *invariant* under *infinite* acceleration-boosts $\xi = \infty$:

$$\Delta X'\Delta P' = 0\times\infty = \Delta X\Delta P(\cosh^2\xi - \sinh^2\xi) = \\ = \Delta X\Delta P = \frac{L_P}{L_P} = 1\,, \tag{161}$$

$$\Delta T'\Delta E' = \infty\times 0 = \Delta T\Delta E(\cosh^2\xi - \sinh^2\xi) = \\ = \Delta T\Delta E = \frac{L_P}{L_P} = 1\,, \tag{162}$$

$$\Delta X'\Delta T' = 0\times\infty = \Delta X\Delta T(\cosh^2\xi - \sinh^2\xi) = \\ = \Delta X\Delta T = (L_P)^2\,, \tag{163}$$

$$\Delta P'\Delta E' = \infty\times 0 = \Delta P\Delta E(\cosh^2\xi - \sinh^2\xi) = \\ = \Delta P\Delta E = \frac{1}{L_P^2}\,. \tag{164}$$

It is important to emphasize that the invariance property of the minimal Planck-scale *Areas* (maximal Tension) is *not* an exclusive property of *infinite* acceleration boosts $\xi = \infty$, but, as a result of the identity $\cosh^2\xi - \sinh^2\xi = 1$, for all values of ξ, the minimal Planck-scale *Areas* are *always* invariant under *any* acceleration-boosts transformations. Meaning physically, in units of $\hbar = c = 1$, that the Maximal Tension (or maximal Force) $b = \frac{1}{L_P^2}$ is a true physical *invariant* universal quantity. Also we notice that the Phase-space areas, or cells, in units of $\hbar$, are also invariant! The pure-acceleration boosts transformations are "symplectic". It can be shown also that areas greater (smaller) than the Planck-area remain greater (smaller) than the invariant Planck-area under acceleration-boosts transformations.

The infinite acceleration-boosts are closely related to the infinite red-shift effects when light signals barely escape Black hole Horizons reaching an asymptotic observer with an infinite red shift factor. The important fact is that the Planck-scale *Areas* are truly maintained invariant under acceleration-boosts. This could reveal very important information about Black-holes Entropy and Holography. The logarithmic corrections to the Black-Hole Area-Entropy relation were obtained directly from Clifford-algebraic methods in C-spaces [21], in addition to the derivation of the maximal Planck temperature condition and the Schwarzschild radius in terms of the Thermodynamics of a gas of p-loop-oscillators quanta represented by area-bits, volume-bits, ... hyper-volume-bits in Planck scale units. Minimal loop-areas, in Planck units, is also one of the most important consequences found in Loop Quantum Gravity long ago [111].

7 Some further important physical applications related to the C-space physics

7.1 Relativity of signature

In previous sections we have seen how Clifford algebra can be used in the formulation of the point particle classical and quantum theory. The metric of spacetime was assumed, as usually, to have the Minkowski signature, and we have used the choice $(+ - - -)$. There were arguments in the literature of why the spacetime signature is of the Minkowski type [113, 43]. But there are also studies in which signature changes are admitted [112]. It has been found out [16, 15, 30] that within Clifford algebra the signature of the underlying space is a matter of choice of basis vectors amongst available Clifford numbers. We are now going to review those important topics.

Suppose we have a 4-dimensional space V_4 with signature

$(++++)$. Let e_μ, $\mu = 0,1,2,3$, be basis vectors satisfying

$$e_\mu \cdot e_\nu \equiv \frac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = \delta_{\mu\nu}\ , \tag{165}$$

where $\delta_{\mu\nu}$ is the *Euclidean signature* of V_4. The vectors e_μ can be used as generators of Clifford algebra $\mathcal{C}_4$ over V_4 with a generic Clifford number (also called polyvector or Clifford aggregate) expanded in term of $e_J = (1, e_\mu, e_{\mu\nu}, e_{\mu\nu\alpha}, e_{\mu\nu\alpha\beta})$, $\mu < \nu < \alpha < \beta$,

$$\begin{aligned} A = a^J e_J = a + a^\mu e_\mu + a^{\mu\nu} e_\mu e_\nu + \\ + a^{\mu\nu\alpha} e_\mu e_\nu e_\alpha + a^{\mu\nu\alpha\beta} e_\mu e_\nu e_\alpha e_\beta\,. \end{aligned} \tag{166}$$

Let us consider the set of four Clifford numbers $(e_0, e_i e_0)$, $i = 1,2,3$, and denote them as

$$\begin{aligned} e_0 &\equiv \gamma_0\ , \\ e_i e_0 &\equiv \gamma_i\,. \end{aligned} \tag{167}$$

The Clifford numbers γ_μ, $\mu = 0,1,2,3$, satisfy

$$\frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu}\ , \tag{168}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the *Minkowski tensor*. We see that the γ_μ behave as basis vectors in a 4-dimensional space $V_{1,3}$ with signature $(+---)$. We can form a Clifford aggregate

$$\alpha = \alpha^\mu \gamma_\mu\ , \tag{169}$$

which has the properties of a *vector* in $V_{1,3}$. From the point of view of the space V_4 the same object α is a linear combination of a vector and bivector:

$$\alpha = \alpha^0 e_0 + \alpha^i e_i e_0\,. \tag{170}$$

We may use γ_μ as generators of the Clifford algebra $\mathcal{C}_{1,3}$ defined over the pseudo-Euclidean space $V_{1,3}$. The basis elements of $\mathcal{C}_{1,3}$ are $\gamma_J = (1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\alpha}, \gamma_{\mu\nu\alpha\beta})$, with $\mu < \nu < \alpha < \beta$. A generic Clifford aggregate in $\mathcal{C}_{1,3}$ is given by

$$\begin{aligned} B = b^J \gamma_J = b + b^\mu \gamma_\mu + b^{\mu\nu} \gamma_\mu \gamma_\nu + \\ + b^{\mu\nu\alpha} \gamma_\mu \gamma_\nu \gamma_\alpha + b^{\mu\nu\alpha\beta} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta\,. \end{aligned} \tag{171}$$

With suitable choice of the coefficients $b^J = (b, b^\mu, b^{\mu\nu}, b^{\mu\nu\alpha}, b^{\mu\nu\alpha\beta})$ we have that B of eq.-(171) is equal to A of eq.-(166). Thus the same number A can be described either with e_μ which generate $\mathcal{C}_4$, or with γ_μ which generate $\mathcal{C}_{1,3}$. The expansions (171) and (166) exhaust all possible numbers of the Clifford algebras $\mathcal{C}_{1,3}$ and $\mathcal{C}_4$. Those expansions are just two different representations of the same set of Clifford numbers (also being called polyvectors or Clifford aggregates).

As an alternative to (167) we can choose

$$\begin{aligned} e_0 e_3 &\equiv \tilde{\gamma}_0\ , \\ e_i &\equiv \tilde{\gamma}_i\ , \end{aligned} \tag{172}$$

from which we have

$$\frac{1}{2}(\tilde{\gamma}_\mu \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \tilde{\gamma}_\mu) = \tilde{\eta}_{\mu\nu} \tag{173}$$

with $\tilde{\eta}_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Obviously $\tilde{\gamma}_\mu$ are basis vectors of a pseudo-Euclidean space $\widetilde{V}_{1,3}$ and they generate the Clifford algebra over $\widetilde{V}_{1,3}$ which is yet another representation of the same set of objects (i. e., polyvectors). The spaces V_4, $V_{1,3}$ and $\widetilde{V}_{1,3}$ are different slices through C-space, and they span different subsets of polyvectors. In a similar way we can obtain spaces with signatures $(+-++)$, $(++-+)$, $(+++-)$, $(-+--)$, $(--+-)$, $(---+)$ and corresponding higher dimensional analogs. But we cannot obtain signatures of the type $(++--)$, $(+-+-)$, etc. In order to obtain such signatures we proceed as follows.

4-space. First we observe that the bivector $\bar{I}$*4-space.* $e_3 e_4$ satisfies $\bar{I}^2 = -1$, commutes with e_1, e_2 and anticommutes with e_3, e_4. So we obtain that the set of Clifford numbers $\gamma_\mu = (e_1 \bar{I}, e_2 \bar{I}, e_3, e_3)$ satisfies

$$\gamma_\mu \cdot \gamma_\nu = \bar{\eta}_{\mu\nu}\ , \tag{174}$$

where $\bar{\eta} = \mathrm{diag}(-1,-1,1,1)$.

8-space. Let e_A be basis vectors of 8-dimensional vector space with signature $(++++++++)$. Let us decompose

$$\begin{aligned} e_A = (e_\mu, e_{\bar{\mu}})\,, \quad \mu &= 0,1,2,3\,, \\ \bar{\mu} &= \bar{0}, \bar{1}, \bar{2}, \bar{3}\,. \end{aligned} \tag{175}$$

The inner product of two basis vectors

$$e_A \cdot e_B = \delta_{AB}\ , \tag{176}$$

then splits into the following set of equations:

$$\begin{aligned} e_\mu \cdot e_\nu &= \delta_{\mu\nu}\ , \\ e_{\bar{\mu}} \cdot e_{\bar{\nu}} &= \delta_{\bar{\mu}\bar{\nu}}\ , \\ e_\mu \cdot e_{\bar{\nu}} &= 0\,. \end{aligned} \tag{177}$$

The number $\bar{I} = e_{\bar{0}} e_{\bar{1}} e_{\bar{2}} e_{\bar{3}}$ has the properties

$$\begin{aligned} \bar{I}^2 &= 1\,, \\ \bar{I} e_\mu &= e_\mu \bar{I}\,, \\ \bar{I} e_{\bar{\mu}} &= -e_{\bar{\mu}} \bar{I}\,. \end{aligned} \tag{178}$$

The set of numbers

$$\begin{aligned} \gamma_\mu &= e_\mu\ , \\ \gamma_{\bar{\mu}} &= e_{\bar{\mu}} \bar{I} \end{aligned} \tag{179}$$

satisfies

$$\begin{aligned} &\gamma_\mu \cdot \gamma_\nu = \delta_{\mu\nu} \,, \\ &\gamma_{\bar\mu} \cdot \gamma_{\bar\nu} = -\delta_{\mu\nu} \,, \\ &\gamma_\mu \cdot \gamma_{\bar\mu} = 0 \,. \end{aligned} \tag{180}$$

The numbers $(\gamma_\mu, \gamma_{\bar\mu})$ thus form a set of basis vectors of a vector space $V_{4,4}$ with signature $(++++----)$.

10-space. Let $e_A = (e_\mu, e_{\bar\mu})$, $\mu = 1,2,3,4,5$; $\bar\mu = \bar 1, \bar 2, \bar 3, \bar 4, \bar 5$ be basis vectors of a 10-dimensional Euclidean space V_{10} with signature $(+++\ldots)$. We introduce $\bar I = e_{\bar 1} e_{\bar 2} e_{\bar 3} e_{\bar 4} e_{\bar 5}$ which satisfies

$$\begin{aligned} &\bar I^2 = 1 \,, \\ &e_\mu \bar I = -\bar I e_\mu \,, \\ &e_{\bar\mu} \bar I = \bar I e_{\bar\mu} \,. \end{aligned} \tag{181}$$

Then the Clifford numbers

$$\begin{aligned} &\gamma_\mu = e_\mu \bar I \,, \\ &\gamma_{\bar\mu} = e_\mu \end{aligned} \tag{182}$$

satisfy

$$\begin{aligned} &\gamma_\mu \cdot \gamma_\nu = -\delta_{\mu\nu} \,, \\ &\gamma_{\bar\mu} \cdot \gamma_{\bar\nu} = \delta_{\bar\mu\bar\nu} \,, \\ &\gamma_\mu \cdot \gamma_{\bar\mu} = 0 \,. \end{aligned} \tag{183}$$

The set $\gamma_A = (\gamma_\mu, \gamma_{\bar\mu})$ therefore spans the vector space of signature $(-----+++++)$.

The examples above demonstrate how vector spaces of various signatures are obtained within a given set of polyvectors. Namely, vector spaces of different signature are different subsets of polyvectors within the same Clifford algebra. In other words, vector spaces of different signature are different subspaces of C-space, i. e., different sections through C-space*.

This has important physical implications. We have argued that physical quantities are polyvectors (Clifford numbers or Clifford aggregates). Physical space is then not simply a vector space (e.g., Minkowski space), but a space of polyvectors, called C-space, a pandimensional continuum of points, lines, planes, volumes, etc., altogether. Minkowski space is then just a subspace with pseudo-Euclidean signature. Other subspaces with other signatures also exist within the pandimensional continuum C and they all have physical significance. If we describe a particle as moving in Minkowski spacetime $V_{1,3}$ we consider only certain physical aspects of the object considered. We have omitted its other physical properties like spin, charge, magnetic moment, etc. We can as well describe the same object as moving in an Euclidean space V_4. Again such a description would reflect only a part of the underlying physical situation described by Clifford algebra.

*What we consider here should not be confused with the well known fact that Clifford algebras associated with vector spaces of different signatures (p, q), with $p + q = n$, are not all isomorphic.

7.2 Clifford space and the conformal group

7.2.1 Line element in C-space of Minkowski spacetime

In 4-dimensional spacetime a polyvector and its square (1) can be written as

$$\mathrm{d}X = \mathrm{d}\sigma + \mathrm{d}x^\mu \gamma_\mu + \frac{1}{2}\,\mathrm{d}x^{\mu\nu}\gamma_\mu \wedge \gamma_\nu + \mathrm{d}\tilde x^\mu\, I\gamma_\mu + \mathrm{d}\tilde\sigma I \,, \tag{184}$$

$$|\mathrm{d}X|^2 = \mathrm{d}\sigma^2 + \mathrm{d}x^\mu \mathrm{d}x_\mu + \frac{1}{2}\,\mathrm{d}x^{\mu\nu}\mathrm{d}x_{\mu\nu} - \mathrm{d}\tilde x^\mu \mathrm{d}\tilde x_\mu - \mathrm{d}\tilde\sigma^2 . \tag{185}$$

The minus sign in the last two terms of the above quadratic form occurs because in 4-dimensional spacetime with signature $(+---)$ we have $I^2 = (\gamma_0\gamma_1\gamma_2\gamma_3)(\gamma_0\gamma_1\gamma_2\gamma_3) = -1$, and $I^\dagger I = (\gamma_3\gamma_2\gamma_1\gamma_0)(\gamma_0\gamma_1\gamma_2\gamma_3) = -1$.

In eq.-(185) the line element $\mathrm{d}x^\mu \mathrm{d}x_\mu$ of the ordinary special or general relativity is replaced by the line element in Clifford space. A "square root" of such a generalized line element is $\mathrm{d}X$ of eq.-(184). The latter object is a *polyvector*, a differential of the coordinate polyvector field

$$X = \sigma + x^\mu \gamma_\mu + \frac{1}{2}\, x^{\mu\nu}\gamma_\mu \wedge \gamma_\nu + \tilde x^\mu I\gamma_\mu + \tilde\sigma I \,, \tag{186}$$

whose square is

$$|X|^2 = \sigma^2 + x^\mu x_\mu + \frac{1}{2}\, x^{\mu\nu} x_{\mu\nu} - \tilde x^\mu \tilde x_\mu - \tilde\sigma^2 \,. \tag{187}$$

The polyvector X contains not only the vector part $x^\mu\gamma_\mu$, but also a *scalar part* σ, *tensor part* $x^{\mu\nu}\gamma_\mu\wedge\gamma_\nu$, *pseudovector part* $\tilde x^\mu I\gamma_\mu$ and pseudoscalar part $\tilde\sigma I$. Similarly for the differential $\mathrm{d}X$.

When calculating the quadratic forms $|X|^2$ and $|\mathrm{d}X|^2$ one obtains in 4-dimensional spacetime with pseudo euclidean signature $(+---)$ the minus sign in front of the squares of the pseudovector and pseudoscalar terms. This is so, because in such a case the pseudoscalar unit square in flat spacetime is $I^2 = I^\dagger I = -1$. In 4-dimensions $I^\dagger = I$ regardless of the signature.

Instead of Lorentz transformations — pseudo rotations in spacetime — which preserve $x^\mu x_\mu$ and $\mathrm{d}x^\mu \mathrm{d}x_\mu$ we have now more general rotations — rotations in C-space — which preserve $|X|^2$ and $|\mathrm{d}X|^2$.

7.2.2 C-space and conformal transformations

From (185) and (187) we see [25] that a subgroup of the Clifford Group, or rotations in C-space is the group $SO(4,2)$. The transformations of the latter group rotate x^μ, σ, $\tilde\sigma$, but leave $x^{\mu\nu}$ and $\tilde x^\mu$ unchanged. Although according to our assumption physics takes place in full C-space, it is very instructive to consider a subspace of C-space, that we shall call *conformal space* whose isometry group is $SO(4,2)$.

Coordinates can be given arbitrary symbols. Let us now use the symbol η^μ instead of x^μ, and η^5,η^6 instead of $\tilde\sigma$, σ. In other words, instead of $(x^\mu, \tilde{\sigma}, \sigma)$ we write $(\eta^\mu, \eta^5, \eta^6) \equiv \eta^a$, $\mu = 0,1,2,3$, $a = 0,1,2,3,5,6$. The quadratic form reads

$$\eta^a \eta_a = g_{ab} \eta^a \eta^b \tag{188}$$

with

$$g_{ab} = \text{diag}(1,-1,-1,-1,-1,1) \tag{189}$$

being the diagonal metric of the flat 6-dimensional space, a subspace of C-space, parametrized by coordinates η^a. The transformations which preserve the quadratic form (188) belong to the group $SO(4,2)$. It is well known [38, 39] that the latter group, when taken on the cone

$$\eta^a \eta_a = 0 \tag{190}$$

is isomorphic to the 15-parameter group of conformal transformations in 4-dimensional spacetime [40].

Let us consider first the rotations of η^5 and η^6 which leave coordinates η^μ unchanged. The transformations that leave $-(\eta^5)^2 + (\eta^6)^2$ invariant are

$$\begin{aligned} \eta'^5 &= \eta^5 \cosh\alpha + \eta^6 \sinh\alpha \\ \eta'^6 &= \eta^5 \sinh\alpha + \eta^6 \cosh\alpha \,, \end{aligned} \tag{191}$$

where α is a parameter of such pseudo rotations.

Instead of the coordinates η^5, η^6 we can introduce [38, 39] new coordinates κ, λ according to

$$\begin{aligned} \kappa &= \eta^5 - \eta^6 \,, \\ \lambda &= \eta^5 + \eta^6 \,. \end{aligned} \tag{192}$$

In the new coordinates the quadratic form (188) reads

$$\eta^a \eta_a = \eta^\mu \eta_\mu - (\eta^5)^2 - (\eta^6)^2 = \eta^\mu \eta_\mu - \kappa\lambda \,. \tag{193}$$

The transformation (191) becomes

$$\kappa' = \rho^{-1} \kappa \,, \tag{194}$$

$$\lambda' = \rho \lambda \,, \tag{195}$$

where $\rho = e^\alpha$. This is just a dilation of κ and the inverse dilation of λ.

Let us now introduce new coordinates x^μ*

$$\eta^\mu = \kappa x^\mu \,. \tag{196}$$

Under the transformation (196) we have

$$\eta'^\mu = \eta^\mu \,, \tag{197}$$

but

$$x'^\mu = \rho x^\mu \,, \tag{198}$$

the latter transformation is *dilatation* of coordinates x^μ.

*These new coordinates x^μ should not be confused with coordinate x^μ used in section 2.

Considering now a line element

$$\mathrm{d}\eta^a \mathrm{d}\eta_a = \mathrm{d}\eta^\mu \mathrm{d}\eta_\mu - \mathrm{d}\kappa \mathrm{d}\,, \lambda \tag{199}$$

we find that *on the cone* $\eta^a \eta_a = 0$ it is

$$\mathrm{d}\eta^a \mathrm{d}\eta_a = \kappa^2 \, \mathrm{d}x^\mu \mathrm{d}x_\mu \tag{200}$$

even if κ is not constant. Under the transformation (194) we have

$$\mathrm{d}\eta'^a \mathrm{d}\eta'_a = \mathrm{d}\eta^a \mathrm{d}\eta_a \,, \tag{201}$$

$$\mathrm{d}x'^\mu \mathrm{d}x'_\mu = \rho^2 \, \mathrm{d}x^\mu \mathrm{d}x_\mu \,. \tag{202}$$

The last relation is a *dilatation* of the 4-dimensional line element related to coordinates x^μ. In a similar way also other transformations of the group $SO(4,2)$ that preserve (190) and (201) we can rewrite in terms of of the coordinates x^μ. So we obtain — besides dilations — translations, Lorentz transformations, and special conformal transformations; altogether they are called *conformal transformations*. This is a well known old observation [38, 39] and we shall not discuss it further. What we wanted to point out here is that conformal group $SO(4,2)$ is a subgroup of the Clifford group.

7.2.3 On the physical interpretation of the conformal group $SO(4,2)$

In order to understand the physical meaning of the transformations (196) from the coordinates η^μ to the coordinates x^μ let us consider the following transformation in 6-dimensional space V_6:

$$\begin{aligned} x^\mu &= \kappa^{-1} \eta^\mu \,, \\ \alpha &= -\kappa^{-1} \,, \\ \Lambda &= \lambda - \kappa^{-1} \eta^\mu \eta_\mu \,. \end{aligned} \tag{203}$$

This is a transformation from the coordinates $\eta^a = (\eta^\mu, \kappa, \lambda)$ to the new coordinates $x^a = (x^\mu, \alpha, \Lambda)$. No extra condition on coordinates, such as (190), is assumed now. If we calculate the line element in the coordinates η^a and x^a, respectively, we find the the following relation [27]

$$\mathrm{d}\eta^\mu \mathrm{d}\eta^\nu \, g_{\mu\nu} - \mathrm{d}\kappa \, \mathrm{d}\lambda = \alpha^{-2} (\mathrm{d}x^\mu \mathrm{d}x^\nu \, g_{\mu\nu} - \mathrm{d}\alpha \mathrm{d}\Lambda) \,. \tag{204}$$

We can interpret a transformation of coordinates *passively* or *actively*. Geometric calculus clarifies significantly the meaning of passive and active transformations. Under a *passive transformation* a vector remains the same, but its components and basis vector change. For a vector $\mathrm{d}\eta = \mathrm{d}\eta^a \gamma_a$ we have

$$\mathrm{d}\eta' = \mathrm{d}\eta'^a \gamma'_a = \mathrm{d}\eta^a \gamma_a = \mathrm{d}\eta \tag{205}$$

with

$$\mathrm{d}\eta'^a = \frac{\partial \eta'^a}{\partial \eta^b} \, \mathrm{d}\eta^b \tag{206}$$

and

$$\gamma'_a = \frac{\partial \eta^b}{\partial \eta'^a} \gamma_b \,. \tag{207}$$

Since the vector is invariant, so it is its square:

$$\mathrm{d}\eta'^2 = \mathrm{d}\eta'^a \gamma'_a \, \mathrm{d}\eta'^b \gamma'_b = \mathrm{d}\eta'^a \mathrm{d}\eta'^b g'_{ab} = \mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab} \,. \tag{208}$$

From (207) we read that the well known relation between new and old coordinates:

$$g'_{ab} = \frac{\partial \eta^c}{\partial \eta'^a} \frac{\partial \eta^d}{\partial \eta'^b} \, g_{cd} \,. \tag{209}$$

Under an *active transformation* a vector changes. This means that in a fixed basis the components of a vector change:

$$\mathrm{d}\eta' = \mathrm{d}\eta'^a \gamma_a \tag{210}$$

with

$$\mathrm{d}\eta'^a = \frac{\partial \eta'^a}{\partial \eta^b} \, \mathrm{d}\eta^b \,. \tag{211}$$

The transformed vector $\mathrm{d}\eta'$ is different from the original vector $\mathrm{d}\eta = \mathrm{d}\eta^a \gamma_a$. For the square we find

$$\mathrm{d}\eta'^2 = \mathrm{d}\eta'^a \mathrm{d}\eta'^b g_{ab} = \frac{\partial \eta'^a}{\partial \eta^c} \frac{\partial \eta'^b}{\partial \eta^d} \, \mathrm{d}\eta^c \mathrm{d}\eta^d g_{ab} \,, \tag{212}$$

i. e., the transformed line element $\mathrm{d}\eta'^2$ is different from the original line element.

Returning now to the coordinate transformation (203) with the identification $\eta'^a = x^a$, we can interpret eq.-(204) passively or actively.

In the *passive interpretation* the metric tensor and the components $\mathrm{d}\eta^a$ change under a transformation, so that in our particular case the relation (208) becomes

$$\begin{aligned} \mathrm{d}x^a \, \mathrm{d}x^b \, g'_{ab} &= \alpha^{-2} (\mathrm{d}x^\mu \mathrm{d}x^\nu \, g_{\mu\nu} - \mathrm{d}\alpha \, \mathrm{d}\Lambda) = \\ &= \mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab} = \mathrm{d}\eta^\mu \mathrm{d}\eta^\nu g_{\mu\nu} - \mathrm{d}\kappa \, \mathrm{d}\lambda \end{aligned} \tag{213}$$

with

$$\begin{aligned} g'_{ab} &= \alpha^{-2} \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \\ g_{ab} &= \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}. \end{aligned} \tag{214}$$

In the above equation the same infinitesimal distance squared is expressed in two different coordinates η^a or x^a.

In *active interpretation*, only $\mathrm{d}\eta^a$ change, whilst the metric remains the same, so that the transformed element is

$$\begin{aligned} \mathrm{d}x^a \, \mathrm{d}x^b \, g_{ab} &= \mathrm{d}x^\mu \mathrm{d}x^\nu \, g_{\mu\nu} - \mathrm{d}\alpha \, \mathrm{d}\Lambda = \\ &= \kappa^{-2} \, \mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab} = \kappa^{-2} (\mathrm{d}\eta^\mu \mathrm{d}\eta^\nu g_{\mu\nu} - \mathrm{d}\kappa \, \mathrm{d}\lambda) \,. \end{aligned} \tag{215}$$

The transformed line element $\mathrm{d}x^a \mathrm{d}x_a$ is physically different from the original line element $\mathrm{d}\eta^a \mathrm{d}\eta_a$ by a factor $\alpha^2 = \kappa^{-2}$.

A rotation (191) in the plane (η^5, η^6) i. e. the transformation (194), (195) of (κ, λ) manifests in the new coordinates x^a as a *dilatation* of the line element $\mathrm{d}x^a \mathrm{d}x_a = \kappa^{-2} \, \mathrm{d}\eta^a \eta_a$:

$$\mathrm{d}x'^a \mathrm{d}x'_a = \rho^2 \mathrm{d}x^a \mathrm{d}x_a \,. \tag{216}$$

All this is true in the full space V_6. On the cone $\eta^a \eta_a = 0$ we have $\Lambda = \lambda - \kappa \eta^\mu \eta_\mu = 0$, $\mathrm{d}\Lambda = 0$ so that $\mathrm{d}x^a \mathrm{d}x_a = = \mathrm{d}x^\mu \mathrm{d}x_\mu$ and we reproduce the relations (202) which is a dilatation of the 4-dimensional line element. It can be interpreted either passively or actively. In general, the pseudo rotations in V_6, that is, the transformations of the 15-parameter group $SO(4,2)$ when expressed in terms of coordinates x^a, assume on the cone $\eta^a \eta_a = 0$ the form of the ordinary conformal transformations. They all can be given the active interpretation [27, 28].

We started from the new paradigm that physical phenomena actually occur not in spacetime, but in a larger space, the so called *Clifford space* or C-space which is a manifold associated with the Clifford algebra generated by the basis vectors γ_μ of spacetime. An arbitrary element of Clifford algebra can be expanded in terms of the objects E_A, $A = 1, 2, \dots, 2^D$, which include, when $D = 4$, the scalar unit $\mathbf{1}$, vectors γ_μ, bivectors $\gamma_\mu \wedge \gamma_\nu$, pseudovectors $I\gamma_\mu$ and the pseudoscalar unit $I \equiv \gamma_5$. C-space contains 6-dimensional subspace V_6 spanned* by $\mathbf{1}$, γ_μ, and γ_5. The metric of V_6 has the signature $(+ - - - - +)$. It is well known that the rotations in V_6, when taken on the conformal cone $\eta^a \eta_a = 0$, are isomorphic to the non linear transformations of the conformal group in spacetime. Thus we have found out that C-space contains — as a subspace — the 6-dimensional space V_6 in which the conformal group acts linearly. From the physical point of view this is an important and, as far as we know, a novel finding, although it might look mathematically trivial. *So far it has not been clear what could be a physical interpretation of the 6 dimensional conformal space*. Now we see that it is just a subspace of Clifford space. The two extra dimensions, parameterized by κ and λ, are not the ordinary extra dimensions; they are coordinates of Clifford space C_4 of the 4-dimensional Minkowski spacetime V_4.

We take C-space seriously as an arena in which physics takes place. The theory is a very natural, although not trivial, extension of the special relativity in spacetime. In special relativity the transformations that preserve the quadratic form

*It is a well known observation that the generators L_{ab} of $SO(4,2)$ can be realized in terms of $\mathbf{1}$, γ_μ, and γ_5. Lorentz generators are $M_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]$, dilatations are generated by $D = L_{65} = -\frac{1}{2}\gamma_5$, translations by $P_\mu = L_{5\mu} + L_{6\mu} = \frac{1}{2}\gamma_\mu(1 - i\gamma_5)$ and the special conformal transformations by $L_{5\mu} - L_{6\mu} = \frac{1}{2}\gamma_\mu(1 + i\gamma_5)$. This essentially means that the generators are $L_{ab} = -\frac{i}{4}[e_a, e_b]$ with $e_a = (\gamma_\mu, \gamma_5, \mathbf{1})$, where care must be taken to replace commutators $[\mathbf{1}, \gamma_5]$ and $[\mathbf{1}, \gamma_\mu]$ with $2\gamma_5$ and $2\gamma_\mu$.

are given an *active interpretation*: they relate the objects or the systems of reference in *relative translational motion*. Analogously also the transformations that preserve the quadratic form (185) or (187) in C-space should be given an active interpretation. We have found that among such transformations (rotations in C-space) there exist the transformations of the group $SO(4,2)$. Those transformations also should be given an active interpretation as the transformations that relate different physical objects or reference frames. Since in the ordinary relativity we do not impose any constraint on the coordinates of a freely moving object so we should not impose any constraint in C-space, or in the subspace V_6. However, by using the projective coordinate transformation (203), without any constraint such as $\eta^a \eta_a = 0$, we arrived at the relation (215) for the line elements. If in the coordinates η^a the line element is constant, then in the coordinates x^a the line element is changing by a scale factor κ which, in general, depends on the evolution parameter τ. The line element need not be one associated between two events along a point particle's worldline: it can be between two arbitrary (space-like or time-like) events within an extended object. We may consider the line element ($\equiv$ distance squared) between two infinitesimally separated events within an extended object such that both events have the same coordinate label Λ so that $\mathrm{d}\Lambda = 0$. Then the 6-dimensional line element $\mathrm{d}x^\mu \mathrm{d}x^\nu\, g_{\mu\nu} - \mathrm{d}\alpha\, \mathrm{d}\Lambda$ becomes the 4-dimensional line element $\mathrm{d}x^\mu \mathrm{d}x^\nu\, g_{\mu\nu}$ and, because of (215) it changes with τ when κ does change. This means that the object changes its *size*, it is moving dilatationally [27, 28]. We have thus arrived at a very far reaching observation that the relativity in C-space implies *scale changes of physical objects as a result of free motion, without presence of any forces or such fields as assumed in Weyl theory*. This was advocated long time ago [27, 28], but without recurse to C-space. However, if we consider the full Clifford space C and not only the Minkowski spacetime section through C, then we arrive at a more general dilatational motion [17] related to the polyvector coordinates $x^{\mu\nu}, x^{\mu\nu\alpha}$ and $x^{0123} \equiv \tilde{\sigma}$ (also denoted s) as reviewed in section 3.

7.3 C-space Maxwell Electrodynamics

Finally, in this section we will review and complement the proposal of ref. [75] to generalize Maxwell Electrodynamics to C-spaces, namely, construct the Clifford algebra-valued extension of the Abelian field strength $F = dA$ associated with ordinary vectors A_μ. Using Clifford algebraic methods we shall describe how to generalize Maxwell's theory of Electrodynamics associated with ordinary point-charges to a generalized Maxwell theory in Clifford spaces involving *extended* charges and p-forms of arbitrary rank, not unlike the couplings of p-branes to antisymmetric tensor fields.

Based on the standard definition of the Abelian field strength $F = dA$ we shall use the same definition in terms of polyvector-valued quantities and differential operators in C-space

$$A = A_N E^N = \phi \underline{1} + A_\mu \gamma^\mu + A_{\mu\nu} \gamma^\mu \wedge \gamma^\nu + \dots \,. \tag{217}$$

The first component in the expansion ϕ is a scalar field; A_μ is the standard Maxwell field A_μ, the third component $A_{\mu\nu}$ is a rank two antisymmetric tensor field… and the last component of the expansion is a pseudo-scalar. The fact that a scalar and pseudo-scalar field appear very naturally in the expansion of the C-space polyvector valued field A_N suggests that one could attempt to identify the latter fields with a dilaton-like and axion-like field, respectively. Once again, in order to match units in the expansion (217), it requires the introduction of suitable powers of a length scale parameter, the Planck scale which is conveniently set to unity.

The differential operator is the generalized Dirac operator

$$d = E^M \partial_M = \underline{1} \partial_\sigma + \gamma^\mu \partial_{x_\mu} + \gamma^\mu \wedge \gamma^\nu \partial_{x_{\mu\nu}} + \dots \tag{218}$$

the polyvector-valued indices $M, N, \dots$ range from $1, 2 \dots 2^D$ since a Clifford algebra in D-dim has 2^D basis elements. The generalized Maxwell field strength in C-space is

$$\begin{aligned} F = dA = E^M \partial_M (E^N A_N) = E^M E^N \partial_M A_N = \\ = \frac{1}{2}\{E^M, E^N\} \partial_M A_N + \frac{1}{2}[E^M, E^N] \partial_M A_N = \\ = \frac{1}{2} F_{(MN)} \{E^M, E^N\} + \frac{1}{2} F_{[MN]} [E^M, E^N] \,, \end{aligned} \tag{219}$$

where one has *decomposed* the Field strength components into a symmetric plus antisymmetric piece by simply writing the Clifford geometric product of two polyvectors $E^M E^N$ as the sum of an anticommutator plus a commutator piece respectively,

$$F_{(MN)} = \frac{1}{2}(\partial_M A_N + \partial_N A_M) \,, \tag{220}$$

$$F_{[MN]} = \frac{1}{2}(\partial_M A_N - \partial_N A_M) \,. \tag{221}$$

Let the C-space Maxwell action (up to a numerical factor) be given in terms of the antisymmetric part of the field strength:

$$I[A] = \int [\mathrm{D}X] F_{[MN]} F^{[MN]} \,, \tag{222}$$

where $[\mathrm{D}X]$ is a C-space measure comprised of all the (holographic) coordinates degrees of freedom

$$\begin{aligned} [\mathrm{D}X] \equiv (d\sigma)(dx^0 dx^1 \dots)(dx^{01} dx^{02} \dots) \dots \\ \dots (dx^{012\dots D}) \,. \end{aligned} \tag{223}$$

Action (222) is invariant under the gauge transformations

$$A'_M = A_M + \partial_M \Lambda \,. \tag{224}$$

The matter-field minimal coupling (interaction term) is:

$$\int A_M dX^M = \int [\mathrm{D}X] J_M A^M \,, \tag{225}$$

where one has reabsorbed the coupling constant, the C-space analog of the electric charge, within the expression for the A field itself. Notice that this term (225) has the same form as the coupling of p-branes (whose world volume is $(p+1)$-dimensional) to antisymmetric tensor fields of rank $p+1$.

The open line integral in C-space of the matter-field interaction term in the action is taken from the polyparticle's proper time interval S ranging from $-\infty$ to $+\infty$ and can be recast via the Stokes law solely in terms of the antisymmetric part of the field strength. This requires closing off the integration contour by a semi-circle that starts at $S = +\infty$, goes all the way to C-space infinity, and comes back to the point $S = -\infty$. The field strength vanishes along the points of the semi-circle at infinity, and for this reason the net contribution to the contour integral is given by the open-line integral. Therefore, by rewriting the $\int A_M dX^M$ via the Stokes law relation, it yields

$$\begin{aligned}\int A_M dX^M = \int F_{[MN]} dS^{[MN]} = \int F_{[MN]} X^M dX^N = \\ = \int dS F_{[MN]} X^M (dX^N/dS)\,,\end{aligned} \tag{226}$$

where in order to go from the second term to the third term in the above equation we have integrated by parts and then used the Bianchi identity for the antisymmetric component $F_{[MN]}$.

The integration by parts permits us to go from a C-space domain integral, represented by the Clifford-value hypersurface S^{MN}, to a C-space boundary-line integral

$$\int dS^{MN} = \frac{1}{2} \int (X^M dX^N - X^N dX^M)\,. \tag{227}$$

The pure matter terms in the action are given by the analog of the proper time integral spanned by the motion of a particle in spacetime:

$$\kappa \int dS = \kappa \int dS \sqrt{\frac{dX^M}{dS} \frac{dX_M}{dS}}\,, \tag{228}$$

where κ is a parameter whose dimensions are mass^{p+1} and S is the polyparticle proper time in C-space.

The Lorentz force relation in C-space is directly obtained from a variation of

$$\int dS F_{[MN]} X^M (dX^N/dS)\,, \tag{229}$$

and

$$\kappa \int dS = \kappa \int \sqrt{dX^M dX_M} \tag{230}$$

with respect to the X^M variables:

$$\kappa \frac{d^2 X_M}{dS^2} = e F_{[MN]} \frac{dX^N}{dS}\,, \tag{231}$$

where we have re-introduced the C-space charge e back into the Lorentz force equation in C-space. A variation of the terms in the action w. r. t the A_M field furnishes the following equation of motion for the A fields:

$$\partial_M F^{[MN]} = J^N\,. \tag{232}$$

By taking derivatives on both sides of the last equation with respect to the X^N coordinate, one obtains due to the symmetry condition of $\partial_M \partial_N$ versus the antisymmetry of $F^{[MN]}$ that

$$\partial_N \partial_M F^{[MN]} = 0 = \partial_N J^N = 0\,, \tag{233}$$

which is precisely the continuity equation for the current.

The continuity equation is essential to ensure that the matter-field coupling term of the action $\int A_M dX^M = \int [\mathrm{D}X]\, J^M A_M$ is also gauge invariant, which can be readily verified after an integration by parts and setting the boundary terms to zero:

$$\begin{aligned}\delta \int [\mathrm{D}X] J^M A_M = \int [\mathrm{D}X] J^M \partial_M \Lambda = \\ = -\int [\mathrm{D}X] (\partial_M J^M) \Lambda = 0.\end{aligned} \tag{234}$$

Gauge invariance also ensures the conservation of the energy-momentum (via Noether's theorem) defined in terms of the Lagrangian density variation. We refer to [75] for further details.

The gauge invariant C-space Maxwell action as given in eq.-(222) is in fact only a part of a more general action given by the expression

$$I[A] = \int [\mathrm{D}X]\, F^\dagger * F = \int [\mathcal{D}X]\; < F^\dagger F >_{scalar}\,. \tag{235}$$

This action can also be written in terms of components, up to dimension-dependent *numerical* coefficients, as [75]:

$$I[A] = \int [\mathcal{D}X]\; (F_{(MN)} F^{(MN)} + F_{[MN]} F^{[MN]})\,. \tag{236}$$

For rigor, one should introduce the numerical coefficients in front of the F terms, noticing that the symmetric combination should have a different dimension-dependent coefficient than the anti-symmetric combination since the former involves contractions of $\{E^M, E^N\}^* \{E_M, E_N\}$ and the latter contractions of $[E^M, E^N]^* [E_M, E_N]$.

The latter action is strictly speaking not gauge invariant, since it contains not only the antisymmetric but also the symmetric part of F. It is invariant under a *restricted* gauge symmetry transformations. It is invariant (up to total derivatives) under *infinitesimal* gauge transformations provided the symmetric part of F is divergence-free $\partial_M F^{(MN)} = 0$ [75]. This divergence-free condition has the same effects as if one were fixing a gauge leaving a *residual* symmetry of *restricted* gauge transformations such that the gauge symmetry parameter obeys the Laplace-like equation $\partial_M \partial^M \Lambda = 0$. Such residual (restricted) symmetries are precisely those that leave invariant the divergence-free condition on the symmetric part of F. Residual, restricted symmetries occur, for example, in the light-cone gauge of p-brane actions leaving a residual symmetry of *volume*-preserving diffs. They also occur in string theory when the conformal gauge is chosen leaving a residual symmetry under conformal reparametrizations; i. e. the so-called Virasoro algebras whose symmetry transformations are given by holomorphic and anti-holomorphic reparametrizations of the string world-sheet.

This Laplace-like condition on the gauge parameter is also the one required such that the action in [75] is invariant under *finite* (restricted) gauge transformations since under such restricted finite transformations the Lagrangian changes by second-order terms of the form $(\partial_M \partial_N \Lambda)^2$, which are total derivatives if, and only if, the gauge parameter is restricted to obey the analog of Laplace equation $\partial_M \partial^M \Lambda = 0$

Therefore the action of eq-(233) is invariant under a *restricted* gauge transformation which bears a resemblance to *volume*-preserving diffeomorphisms of the p-branes action in the light-cone gauge. A lesson that we have from these considerations is that the C-space Maxwell action written in the form (235) automatically contains a gauge fixing term. Analogous result for *ordinary* Maxwell field is known from Hestenes work [1], although formulated in a slightly different way, namely by directlty considering the field equations without employing the action.

It remains to be seen if this construction of C-space generalized Maxwell Electrodynamics of p-forms can be generalized to the non-Abelian case when we replace ordinary derivatives by gauge-covariant ones:

$$F = dA \rightarrow F = DA = (dA + A \bullet A). \tag{237}$$

For example, one could define the graded-symmetric product $E_M \bullet E_N$ based on the graded commutator of Superalgebras:

$$[A, B] = AB - (-1)^{s_A s_B} BA\,, \tag{238}$$

s_A, s_B is the grade of A and B respectively. For bosons the grade is even and for fermions is odd. In this fashion the graded commutator captures both the anti-commutator of two fermions and the commutator of two bosons in one stroke. One may extend this graded bracket definition to the graded structure present in Clifford algebras, and define

$$E_M \bullet E_N = E_M E_N - (-1)^{s_M s_N} E_N E_M\,, \tag{239}$$

s_M, s_N is the grade of E_M and E_N respectively. Even or odd depending on the grade of the basis elements.

One may generalize Maxwell's theory to Born-Infeld nonlinear Electrodynamics in C-spacesbased on this extension of Maxwell Electrodynamics in C-spaces and to couple a C-space version of a Yang-Mills theory to C-space gravity, a higher derivative gravity with torsion, this will be left for a future publication. Clifford algebras have been used in the past [62] to study the Born-Infeld model in ordinary spacetime and to write a nonlinear version of the Dirac equation. The natural incorporation of monopoles in Maxwell's theory was investigated by [89] and a recent critical analysis of "unified" theories of gravity with electromagnetism has been presented by [90]. Most recently [22] has studied the covariance of Maxwell's theory from a Clifford algebraic point of view.

8 Concluding remarks

We have presented a brief review of some of the most important features of the Extended Relativity theory in Clifford-spaces (C-spaces). The "coordinates" X are non-commutating Clifford-valued quantities which incorporate the lines, areas, volumes, ... degrees of freedom associated with the collective particle, string, membrane, ... dynamics underlying the center-of-mass motion and holographic projections of the p-loops onto the embedding target spacetime backgrounds. C-space Relativity incorporates the idea of an invariant length, which upon quantization, should lead to the notion of minimal Planck scale [23]. Other relevant features are those of maximal acceleration [52], [49]; the invariance of Planck-areas under acceleration boosts; the resolution of ordering ambiguities in QFT; supersymmetry; holography [119]; the emergence of higher derivative gravity with torsion; and the inclusion of variable dimensions/signatures that allows to study the dynamics of all (closed) p-branes, for all values of p, in one single unified footing, by starting with the C-space brane action constructed in this work.

The Conformal group construction presented in sect. 7, as a natural subgroup of the Clifford group in four-dimensions, needs to be generalized to other dimensions, in particular to two dimensions where the Conformal group is infinite-dimensional. Kinani [130] has shown that the Virasoro algebra can be obtained from generalized Clifford algebras. The construction of area-preserving diffs algebras, like w_∞ and $su(\infty)$, from Clifford algebras remains an open problem. Area-preserving diffs algebras are very important in the study of membranes and gravity since Higher-dim Gravity in $(m+n)$-dim has been shown a while ago to be equivalent to a lower m-dim Yang-Mills-like gauge theory of diffs of an internal n-dim space [120] and that amounts to another explanation of the holographic principle behind the AdS/CFT duality conjecture [121]. We have shown how C-space

Relativity involves scale changes in the sizes of physical objects, in the absence of forces and Weyl'gauge field of dilations. The introduction of scale-motion degrees of freedom has recently been implemented in the wavelet-based regularization procedure of QFT by [87]. The connection to Penrose's Twistors program is another interesting project worthy of investigation.

The quantization and construction of QFTs in C-spaces remains a very daunting task since it may involve the construction of QM in Noncommutative spacetimes [136], braided Hopf quantum Clifford algebras [86], hypercomplex extensions of QM like quaternionic and octonionic QM [99], [97], [98], exceptional group extensions of the Standard Model [85], hyper-matrices and hyper-determinants [88], multi-symplectic mechanics, the de Donde-Weyl formulations of QFT [82], to cite a few, for example. The quantization program in C-spaces should share similar results as those in Loop Quantum Gravity [111], in particular the minimal Planck areas of the expectation values of the area-operator.

Spacetime at the Planck scale may be discrete, fractal, fuzzy, noncommutative... The original Scale Relativity theory in fractal spacetime [23] needs to be extended further to incorporate the notion of fractal "manifolds". A scale-fractal calculus and a fractal-analysis construction that are esential in building the notion of a fractal "manifold" has been initiated in the past years by [129]. It remains yet to be proven that a scale-fractal calculus in fractal spacetimes is another realization of a Connes Noncommutative Geometry. Fractal strings/branes and their spectrum have been studied by [104] that may require generalized Statistics beyond the Boltzmann-Gibbs, Bose-Einstein and Fermi-Dirac, investigated by [105], [103], among others.

Non-Archimedean geometry has been recognized long ago as the natural one operating at the minimal Planck scale and requires the use p-adic numbers instead of ordinary numbers [101]. By implementing the small/large scale, ultraviolet/infrared duality principle associated with QFTs in Noncommutative spaces, see [125] for a review, one would expect an upper maximum scale [23] and a maximum temperature [21] to be operating in Nature. Non-Archimedean Cosmologies based on an upper scale has been investigated by [94].

An upper/lower scale can be accomodated simultaneously and very naturally in the q-Gravity theory of [114], [69] based on bicovariant quantum group extensions of the Poincaré, Conformal group, where the q deformation parameter could be equated to the quantity $e^{\Lambda/L}$, such that both $\Lambda = 0$ and $L = \infty$, yield the same classical $q = 1$ limit. For a review of q-deformations of Clifford algebras and their generalizations see [86], [128].

It was advocated long ago by Wheeler and others, that information theory [106], set theory and number theory, may be the ultimate physical theory. The important role of Clifford algebras in information theory have been known for some time [95]. Wheeler's spacetime foam at the Planck scale may be the background source generation of Noise in the Parisi-Wu stochastic quantization [47] that is very relevant in Number theory [100]. The pre-geometry cellular-networks approach of [107] and the quantum-topos views based on gravitational quantum causal sets, noncommutative topology and category theory [109], [110], [124] deserves a further study within the C-space Relativity framework, since the latter theory also invokes a Category point of view to the notion of dimensions. C-space is a pandimensional continuum [14], [8]. Dimensions are topological invariants and, since the dimensions of the extended objects change in C-space, topology-change is another ingredient that needs to be addressed in C-space Relativity and which may shed some light into the physical foundations of string/M theory [118]. It has been speculated that the universal symmetries of string theory [108] may be linked to Borcherds Vertex operator algebras (the Monstruous moonshine) that underline the deep interplay between Conformal Field Theories and Number theory. A lot remains to be done to bridge together these numerous branches of physics and mathematics. Many surprises may lie ahead of us. For a most recent discussion on the path towards a Clifford-Geometric Unified Field theory of all forces see [138], [140]. The notion of a Generalized Supersymmetry in Clifford Superspaces as extensions of M, F theory algebras was recently advanced in [139].

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Rational Numbers Distribution and Resonance

Kyril Dombrowski*

This study solves a problem on the distribution of rational numbers along the number plane and number line. It is shown that the distribution is linked to resonance phenomena and also to stability of oscillating systems.

"God created numbers, all the rest has been created by Man...". With greatest esteem to Leopold Kronecker, one of the founders of the contemporary theory of numbers, it is impossible to agree with him in both the divine origin of number and Man's creation of mathematics. I propound herein the idea that numbers, their relations, and all mathematics in general are objective realities of our world. A part of science is not only understanding things, but also studying the relations that are objective realities in nature.

In this work I am going to consider a problem concerning the distribution of rational numbers along the number line and also in the number plane, and the relation of this distribution to resonance phenomena and stability of oscillating systems in low linear perturbations.

Any oscillating process involving at least two interacting oscillators is necessarily linked to abstract numbers — ratios between the oscillation periods. This fact displays a close relationship between such sections of science as the physical theory of oscillations and the abstract theory of numbers.

As is well known, the rational numbers are distributed on the number line everywhere compactly, so this problem statement that a function of their distribution exists might be thought false, as the case of prime numbers. But, as we will see below, it is not false — a rational numbers distribution function has an objective reality, manifest in numerous physical phenomena of Nature. This thesis will become clearer if we consider the "number lattice" introduced by Minkowski (Fig. 1). Therein are given all points of coordinates p and q which are related to numerators and denominators, respectively. If we exclude all points of the Minkowski lattice with coordinates have a common divisor different from unity, this plane will contain only "rational points" p/q (the non-cancelled fractions). Their distribution in the plane is defined by a sequence of numbers forming a rational series (Fig. 1).

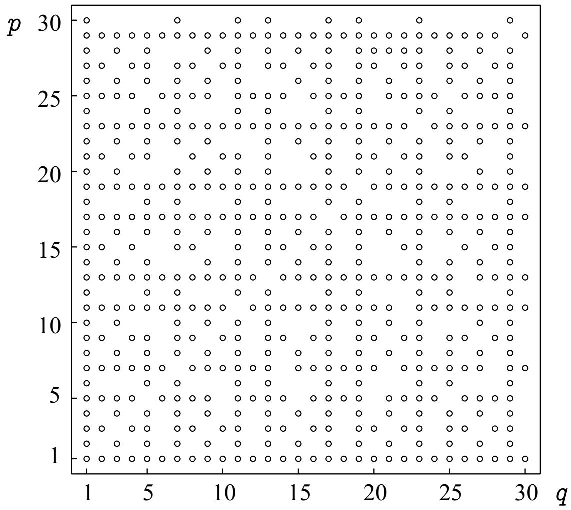

Fig. 1: The lattice of numbers (Minkowski's lattice).

This simplest drawing shows that rational numbers are distributed *inhomogeneously* in the Minkowski number plane. It is easy to see that this distribution is symmetric with respect to the axis $p = q$. Numbers of columns (and rows) in intervals, limited by this axis and one of the coordinate axes, are equal to Euler functions — the numbers less than m and relatively prime with m. Therefore, if we expand the number lattice infinitely, the *average density* of rational numbers in the plane (the ratio between the number of rational numbers and the number of all possible pairs of natural numbers the points of the lattice) approaches the limit

$$\lim_{N\to\infty} \frac{\mathrm{Ra}\left(N^2\right)}{N^2} = \lim_{N\to\infty} \frac{2}{N^2} \sum_{m=1}^{\infty} \varphi\left(m\right) = $$

$$= \frac{1}{\zeta\left(2\right)} = \frac{6}{\pi^2} = \left(\sum_{m=1}^{\infty} \frac{1}{n^2}\right)^{-1},$$

where N is the number of rational numbers, $\mathrm{Ra}\left(N^2\right)$ is the number of rational numbers located inside the square whose elements are of length equal to N, $\varphi\left(m\right)$ is Euler's function, $\zeta\left(n\right)$ is Riemann's zeta function, m and n are natural numbers. In particular, we can conclude from this that when $N < \infty$ the average density of rational numbers located in the plane is restricted to a very narrow interval of numerical values. It is possible to this verify by very simple calculations.

To study the problem of what is common to the rational number distribution and resonance phenomena it is necessary to have a one-dimensional picture of the function $\mathrm{Ra}\left(x\right)$ on the number line. In this problem, because the set of rational numbers is infinitely dense, we need to give a criterion for selecting a finite number of rational numbers which could give an objective picture of their distribution on the number line. We can do this in two ways. First, we can study, for instance, the distribution of rational number rays, drawn from the origin of coordinates in the Minkowski lattice. This

*Translated from the Russian by D. Rabounski and S. J. Crothers.

is a contemporary development of the method created by Klein [1]. Second, we can employ continued fractions, taking into account Khinchin's remark that "continued fractions... in their pure form display properties of the numbers they represent" [2]. So we can employ the mathematical apparatus of continued fractions as a systematic ground in order to find an analogous result that had been previously obtained by a purely arithmetical way.

We will use the second option because it is easier (although it is more difficult to imagine). So, let us plot points by writing a single-term continued fraction $1/n$ (so these are the numbers $1/1$, $1/2$, $1/3$, ...) inside an interval of unit length. We obtain thereby the best approximations of these numbers. This could be done inside every interval $1/(n+1) < x < 1/n$ by plotting points which are numerical values of a two-term continued fraction

$$\frac{1}{m+\frac{1}{n}} = \frac{n}{mn+1}\,.$$

These points, according to the theory of continued fractions, are the best approximations of the numbers $1/n$ from the left side.We then get the best approximations of the numbers $1/n$ from the right side, expressed by the fractions

$$\frac{1}{l+\frac{1}{1+\frac{1}{n}}} = \frac{n+1}{l\,(n+1)+n}\,, \qquad l, n, m = 1, 2, 3\ldots\;.$$

We will call the approximation obtained the *first order approximation* (the second and third rank approximation in Khinchin's terminology). It is evident that every rational point of k-th order obtained in this way has analogous sequences of the $(k+1)$-th rank and higher. Such sequences fill the whole set of rational numbers.

To consider the simplest cases of resonance it would be enough to take the first order approximation, but to consider numerous processes such as colour vision, musical harmony, or Bohr's orbit distribution in atoms, requires a high order distribution function for rational numbers.

To obtain the function $\mathrm{Ra}\,(x)$ as a regular diagram we define this function (meaning the finite approximation order, the first order in this case) as a quantity in reverse to the interval between the neighbouring rational points located on the number line, where the points are plotted in the fashion of Khinchin, mentioned above. If the numerical values of the numbers l, m, n are limited, this interval is finite (see Fig. 2a). Such a drawing gives a possibility for estimating the structure of rational number distribution along the number line. In Fig. 2a we consider the distribution structure of rational numbers derived from a three-component continued fraction. For the purpose of comparison, Fig. 2b depicts a voltage function dependent on the stimulating frequency in an oscillating contour (drawn in the same scale as that in Fig. 2a). In this case an alternating signal frequency at a constant voltage was applied to the input of a resonance

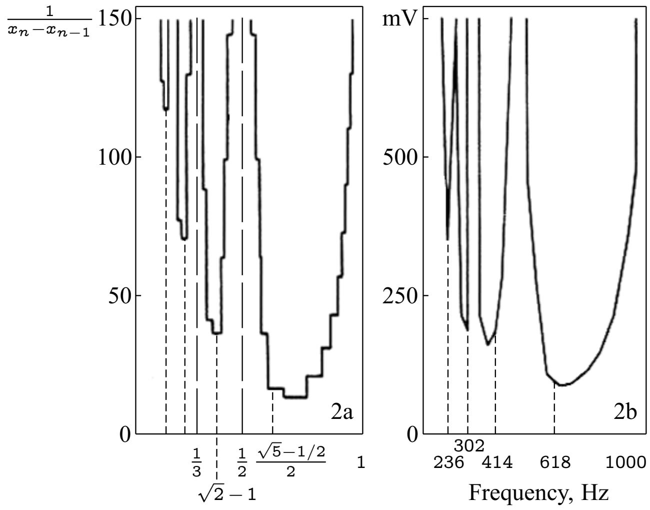

Fig. 2: The rational numbers distribution.

amplifier (an active LC-filter having a frequency of 1 kHz and the quality $Q = 17$). The frequency at the input was varied within the interval 200–1000 Hz through steps of 25 Hz. The average numerical value of the outgoing voltage was measured for two different voltages of the incoming signal — 0.75V and 1.25V.

The apexes of both functions shown in the diagrams are located at the points plotted by the fractions $1/n$. This fact is trivial, because both apexes are actually analogous to Fourier-series expansions of white noise. Such experimental diagrams could be obtained in a purely theoretical way.

Much more interesting is the problem of the minimum numerical values of both functions. The classical theory of oscillations predicts that the minimum points should coincide with the minimum amplitude of forced oscillations, while according to the theory of continued fractions the minimum points should coincide with irrational numbers which, being the roots of the equation $x^2 \pm px - 1 = 0$ for all p, are approximated by rational numbers less accurately than by other numbers [2].

Direct calculations give the following numbers

$$M_1 = \frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}} = \frac{\sqrt{5}\mp 1}{2} = (0.6180339\ldots)^{\pm 1},$$

$$M_2 = \frac{1}{2+\frac{1}{2+\frac{1}{2+\ldots}}} = \frac{\sqrt{8}\mp 1}{2} = (0.4142135\ldots)^{\pm 1},$$

....

$$M_n = \frac{1}{n+\frac{1}{n+\frac{1}{n+\ldots}}} = \frac{\sqrt{n^2+4}\mp n}{2}\,.$$

In other words, the first conclusion is that the distribution of rational numbers, represented by continued fractions with

Table 1: Orbital radii of planets in the solar system in comparison with the calculated values of the radii $R_k = \left(\sqrt{n^2+4} \mp n\right)/2$

Planet	Real R_k	Calculated R_k	n	$\frac{R_{k(\mathrm{calc})}}{R_{k(\mathrm{real})}}$
Mercury	0.0744	0.0765	−13	1.0282
Venus	0.1390	0.1401	−7	1.0079
Earth	0.1922	0.1926	−5	1.0021
Mars	0.2929	0.3028	−3	1.0338
Asteroids	0.6180	0.6180	−1	1.0000
Jupiter	1.0000	1.0000	0	1.0000
Saturn	1.8334	1.6180	1	0.8825
Uranus	3.6883	3.3028	3	0.8955
Neptune	5.7774	5.1926	5	0.8988
Pluto	7.6398	7.1401	7	0.9346

Table 2: Orbital periods of planets in the solar system in comparison with the calculated values of the periods $T_k = \left(\sqrt{n^2+4} \mp n\right)/2$

Planet	Real T_k	Calculated T_k	n	$\frac{T_{k(\mathrm{calc})}}{T_{k(\mathrm{real})}}$
Mercury	0.0203	0.0203	−49	1.0000
Venus	0.0519	0.0524	−19	1.0096
Earth	0.0843	0.0828	−12	0.9822
Mars	0.1586	0.1623	−6	1.0233
Asteroids	0.4877	0.4142	−2	0.8493
Jupiter	1.0000	1.0000	0	1.0000
Saturn	2.4834	2.4142	2	0.9721
Uranus	7.0827	7.1378	7	1.0077
Neptune	13.8922	14.0711	13	1.0129
Pluto	21.1166	21.0475	21	0.9967

Note: Here the measurement units are the orbital radius and period of Jupiter. For asteroids the overall average orbit is taken, its radius 3.215 astronomical units and period 5.75 years are the average values between asteroids.

a limited number of elements, takes its minimum density at the points of a unit interval on number line as shown by the aforementioned numbers. The second conclusion is that if these numbers express ratios between interacting frequencies, the amplitude of the forced oscillations takes its minimum numerical value.

It is evident that an oscillating system, where the oscillation parameters undergo changes due to interactions inside the system, will be maximally stable in that case where the forced oscillation amplitude will be a minimum.

The simplest verification of this thesis is given by the solar system. As we know it Laplace's classic works, the whole solar system (the planet orbits on the average) are stable under periodic gravitational perturbations only if the ratios between the orbital parameters are expressed by irrational numbers. If we will take this problem forward, proceeding from the viewpoint proposed above, the ratios between the orbital periods T_k/T_0 or, alternatively, the ratios between their functions (the average orbital radii R_k/R_0) will be close to those numbers that correspond to the minima of the rational numbers density on number line

$$\frac{T_k}{T_0}; \frac{R_k}{R_0} \approx M_n = \frac{\left(\sqrt{n^2+4} \mp n\right)}{2}.$$

The truth or falsity of this can be decided by using Table 1 and Table 2.

As a matter of fact, all that has been said on the distribution of rational numbers on a unit interval could be extrapolated for the entire number line (proceeding from the above mentioned concept).

All that has been said gives a possibility to formulate the next conclusions:

1. Rational numbers having limited numerator and denominator are distributed inhomogeneously along the number line;
2. Oscillating systems, having a peculiarity to change their own parameters because of interactions inside the systems, have a tendency to reach a stable state where the separate oscillators frequencies are interrelated by specific numbers — minima of the rational number density on number line.

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On the General Solution to Einstein's Vacuum Field and Its Implications for Relativistic Degeneracy

Stephen J. Crothers

Australian Pacific College, Lower Ground, 189 Kent St., Sydney, 2000, Australia

E-mail: thenarmis@yahoo.com

The general solution to Einstein's vacuum field equations for the point-mass in all its configurations must be determined in such a way as to provide a means by which an infinite sequence of particular solutions can be readily constructed. It is from such a solution that the underlying geometry of Einstein's universe can be rightly explored. I report here on the determination of the general solution and its consequences for the theoretical basis of relativistic degeneracy, i. e. gravitational collapse and the black hole.

1 Introduction

A serious misconception prevails that the so-called "Schwarzschild solution" is a solution for the vacuum field. Not only is this incorrect, it is not even Schwarzschild's solution. The aforesaid solution was obtained by David Hilbert [1], a full year after Karl Schwarzschild [2] obtained his original solution. Moreover, Hilbert's metric is a corruption of the solution first found by Johannes Droste [3], and subsequently by Hermann Weyl [4] by a different method.

The orthodox concepts of gravitational collapse and the black hole owe their existence to a confusion as to the true nature of the r-parameter in the metric tensor for the gravitational field.

The error in the conventional analysis of Hilbert's solution is twofold in that two tacit and invalid assumptions are made:

(a) r is a coordinate and radius (of some kind) in the gravitational field;

(b) The regions $0 < r < \alpha = 2m$ and $\alpha < r < \infty$ are valid.

Contrary to the conventional analysis the nature and range or the r-parameter must be determined by rigorous mathematical means, *not* by mere assumption, tacit or otherwise. When the required mathematical rigour is applied it is revealed that $r_0 = \alpha$ denotes a point, not a 2-sphere, and that $0 < r < \alpha$ is undefined on the Hilbert metric. The consequence of this is that gravitational collapse, if it occurs in Nature at all, cannot produce a relativistic black hole under any circumstances. Since the Michell-Laplace dark body is not a black hole either, there is no theoretical basis for it whatsoever. Furthermore, the conventional conception of gravitational collapse is demonstrably false.

The sought for general solution must not only result in a means for construction of an infinite sequence of particular solutions, it must also naturally produce the solutions due to Schwarzschild, Droste and Weyl, and M. Brillouin [5]. To obtain the general solution the general conditions that the required solution must satisfy must be established. Abrams [9] has determined these conditions. I obtain them by other arguments, and therefrom construct the general solution, from which the original Schwarzschild solution, the Droste/Weyl solution, and the Brillouin solution all arise quite naturally. It will be evident that the black hole is theoretically unsound. Indeed, it never arose in the solutions of Schwarzschild, Droste and Weyl, and Brillouin. It comes solely from the mathematically inadmissible assumptions conventionally imposed upon the Hilbert metric.

I provide herein a derivation of the general solution for the simple point-mass and briefly discuss its geometry. Although I have obtained the complete solution up to the rotating point-charge I reserve its derivation to a subsequent paper and similarly a full discussion of the geometry to a third paper. However, I include the expression for the overall general solution as a prelude to my following papers.

2 The general solution for the simple point-mass and its basic geometry

A general metric for the static, time-symmetric, cento-symmetric configuration of energy or matter in quasi-Cartesian coordinates is,

$$\begin{aligned} ds^2 &= L(r)dt^2 - M(r)(dx^2 + dy^2 + dz^2) - \\ &\quad - N(r)(xdx + ydy + zdz)^2, \\ r &= \sqrt{x^2 + y^2 + z^2}, \end{aligned} \qquad (1)$$

where, $\forall\, t$, L, M, N are analytic functions such that,

$$L, M, N > 0\,. \qquad (2)$$

In polar coordinates (1) becomes,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (3)$$

where analytic $A, B, C > 0$ owing to (2).

Transform (3) by setting

$$r^* = \sqrt{C(r)}\,, \tag{4}$$

then

$$ds^2 = A^*(r^*)dt^2 - B^*(r^*)dr^{*2} - r^{*2}(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{5}$$

from which one obtains in the usual way,

$$ds^2 = \left(\frac{r^* - \alpha}{r^*}\right)dt^2 - \left(\frac{r^*}{r^* - \alpha}\right)dr^{*2} - r^{*2}(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{6}$$

Substituting (4) gives

$$ds^2 = \left(\frac{\sqrt{C} - \alpha}{\sqrt{C}}\right)dt^2 - \left(\frac{\sqrt{C}}{\sqrt{C} - \alpha}\right)\frac{C'^2}{4C}dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{7}$$

Thus, (7) is a general metric in terms of one unknown function $C(r)$. The following arguments are coordinate independent since $C(r)$ in (7) is an arbitrary function.

The general metric for Special Relativity is,

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2 d\varphi^2\right), \tag{8}$$

and the radial distance (the *proper distance*) between two *points* is,

$$d = \int_{r_0}^{r} dr = r - r_0\,. \tag{9}$$

Let a test particle be located at each of the points r_0 and $r > r_0$ (owing to the isotropy of space there is no loss of generality in taking $r \geqslant r_0 \geqslant 0$). Then by (9) the distance between them is given by

$$d = r - r_0\,,$$

and if $r_0 = 0,\ d \equiv r$ in which case the distance from $r_0 = 0$ is the same as the radius (the *curvature radius*) of a great circle, the circumference χ of which is from (8),

$$\chi = 2\pi\sqrt{r^2} = 2\pi r\,. \tag{10}$$

In other words, the curvature radius and the proper radius are identical, owing to the pseudo-Euclidean nature of (8). Furthermore, d gives the radius of a sphere centred at the *point* r_0. Let the test particle at r_0 acquire mass. This produces a gravitational field centred at the *point* $r_0 \geqslant 0$. The geometrical relations between the components of the metric tensor of General Relativity must be precisely the same in the metric of Special Relativity. Therefore the distance between r_0 and $r > r_0$ is no longer given by (9) and the curvature radius no longer by (10). Indeed, the proper radius R_p, in keeping with the geometrical relations on (8), is now given by,

$$R_p = \int_{r_0}^{r} \sqrt{-g_{11}}dr\,, \tag{11}$$

where from (7),

$$-g_{11} = \left(1 - \frac{\alpha}{\sqrt{C(r)}}\right)^{-1} \frac{[C'(r)]^2}{4C(r)}\,. \tag{12}$$

Equation (11) with (12) gives the mapping of d from the flat spacetime of Special Relativity into the curved spacetime of General Relativity, thus,

$$R_p(r) = \int \sqrt{\frac{\sqrt{C}}{\sqrt{C} - \alpha}\,\frac{C'}{2\sqrt{C}}}\,dr = \sqrt{\sqrt{C(r)}\left(\sqrt{C(r)} - \alpha\right)} + \alpha\ln\left|\frac{\sqrt{\sqrt{C(r)}} + \sqrt{\sqrt{C(r)} - \alpha}}{K}\right|, \quad K = \text{const}\,. \tag{13}$$

The relationship between r and R_p is

$$r \to r_0 \Rightarrow R_p \to 0\,,$$

so from (13) it follows,

$$r \to r_0 \ \Rightarrow C(r_0) = \alpha^2,\ K = \sqrt{\alpha}\,.$$

So (13) becomes,

$$R_p(r) = \sqrt{\sqrt{C(r)}\left(\sqrt{C(r)} - \alpha\right)} + \alpha\ln\left|\frac{\sqrt{\sqrt{C(r)}} + \sqrt{\sqrt{C(r)} - \alpha}}{\sqrt{\alpha}}\right|. \tag{14}$$

Therefore (7) is singular only at $r = r_0$, where $C(r_0) = \alpha^2$ and $g_{00} = 0\ \forall\ r_0$, irrespective of the value of r_0. $C(r_0) = \alpha^2$ emphasizes the true meaning of α, viz., α is a scalar invariant which fixes the spacetime for the point-mass from an infinite number of mathematically possible forms, as pointed out by Abrams. Moreover, α embodies the effective gravitational mass of the source of the field, and fixes a boundary to an otherwise incomplete spacetime. Furthermore, one can see from (13) and (14) that r_0 is arbitrary, i.e. the point-mass can be located at any point and its location has no intrinsic meaning. Furthermore, the condition $g_{00} = 0$ is clearly equivalent to the boundary condition $r \to r_0 \Rightarrow R_p \to 0$, from which it follows that $g_{00} = 0$ is the *end result* of gravitational collapse. There exists no value of r making $g_{11} = 0$.

If $C' = 0$ for $r > r_0$ the structure of (7) is destroyed: $g_{11} = 0$ for $r > r_0 \Rightarrow B(r) = 0$ for $r > r_0$ in violation of (3). Therefore $C' \neq 0$. For (7) to be spatially asymptotically flat,

$$\lim_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1\,. \tag{15}$$

Since $C(r)$ must behave like $(r - r_o)^2$ and make (7) singular only at $r = r_0$, $C(r)$ must be a strictly monotonically increasing function. Then by virtue of (15) and the fact that $C' \neq 0$, it follows that $C' > 0$ for $r > r_0$. Thus the necessary conditions that must be imposed upon $C(r)$ to render a solution to (3) are:

1. $C'(r) > 0$ for $r > r_0$;
2. $\lim\limits_{r\to\infty} \dfrac{C(r)}{\left(r - r_0\right)^2} = 1$;
3. $C(r_0) = \alpha^2$.

I call the foregoing the Metric Conditions of Abrams for the point-mass (MCA) since when $r_0 = 0$ they are precisely the conditions he determined by his use of (3) and the field equations. In addition to MCA any admissible function $C(r)$ must reduce (7) to the metric of Special Relativity when $\alpha = 2m = 0$.

The invalid conventional assumptions that $0 < r < \alpha$ and that r is a radius of sorts in the gravitational field lead to the incorrect conclusion that $r = \alpha$ is a 2-sphere in the gravitational field of the point-mass. The quantity $r = \alpha$ does not describe a 2-sphere; it does not yield a Schwarzschild sphere; it is actually a *point*. Stavroulakis [10, 8, 9] has also remarked upon the true nature of the r-parameter (coordinate radius). Since MCA must be satisfied, admissible systems of coordinates are restricted to a particular (infinite) class. To satisfy MCA, and therefore (3), and (7), the form that $C(r)$ can take must be restricted to,

$$C_n(r) = \left[(r - r_0)^n + \alpha^n\right]^{\frac{2}{n}}\,, \tag{16}$$

$$r_0 \in (\Re - \Re^-),\ n \in \Re^+,$$

where n and r_0 are arbitrary. I call equations (16) Schwarzschild forms. The value of n in (16) fixes a set of coordinates, and the infinitude of such reflects the fact that no set of coordinates is privileged in General Relativity.

The general solution for the simple point-mass is therefore,

$$ds^2 = \left(\frac{\sqrt{C_n} - \alpha}{\sqrt{C_n}}\right) dt^2 - \left(\frac{\sqrt{C_n}}{\sqrt{C_n} - \alpha}\right) \frac{C_n'^{\,2}}{4C_n} dr^2 - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{17}$$

$$C_n(r) = \left[(r - r_0)^n + \alpha^n\right]^{\frac{2}{n}}\,,\ n \in \Re^+,$$

$$r_0 \in (\Re - \Re^-)\,,$$

$$r_0 < r < \infty\,,$$

where n and r_0 are arbitrary. Therefore with r_0 arbitrary, (17) reduces to the metric of Special Relativity when $\alpha = 2m = 0$.

From (17), with $r_0 = 0$ and n taking integer values, the following infinite sequence obtains:

$C_1(r) = (r + \alpha)^2$ (Brillouin's solution)
$C_2(r) = (r^2 + \alpha^2)$
$C_3(r) = (r^3 + \alpha^3)^{\frac{2}{3}}$ (Schwarzschild's solution)
$C_4(r) = (r^4 + \alpha^4)^{\frac{1}{2}}$, etc.

Hilbert's solution is rightly obtained when $r_0 = \alpha$, i. e. when $r_0 = \alpha$ and the values of n take integers, the infinite sequence of particular solutions is then given by,

$C_1(r) = r^2$ [Droste/Weyl/(Hilbert) solution]
$C_2(r) = (r - \alpha)^2 + \alpha^2$,
$C_3(r) = \left[(r - \alpha)^3 + \alpha^3\right]^{\frac{2}{3}}$,
$C_4(r) = \left[(r - \alpha)^4 + \alpha^4\right]^{\frac{1}{2}}$, etc.

The curvature $f = R^{ijkm}R_{ijkm}$ is finite everywhere, including $r = r_0$. Indeed, for metric (17) the Kretschmann scalar is,

$$f = \frac{12\alpha^2}{C_n^3} = \frac{12\alpha^2}{\left[\left(r - r_0\right)^n + \alpha^n\right]^{\frac{6}{n}}}\,. \tag{18}$$

Gravitational collapse does not produce a curvature singularity in the gravitational field of the point-mass. The scalar invariance of $f(r_0) = \frac{12}{\alpha^4}$ is evident from (18).

All the particular solutions of (17) are inextendible, since the singularity when $r = r_0$ is quasiregular, irrespective of the values of n and r_0. Indeed, the circumference χ of a great circle becomes,

$$\chi = 2\pi\sqrt{C(r)}. \tag{19}$$

Then the ratio

$$\lim_{r\to r_0} \frac{\chi}{R_p} \to \infty, \tag{20}$$

shows that $R_p(r_0) \equiv 0$ is a quasiregular singularity and cannot be extended.

Equation (19) shows that $\chi = 2\pi\alpha$ is also a scalar invariant for the point-mass.

It is plain from the foregoing that the Kruskal-Szekeres extension is meaningless, that the "Schwarzschild radius" is meaningless, that the orthodox conception of gravitational collapse is incorrect, and that the black hole is not consistent at all with General Relativity. All arise wholely from a bungled analysis of Hilbert's solution.

3 Implications for gravitational collapse

As is well known the gravitational potential Φ for an arbitrary metric is

$$g_{00} = \left(1 - \Phi\right)^2, \tag{21}$$

from which it is concluded that gravitational collapse occurs at $\Phi = 1$. Physically, the conventional process of collapse involves Newtonian gravitation down to the so-called "gravitational radius". Far from the source, the alleged weak field potential is,

$$\Phi = \frac{m}{r},$$

and so

$$g_{00} = 1 - \frac{\alpha}{r}, \qquad (22)$$

$$\alpha = 2m.$$

The scalar α is conventionally called the "gravitational radius", or the "Schwarzschild radius", or the "event horizon". However, as I have shown, neither α nor the coordinate radius r are radii in the gravitational field. In the case of the Hilbert metric, $r_0 = \alpha$ is a *point*, not a 2-sphere. It is the location of the point-mass. In consequence of this $g_{00} = 0$ is the end result of gravitational collapse. It therefore follows that in the vacuum field,

$$0 < g_{00} < 1, \qquad 1 < |g_{11}| < \infty,$$

$$\alpha < \sqrt{C(r)}.$$

In the case of the Hilbert metric, $C(r) = r^2$, so

$$0 < g_{00} < 1, \qquad 1 < |g_{11}| < \infty,$$

$$\alpha < r.$$

In the case of Schwarzschild's metric we have $C(r) = \left(r^3 + \alpha^3\right)^{\frac{2}{3}}$, so

$$0 < g_{00} < 1, \qquad 1 < |g_{11}| < \infty,$$

$$0 < r.$$

It is unreasonable to expect the weak field potential function to be strictly Newtonian. Only in the infinitely far field is Newton's potential function to be recovered. Consequently, the conventional weak field expression (22) cannot be admitted with the conventional interpretation thereof. The correct potential function must contain the arbitrary location of the point-mass. From (21),

$$\Phi = 1 - \sqrt{g_{00}} = 1 - \sqrt{1 - \frac{\alpha}{\sqrt{C(r)}}},$$

so in the weak far field,

$$\Phi \approx 1 - \left(1 - \frac{\alpha}{2\sqrt{C}}\right) = \frac{m}{\sqrt{C}},$$

and so

$$g_{00} = 1 - \frac{\alpha}{\sqrt{C(r)}} = 1 - \frac{\alpha}{\left[(r - r_0)^n + \alpha^n\right]^{\frac{1}{n}}}, \qquad (23)$$

$$r_0 \in (\Re - \Re^-),\ n \in \Re^+.$$

Then

$$\text{as } r \to \infty,\ g_{00} \to 1 - \frac{\alpha}{r - r_0},$$

and Newton is recovered at infinity.

According to (23), at $r = r_0$, $g_{00} = 0$ and $\Phi = \frac{1}{2}$. The weak field potential approaches a finite maximum of $\frac{1}{2}$ (i. e. $\frac{1}{2}c^2$), in contrast to Newton's potential. The conventional concept of gravitational collapse at $r_s = \alpha$ is therefore meaningless.

Similarly, it is unreasonable to expect Kepler's 3rd Law to be unaffected by general relativity, contrary to the conventional analysis. Consider the Lagrangian,

$$\begin{aligned} L &= \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{d\tau}\right)^2\right] - \\ &- \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2\right] - \\ &- \frac{1}{2}\left[C_n\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right)\right], \\ C_n(r) &= \left[\left(r - r_0\right)^n + \alpha^n\right]^{\frac{2}{n}},\ n \in \Re^+, \\ r_0 &\in (\Re - \Re^-),\ r_0 < r < \infty, \end{aligned} \qquad (24)$$

where τ is the proper time.

Restricting motion, without loss of generality, to the equatorial plane, $\theta = \frac{\pi}{2}$, the Euler-Lagrange equations for (24) are,

$$\begin{aligned} &\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\frac{d^2\sqrt{C_n}}{d\tau^2} + \frac{\alpha}{2C_n}\left(\frac{dt}{d\tau}\right)^2 - \\ &-\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-2}\frac{\alpha}{2C_n}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - \sqrt{C_n}\left(\frac{d\varphi}{d\tau}\right)^2 = 0, \end{aligned} \qquad (25)$$

$$\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\frac{dt}{d\tau} = \text{const} = k, \qquad (26)$$

$$C_n\frac{d\varphi}{d\tau} = \text{const} = h, \qquad (27)$$

and $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ becomes,

$$\begin{aligned} &\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)\left(\frac{dt}{d\tau}\right)^2 - \\ &-\left(1 - \frac{\alpha}{\sqrt{C_n}}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - C_n\left(\frac{d\varphi}{d\tau}\right)^2 = 1. \end{aligned} \qquad (28)$$

Using the foregoing equations it readily follows that the angular velocity is,

$$\omega = \sqrt{\frac{\alpha}{2C_n^{\frac{3}{2}}}}. \qquad (29)$$

Then,

$$\lim_{r \to r_0} \omega = \frac{1}{\alpha\sqrt{2}} \tag{30}$$

is a scalar invariant which shows that the angular velocity approaches a finite limit, in contrast to Newton's theory where it becomes unbounded. Schwarzschild obtained this result for his particular solution. Equation (29) is the General Relativistic modification of Kepler's 3rd Law.

For a falling particle in a true Schwarzschild field,

$$d\tau = \sqrt{g_{00}}\, dt = \sqrt{1 - \frac{\alpha}{\sqrt{C(r)}}}\, dt\,.$$

Therefore, as a neutral test particle approaches the field source at r_0 along a radial geodesic, $d\tau \to 0$. Thus, according to an external observer, it takes an infinite amount of coordinate time for a test particle to reach the source. Time stops at the Schwarzschild point-mass. The conventional concepts of the Schwarzschild sphere and its interior are meaningless.

Doughty [10] has shown that the acceleration of a test particle approaching the point-mass along a radial geodesic is given by,

$$a = \frac{\sqrt{-g_{11}}\left(-g^{11}\right)\left|g_{00,1}\right|}{2g_{00}}. \tag{31}$$

By (17),

$$a = \frac{\alpha}{2C^{\frac{3}{4}}\left(\sqrt{C} - \alpha\right)^{\frac{1}{2}}}\,.$$

Clearly, as $r \to r_0$, $a \to \infty$, independently of the value of r_0. In the case of $C(r) = r^2$, where $r_0 = \alpha$,

$$a = \frac{\alpha}{2r^{\frac{3}{2}}\sqrt{r - \alpha}}\,, \tag{32}$$

so $a \to \infty$ as $r \to r_0 = \alpha$.

Applying (31) to the Kruskal-Szekeres extension gives rise to the absurdity of an infinite acceleration at $r = \alpha$ where it is conventionally claimed that there is no matter and no singularity. It is plainly evident that gravitational collapse terminates at a Schwarzschild simple point-mass, not in a black hole. Also, one can readily see that the alleged interchange of the spatial and time coordinates "inside" the "Schwarzschild sphere" is nonsensical. To amplify this, in (17), suppose $\sqrt{C(r)} < \alpha$, then

$$ds^2 = -\left(\frac{\alpha}{\sqrt{C}} - 1\right) dt^2 + \left(\frac{\alpha}{\sqrt{C}} - 1\right)^{-1} \frac{C'^2}{4C} dr^2 - C\left(d\theta^2 + \sin^2 d\varphi^2\right). \tag{33}$$

Let $r = \tilde{t}$ and $t = \tilde{r}$, then

$$ds^2 = \left(\frac{\alpha - \sqrt{C}}{\sqrt{C}}\right)^{-1} \frac{\dot{C}^2}{4C} d\tilde{t}^2 - \left(\frac{\alpha - \sqrt{C}}{\sqrt{C}}\right) d\tilde{r}^2 - C(\tilde{t})\left(d\theta^2 + \sin^2 d\varphi^2\right). \tag{34}$$

This is a time dependent metric which does not have any relationship to the original static problem. It does not extend (17) at all, as also noted by Brillouin in the particular solution given by him. Equation (34) is meaningless.

It is noteworthy that Hagihara [11] has shown that all geodesics that do not run into the Hilbert boundary at $r_0 = \alpha$ are complete. His result is easily extended to any $r_0 \geqslant 0$ in (17).

The correct conclusion is that gravitational collapse terminates at the point-mass without the formation of a black hole in all general relativistic circumstances.

4 Generalization of the vacuum solution for charge and angular momentum

The foregoing analysis can be readily extended to include the charged and rotating point-mass. In similar fashion it follows that the Reissner-Nordstrom, Carter, Graves-Brill, Kerr, and Kerr-Newman black holes are all inconsistent with General Relativity.

In a subsequent paper I shall derive the following overall general solution for the point-mass when $\Lambda = 0$,

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(C_n + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}\frac{C_n'^{\,2}}{4C_n} dr^2 - \rho^2 d\theta^2\,,$$

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}},\ r_0 \in \left(\Re - \Re^-\right),$$

$$n \in \Re^+,\ a = \frac{L}{m},\ \rho^2 = C_n + a^2\cos^2\theta\,,$$

$$\Delta = C_n - \alpha\sqrt{C_n} + q^2 + a^2\,,$$

$$\beta = m + \sqrt{m^2 - q^2 - a^2\cos^2\theta},\ a^2 + q^2 < m^2,$$

$$r_0 < r < \infty\,.$$

The different configurations for the point-mass are easily extracted from this set of equations by the setting of the values of the parameters in the obvious way.

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

Epilogue

My interest in the problem of the black hole was aroused by coming across the papers of the American physicist Leonard S. Abrams, and subsequently to the original papers of Schwarzschild, Droste, Weyl, Hilbert, and Brillouin. I was drawn to the logic of Abrams' approach in his determination of the required metric in terms of a single generalised function and the conditions that this function must satisfy to render a solution for the point-mass. It was not until I read Abrams that I became aware of the startling facts that the "Schwarzschild solution" is not due to Schwarzschild, that Schwarzschild did not predict the black hole and made none of the claims about black holes that are invariably attributed to him in the textbooks and almost invariably in the literature. These facts alone give cause for disquiet and reading of the original papers gives cause for serious concern about how modern science is reported.

Dr. Leonard S. Abrams was born in Chicago in 1924 and died on December 28, 2001, in Los Angeles at the age of 77. He received a B. S. in Mathematics from the California Institute of Technology and a Ph. D. in physics from the University of California at Los Angeles at the age of 45. He spent almost all of his career working in the private sector, although he taught at a variety of institutions including California State University at Dominguez Hills and at the University of Southern California. He was a pioneer in applying game theory to business problems and was an expert in noise theory, but his first love always was general relativity. His principle theoretical contributions focused on non-black hole solutions to Einstein's equations and on the inextendability of the "Schwarzschild" solution. Dr. Abrams is survived by his wife and two children.

Dr. Abrams encountered great resistance to publication of his work on General Relativity. Nonetheless he continued with his work and managed to publish several important papers despite the obstacles placed in his way by the mainstream authorities.

I extend my thanks to Diana Abrams for providing me with information about her late husband.

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On the Ramifications of the Schwarzschild Space-Time Metric

Stephen J. Crothers

Australian Pacific College, Lower Ground, 189 Kent St., Sydney, 2000, Australia

E-mail: thenarmis@yahoo.com

In a previous paper I derived the general solution for the simple point-mass in a true Schwarzschild space. I extend that solution to the point-charge, the rotating point-mass, and the rotating point-charge, culminating in a single expression for the general solution for the point-mass in all its configurations when $\Lambda = 0$. The general exact solution is proved regular everywhere except at the arbitrary location of the source of the gravitational field. In no case does the black hole manifest. The conventional solutions giving rise to various black holes are shown to be inconsistent with General Relativity.

1 Introduction

In a previous paper [1] I showed that the general solution of the vacuum field for the simple point-mass is regular everywhere except at the arbitrary location of the source of the field, $r = r_0$, $r_0 \in (\Re - \Re^-)$, where there is a quasiregular singularity. I extend herein the general solution to the rotating and charged configurations of the point-mass and show that they too are regular everywhere except at $r = r_0$, obviating the formation of the Reissner-Nordstrom, Kerr, and Kerr-Newman black holes. Consequently, there is no basis in General Relativity for the black hole.

The sought for complete solution for the point-mass must reduce to the general solution for the simple point-mass in a natural way, give rise to an infinite sequence of particular solutions in each particular configuration, and contain a scalar invariant which embodies all the factors that contribute to the effective gravitational mass of the field's source for the respective configurations.

2 The vacuum field of the point-charge

The general metric, in polar coordinates, for the vacuum field is, in relativistic units,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad (1)$$

where analytic $A, B, C > 0$. The general solution to (1) for the simple point-mass is,

$$ds^2 = \left[\frac{(\sqrt{C_n} - \alpha)}{\sqrt{C_n}}\right] dt^2 - \left[\frac{\sqrt{C_n}}{(\sqrt{C_n} - \alpha)}\right] \frac{C_n'^{\,2}}{4C_n} dr^2 - C_n(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad (2)$$

$$C_n(r) = \left[\left(r - r_0\right)^n + \alpha^n\right]^{\frac{2}{n}}, \; \alpha = 2m\,, \;\; r_0 \in (\Re - \Re^-)\,,$$

$$n \in \Re^+\,, \quad r_0 < r < \infty\,,$$

where $C_n(r)$ satisfies the Metric conditions of Abrams (MCA) [2]* for the simple point-mass,

1. $C_n'(r) > 0$, $r > r_0$;
2. $\lim\limits_{r\to\infty} \dfrac{C_n(r)}{\left(r - r_0\right)^2} = 1$;
3. $C_n(r_0) = \alpha^2$.

The Reissner-Nordstrom [3] solution is,

$$ds^2 = \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \quad (3)$$

which is conventionally taken to be valid for all $\frac{q^2}{m^2}$. It is also alleged that (3) can be extended down to $r = 0$, giving rise to the so-called Reissner-Nordstrom black hole. These conventional allegations are demonstrably false.

The conventional analysis simply looks at (3) and makes two mathematically invalid assumptions, viz.,

1. *The parameter r is a radius of some kind in the gravitational field*;
2. *r down to $r = 0$ is valid.*

The nature and range of the r-parameter must be established by mathematical rigour, *not* by mere assumption.

Transform (1) by the substitution

$$r^* = \sqrt{C(r)}. \quad (4)$$

*Abrams' equation (A.1) should read:

$$-8\pi T_1^1 = \frac{-1}{C} + \frac{C'^2}{4BC^2} + \frac{A'C'}{2ABC} = 0\,,$$

and his equation (A.6),

$$\frac{2C''}{C'} - [\ln(ABC)]' = 0\,.$$

The errors are apparently escapees from the proof reading.

Equation (4) carries (1) into

$$ds^2 = A^*(r^*)dt^2 - B^*(r^*)dr^{*2} - r^{*2}(d\theta^2 + \sin^2\theta d\varphi^2)\,. \quad (5)$$

Using (5) to determine the Maxwell stress-energy tensor, and substituting the latter into the Einstein-Maxwell field equations in the usual way, yields,

$$\begin{aligned} ds^2 \;&= \left(1 - \frac{\alpha}{r^*} + \frac{q^2}{r^{*2}}\right) dt^2 - \\ &- \left(1 - \frac{\alpha}{r^*} + \frac{q^2}{r^{*2}}\right)^{-1} dr^{*2} - r^{*2}(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \quad (6)$$

Substituting (4) into (6),

$$\begin{aligned} ds^2 \;&= \left(1 - \frac{\alpha}{\sqrt{C}} + \frac{q^2}{C}\right) dt^2 - \left(1 - \frac{\alpha}{\sqrt{C}} + \frac{q^2}{C}\right)^{-1} \times \\ &\times \frac{C'^2}{4C} dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \quad (7)$$

The proper radius R_p on (1) is,

$$R_p(r) = \int \sqrt{B(r)}\, dr\,. \quad (8)$$

The parameter r therefore does not lie in the spacetime M_q of the point-charge.

Taking $B(r)$ from (7) into (8) gives the proper distance in M_q,

$$\begin{aligned} R_p(r) &= \int \left(1 - \frac{\alpha}{\sqrt{C(r)}} + \frac{q^2}{C(r)}\right)^{-\frac{1}{2}} \frac{C'(r)}{2\sqrt{C(r)}} dr = \\ &= \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2} + \\ &+ m \ln \left| \frac{\sqrt{C(r)} - m + \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2}}{K} \right|, \end{aligned} \quad (9)$$

$$K = \text{const.}$$

The valid relationship between r and $R_p(r)$ is,

$$\text{as} \;\; r \to r_0\,, \quad R_p(r) \to 0\,,$$

so by (9),

$$r \to r_0 \;\Rightarrow \sqrt{C(r_0)} = m \pm \sqrt{m^2 - q^2}\,,$$

$$K = \pm\sqrt{m^2 - q^2}\,.$$

When $q = 0$, (9) must reduce to the Droste/Weyl [4, 5] solution, so it requires,

$$\sqrt{C(r_0)} = m + \sqrt{m^2 - q^2}\,. \quad (10)$$

Then by (9),

$$K = \sqrt{m^2 - q^2}\,,\; q^2 < m^2\,. \quad (11)$$

Clearly, r_0 is the lower bound on r.

Putting (11) into (9) gives,

$$\begin{aligned} R_p(r) &= \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2} + \\ &+ m \ln \left| \frac{\sqrt{C(r)} - m + \sqrt{C(r) - \alpha\sqrt{C(r)} + q^2}}{\sqrt{m^2 - q^2}} \right|. \end{aligned} \quad (12)$$

Equation (7) is therefore singular *only* when $r = r_0$ in which case $g_{00} = 0$. Hence, the condition $r \to r_0 \Rightarrow R_p \to 0$ is equivalent to $r = r_0 \Rightarrow g_{00} = 0$.

If $C' = 0$ the structure of (7) is destroyed, since $g_{11} = 0 \;\forall\; r > r_0 \Rightarrow B(r) = 0 \;\forall\; r > r_0$ in violation of (1). Therefore $C'(r) \neq 0$ for $r > r_0$.

For (7) to be asymptotically flat,

$$r \to \infty \Rightarrow \frac{C(r)}{\left(r - r_0\right)^2} \to 1\,. \quad (13)$$

Therefore,

$$\lim_{r \to \infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1\,. \quad (14)$$

Since $C(r)$ behaves like $\left(r - r_0\right)^2$, must make (7) singular only at $r = r_0$, and $C'(r) \neq 0$ for $r > r_0$, $C(r)$ is strictly monotonically increasing, therefore, $C'(r) > 0$ for $r > r_0$. Thus, to satisfy (1) and (7), $C(r)$ must satisfy,

1. $C'(r) > 0$, $r > r_0$;
2. $\lim_{r \to \infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1$;
3. $\sqrt{C(r_0)} = \beta = m + \sqrt{m^2 - q^2}$, $q^2 < m^2$.

I call the foregoing the Metric Conditions of Abrams (MCA) for the point-charge. Abrams [6] obtained them by a different method — using (1) and the field equations directly.

In the absence of charge (7) must reduce to the general Schwarzschild solution for the simple point-mass (2). The only functions that satisfy this requirement and MCA are,

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}}\,,$$

$$\beta = m + \sqrt{m^2 - q^2}, \quad q^2 < m^2,$$

$$n \in \Re^+, \quad r_0 \in (\Re - \Re^-)\,,$$

where n and r_0 are arbitrary. Therefore, the general solution for the point-charge is,

$$\begin{aligned} ds^2 \;&= \left(1 - \frac{\alpha}{\sqrt{C}} + \frac{q^2}{C}\right) dt^2 - \left(1 - \frac{\alpha}{\sqrt{C}} + \frac{q^2}{C}\right)^{-1} \times \\ &\times \frac{C'^2}{4C} dr^2 - C(d\theta^2 + \sin^2\theta d\varphi^2)\,, \end{aligned} \quad (15)$$

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}},$$

$$\beta = m + \sqrt{m^2 - q^2}, \quad q^2 < m^2,$$

$$n \in \Re^+, \quad r_0 \in (\Re - \Re^-),$$

$$r_0 < r < \infty.$$

When $n = 1$ and $r_0 = 0$, Abrams' [6] solution for the point-charge results.

Equation (15) is regular $\forall\ r > r_0$. There is no event horizon and therefore no Reissner-Nordstrom black hole. Furthermore, the Graves-Brill black hole and the Carter black hole are also invalid.

By (15) the correct rendering of (3) is,

$$ds^2 = \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{16}$$

$$q^2 < m^2, \quad m + \sqrt{m^2 - q^2} < r < \infty,$$

so Nordstrom's assumption that $\sqrt{C(0)} = 0$ is invalid.

The scalar curvature $f = R_{ijkm}R^{ijkm}$ for (1) with charge included is,

$$f = \frac{8\left[6\left(m\sqrt{C} - q^2\right)^2 + q^4\right]}{C^4}.$$

Using (15) the curvature is,

$$f = \frac{8\left[6\left(m\left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{1}{n}} - q^2\right)^2 + q^4\right]}{\left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{8}{n}}}.$$

The curvature is always finite, even at r_0. No curvature singularity can arise in the gravitational field of the point-charge. Furthermore,

$$f(r_0) = \frac{8\left[6\left(m\beta - q^2\right)^2 + q^4\right]}{\beta^8},$$

where $\beta = m + \sqrt{m^2 - q^2}$. Thus, $f(r_0)$ is a scalar invariant for the point-charge. When $q = 0$, $f(r_0) = \frac{12}{\alpha^4}$, which is the scalar curvature invariant for the simple point-mass.

From (15) the circumference χ of a great circle is given by,

$$\chi = 2\pi\sqrt{C(r)}.$$

The proper radius is given by (12). Then the ratio $\frac{\chi}{R_p} > 2\pi$ for finite r and,

$$\lim_{r\to\infty} \frac{\chi}{R_p} = 2\pi,$$

$$\lim_{r\to r_0} \frac{\chi}{R_p} \to \infty,$$

which shows that $R_p(r_0)$ is a quasiregular singularity and cannot be extended.

Consider the Lagrangian,

$$L = \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)\left(\frac{dt}{d\tau}\right)^2\right] - \frac{1}{2}\left[\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2\right] - \frac{1}{2}\left[C_n\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right)\right]. \tag{17}$$

Restricting motion to the equatorial plane without loss of generality, the Euler-Lagrange equations from (17) are,

$$\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)\frac{d^2\sqrt{C_n}}{d\tau^2} + \left(\frac{\alpha}{2C_n} - \frac{q^2}{C_n^{\frac{3}{2}}}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{\alpha}{2C_n} - \frac{q^2}{C_n^{\frac{3}{2}}}\right)\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)^{-2}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - \sqrt{C_n}\left(\frac{d\varphi}{d\tau}\right)^2 = 0, \tag{18}$$

$$\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)\frac{dt}{d\tau} = k = \text{const}, \tag{19}$$

$$C_n\frac{d\varphi}{d\tau} = h = \text{const}. \tag{20}$$

Also, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ becomes,

$$\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right)^{-1}\left(\frac{d\sqrt{C_n}}{d\tau}\right)^2 - C_n\left(\frac{d\varphi}{d\tau}\right)^2 = 1. \tag{21}$$

It follows from these equations that the angular velocity ω of a test particle is,

$$\omega^2 = \left(\frac{\alpha}{2C_n^{\frac{3}{2}}} - \frac{q^2}{C_n^2}\right) = \left[\frac{\alpha}{2\left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{3}{n}}} - \frac{q^2}{\left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{4}{n}}}\right]. \tag{22}$$

Then,

$$\lim_{r\to r_0} \omega = \sqrt{\frac{\alpha}{2\beta^3} - \frac{q^2}{\beta^4}}, \tag{23}$$

where $\beta = m + \sqrt{m^2 - q^2}$, $q^2 < m^2$.

Equation (22) is Kepler's 3rd Law for the point-charge. It obtains the finite limit given in (23), which is a scalar invariant for the point-charge. When $q = 0$, equations (22) and (23) reduce to those for the simple point-mass,

$$\omega = \sqrt{\frac{\alpha}{2C_n^{\frac{3}{2}}}}\,,$$

$$\lim_{r \to r_0} \omega = \frac{1}{\alpha\sqrt{2}}\,.$$

In the case of a photon in circular orbit about the point-charge, (21) yields,

$$\omega^2 = \frac{1}{C_n}\left(1 - \frac{\alpha}{\sqrt{C_n}} + \frac{q^2}{C_n}\right), \tag{24}$$

and (18) yields,

$$\omega^2 = \frac{1}{\sqrt{C_n}}\left(\frac{\alpha}{2C_n} - \frac{q^2}{C_n^{\frac{3}{2}}}\right). \tag{25}$$

Equating the two, denoting the stable photon radial coordinate by r_{ph}, and solving for the curvature radius $\sqrt{C_{ph}} = \sqrt{C_n(r_{(ph)})}$, gives (since when $q = 0$, $\sqrt{C_{ph}} \neq 0$),

$$\sqrt{C_{ph}} = \sqrt{C_n(r_{(ph)})} = \frac{3\alpha + \sqrt{9\alpha^2 - 32q^2}}{4}\,, \tag{26}$$

which is a scalar invariant. In terms of coordinate radii,

$$r_{ph} = \left[\frac{\left(3\alpha + \sqrt{9\alpha^2 - 32q^2}\right)^n}{4^n} - \beta^n\right]^{\frac{1}{n}} + r_0\,, \tag{27}$$

which depends upon the values of n and r_0.

When $q = 0$ equations (26) and (27) reduce to the corresponding equations for the simple point-mass,

$$\sqrt{C_n(r_{ph})} = \frac{3\alpha}{2}, \tag{28}$$

$$r_{ph} = \left[\left(\frac{3\alpha}{2}\right)^n - \alpha^n\right]^{\frac{1}{n}} + r_0\,. \tag{29}$$

The proper radius associated with (28) and (29) is,

$$R_{p(ph)} = \frac{\alpha\sqrt{3}}{2} + \alpha \ln\left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right), \tag{30}$$

which is a scalar invariant for the simple point-mass. Putting (26) into (12) gives the invariant proper radius for a stable photon orbit about the point-charge.

3 The vacuum field of the rotating point-mass

The Kerr solution, in Boyer-Lindquist coordinates and relativistic units is,

$$\begin{gathered} ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ - \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2, \end{gathered} \tag{31}$$

$$a = \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta\,,$$

$$\Delta = r^2 - r\alpha + a^2, \quad 0 < r < \infty\,,$$

where L is the angular momentum.

If $a = 0$, equation (31) reduces to Hilbert's [7] solution for the simple point-mass,

$$\begin{gathered} ds^2 = \left(1 - \frac{\alpha}{r}\right)dt^2 - \left(1 - \frac{\alpha}{r}\right)^{-1}dr^2 - \\ - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \end{gathered} \tag{32}$$

$$0 < r < \infty\,.$$

However, according to the general formula (2) the correct range for r in (32) is,

$$\sqrt{C(r_0)} < r < \infty\,,$$

where $\sqrt{C(r_0)} = \alpha$. Therefore (32) should be,

$$\begin{gathered} ds^2 = \left(1 - \frac{\alpha}{r}\right)dt^2 - \left(1 - \frac{\alpha}{r}\right)^{-1}dr^2 - \\ - r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \end{gathered} \tag{33}$$

$$\alpha < r < \infty\,.$$

Equation (33) is the Droste/Weyl solution.

Since the r that appears in (32) is the same r appearing in (31) and (33), taking (4) into account, the correct general form of (31) is,

$$\begin{gathered} ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ - \frac{\sin^2\theta}{\rho^2}\left[\left(C + a^2\right)d\varphi - a dt\right]^2 - \\ - \frac{\rho^2}{\Delta}\frac{C'^2}{4C}dr^2 - \rho^2 d\theta^2\,, \end{gathered} \tag{34}$$

$$a = \frac{L}{m}\,, \quad \rho^2 = C + a^2\cos^2\theta\,,$$

$$\Delta = C - \alpha\sqrt{C} + a^2\,, \quad r_0 < r < \infty\,.$$

When $a = 0$, (34) must reduce to (2).

If $C' = 0$ the structure of (34) is destroyed, since then $g_{11} = 0\ \forall\ r > r_0 \Rightarrow B(r) = 0$ in violation of (1). Therefore $C' \neq 0$. Equation (34) must have a global arrow for time, whereupon $g_{00}(r_0 = 0$, so

$$\Delta(r_0) = C(r_0) - \alpha\sqrt{C(r_0)} + a^2 = a^2\sin^2\theta\,. \tag{35}$$

Solving (35) for $\sqrt{C(r_0)}$ gives,

$$\beta = \sqrt{C(r_0)} = m \pm \sqrt{m^2 - a^2\cos^2\theta}\,, \qquad (36)$$

having used $\alpha = 2m$. When $a = 0$ (36) must reduce to the value for Schwarzschild's [8] original solution, i. e. $\sqrt{C(r_0)} = \alpha = 2m$, therefore the plus sign must be taken in (36). Since the angular momentum increases the gravitational mass, and since there can be no angular momentum without mass, $a^2 < m^2$. Thus, there exists no spacetime for $a^2 \geqslant m^2$. To reduce to (2) equation (36) becomes,

$$\beta = \sqrt{C(r_0)} = m + \sqrt{m^2 - a^2\cos^2}\,, \qquad (37)$$

$$a^2 < m^2\,.$$

Equation (34) must be asymptotically flat, so

$$r \to \infty \Rightarrow \frac{C(r)}{\left(r - r_0\right)^2} \to 1\,. \qquad (38)$$

Therefore,

$$\lim_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1\,. \qquad (39)$$

Since $C(r)$ behaves like $\left(r - r_0\right)^2$, must make (34) singular only at $r = r_0$, and $C'(r) > 0 \;\forall\; r > r_0$, $C(r)$ is strictly monotonically increasing, so

$$C'(r) > 0,\; r > r_0\,. \qquad (40)$$

Consequently, the conditions that $C(r)$ must satisfy to render a solution to (34) are:

1. $C'(r) > 0,\;\; r > r_0$;
2. $\lim_{r\to\infty} \frac{C(r)}{\left(r - r_0\right)^2} = 1$;
3. $\sqrt{C(r_0)} = \beta = m + \sqrt{m^2 - a^2\cos^2\theta},\; a^2 < m^2$.

I call the foregoing the Metric Conditions of Abrams (MCA) for the rotating point-mass.

The only form admissible for $C(r)$ in (34) that satisfies MCA and is reducible to (2) is,

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}}, \qquad (41)$$

$$\beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,, \quad a^2 < m^2\,,$$

$$r_0 \in (\Re - \Re^-) \quad n \in \Re^+\,.$$

Associated with (31) are the so-called "horizons" and "static limits" given respectively by,

$$r_h = m \pm \sqrt{m^2 - a^2}\,, \quad r_b = m \pm \sqrt{m^2 - a^2\cos^2\theta}\,, \qquad (42)$$

where r_h is obtained from (31) by setting its $\Delta = 0$, and r_b by setting its $g_{00} = 0$. Conventionally equations (42) are rather arbitrarily restricted to,

$$r_h = m + \sqrt{m^2 - a^2}\,, \quad r_b = m + \sqrt{m^2 - a^2\cos^2\theta}\,, \qquad (43)$$

$$a^2 < m^2\,.$$

For (34), $\Delta \geq 0$ and so there is no static limit, since by (41),

$$C_n(r_0) = \beta^2 \Rightarrow\,, \qquad (44)$$

$$\Rightarrow \Delta(r_0) = \beta^2 - \alpha\beta + a^2\,.$$

Solving (41) i .e .

$$\sqrt{C_n(r)} = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{1}{n}}, \qquad (45)$$

gives the r-parameter location of a spacetime event,

$$r = \left[C_n(r)^{\frac{1}{2n}} - \beta^n\right]^{\frac{1}{n}} + r_0\,. \qquad (46)$$

When $a = 0$, equation (46) reduces to $r_0 = \alpha$, as expected for the non-rotating point-mass.

From (46) it is concluded that there exists no spacetime drag effect for the rotating point-mass and no ergosphere.

The generalisation of equation (34) is then,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(C + a^2\right)d\varphi - a dt\right]^2 - \\ &- \frac{\rho^2}{\Delta}\frac{C'^2}{4C}dr^2 - \rho^2 d\theta^2\,, \end{aligned} \qquad (47)$$

$$C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}}, \quad n \in \Re^+,$$

$$r_0 \in (\Re - \Re^-)\,, \quad \beta = m + \sqrt{m^2 - a^2\cos^2\theta}\,, \quad a^2 < m^2\,,$$

$$a = \frac{L}{m}\,, \quad \rho^2 = C_n + a^2\cos^2\theta\,,$$

$$\Delta = C_n - \alpha\sqrt{C_n} + a^2\,,$$

$$r_0 < r < \infty\,.$$

Equation (47) is regular $\forall\, r > r_0$, and $g_{00} = 0$ only when $r = r_0$. There is no event horizon and therefore no Kerr black hole.

By (47) the correct expression for the Kerr solution (31) is,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2\,, \end{aligned} \qquad (48)$$

$$\Delta = r^2 - r\alpha + a^2, \quad a = \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta,$$
$$a^2 < m^2, \quad m + \sqrt{m^2 - a^2\cos^2\theta} < r < \infty .$$

When $a = 0$ in (48) the Droste/Weyl solution (33) is recovered.

4 The vacuum field of the rotating point-charge

The Kerr-Newman solution is, in relativistic units,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\varphi - adt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 , \end{aligned} \tag{49}$$

$$a = \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - r\alpha + a^2 + q^2,$$
$$0 < r < \infty .$$

By applying the analytic technique of section 3, the general solution for the rotating point-charge is found to be,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(C_n + a^2\right)d\varphi - adt\right]^2 - \frac{\rho^2}{\Delta}\frac{C_n'^{\,2}}{4C_n}dr^2 - \rho^2 d\theta^2 , \\ & C_n(r) = \left[\left(r - r_0\right)^n + \beta^n\right]^{\frac{2}{n}}, \quad n \in \Re^+ , \\ & r_0 \in (\Re - \Re^-), \quad \beta = m + \sqrt{m^2 - (q^2 + a^2\cos^2\theta)}, \\ & a^2 + q^2 < m^2, \quad a = \frac{L}{m}, \quad \rho^2 = C_n + a^2\cos^2\theta, \\ & \Delta = C_n - \alpha\sqrt{C_n} + q^2 + a^2 , \\ & r_0 < r < \infty . \end{aligned} \tag{50}$$

Equations (50) give the overall general solution to Einstein's vacuum field when $\Lambda = 0$. The associated Metric Conditions of Abrams (MCA) for the rotating point-charge are,

1. $C_n'(r) > 0, \ r > r_0$;
2. $\lim\limits_{r\to\infty} \dfrac{C_n(r)}{\left(r - r_0\right)^2} = 1$;
3. $\sqrt{C_n(r_0)} = \beta = m + \sqrt{m^2 - (q^2 + a^2\cos^2\theta)}$, $a^2 + q^2 < m^2$.

From (50) it is concluded that there exists no spacetime drag effect for the rotating point-charge, and no ergosphere.

Equation (50) is regular $\forall\, r > r_0$, and $g_{00} = 0$ only when $r = r_0$; $r_h \equiv r_0$. When $a = 0$ in (50) the general solution for the point-charge (15) is recovered. If both $a = 0$ and $q = 0$ in (50) the general solution (2) for the simple Schwarzschild point-mass is recovered. There is no event horizon and therefore no Kerr-Newman black hole.

By (50) the correct expression for the Kerr-Newman solution (49) is,

$$\begin{aligned} ds^2 &= \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \\ &- \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\varphi - adt\right]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 , \\ & a = \frac{L}{m}, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - r\alpha + a^2 + q^2, \\ & q^2 + a^2 < m^2, \quad m + \sqrt{m^2 - (q^2 + a^2\cos^2\theta)} < r < \infty . \end{aligned} \tag{51}$$

If $a = 0$ in (51) the correct expression for the Reissner-Nordstrom solution (16) is recovered. If $q = 0$ in (51) the correct expression for the Kerr solution (48) is recovered. If both $a = 0$ and $q = 0$ in (51) the correct expression for Hilbert's (i. e. the Droste/Weyl) solution (33) is recovered.

5 The Einstein-Rosen Bridge

The Einstein-Rosen Bridge [9] is obtained by substituting into the Droste/Weyl solution (33) the transformation,

$$u^2 + \alpha = r , \tag{52}$$

which carries (33) into,

$$\begin{aligned} ds^2 &= \left[\frac{u^2}{(u^2 + \alpha)}\right] dt^2 - \\ &- 4\left(u^2 + \alpha\right)du^2 - \left(u^2 + \alpha\right)^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \\ & -\infty < u < \infty . \end{aligned} \tag{53}$$

Metric (53) is singular nowhere, and as u runs $-\infty$ to 0 and 0 to $+\infty$, r runs $+\infty$ to α then α to $+\infty$, thereby allegedly removing the singularity at $r = \alpha$. However, (53) is inadmissible by (2): (52) is not a valid form for $C_n(r)$ for the simple point-mass. This manifests in a violation of MCA. Indeed,

$$\lim_{u\to\infty} \frac{C(u)}{u^2} = \lim_{u\to\infty} \frac{\left(u^2 + \alpha\right)^2}{u^2} \to \infty , \tag{54}$$

so the far field is not flat. The Einstein-Rosen Bridge is therefore invalid.

6 Interacting black holes and the Michell-Laplace dark body

It is quite commonplace for black holes to be posited as members of binary systems, either as a hole and a star, or as two holes. Even colliding black holes are frequently alleged (see e. g. [10]). Such ideas are inadmissible, even if the existence of black holes were allowed. All solutions to the Einstein field equations involve a single gravitating body and a test particle. No solutions are known that address two bodies of comparable mass. It is not even known if solutions to such configurations exist. One simply cannot talk of black hole binaries or colliding black holes unless it can be shown, as pointed out by McVittie [11], that Einstein's field equations admit of solutions for such configurations. Without such an existence theorem these ideas are without any theoretical basis. McVittie's existence theorem however, does not exist, because the black hole does not exist in the formalism of General Relativity. It is also commonly claimed that the Michell-Laplace dark body is a kind of black hole or an anticipation of the black hole [10, 12]. This claim is utterly false as there always exists a class of observers who can see a Michell-Laplace dark body [11]: ipso facto, it is not a black hole. Consequently, there is no theoretical basis whatsoever for the existence of black holes. If such an object is ever detected then both Newton and Einstein would be invalidated.

Dedication

I dedicate this paper to the memory of Dr. Leonard S. Abrams: (27 Nov. 1924 — 28 Dec. 2001).

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Experiments with Rotating Collimators Cutting out Pencil of α-Particles at Radioactive Decay of ^{239}Pu Evidence Sharp Anisotropy of Space

Simon E. Shnoll*, Ilia A. Rubinshtein†, Konstantin I. Zenchenko*, Vladimir A. Shlekhtarev*, Alexander V. Kaminsky, Alexander A. Konradov‡, Natalia V. Udaltsova*

*Institute of Theoretical and Experimental Biophysics, Russian Academy of Sciences, Pushchino, Moscow Region, 142290, Russia
E-mail: shnoll@iteb.ru

†Skobeltsyn Inst. of Nuclear Physics, Moscow State Univ., Moscow, 119992, Russia
‡Institute of Biochemical Physics, Russian Academy of Sciences, Moscow, Russia

As shown in our previous experiments fine structure of histograms of α-activity measurements serve as a sensitive tool for investigation of cosmo-physical influences. Particularly, the histograms structure is changed with the period equal to sidereal (1436 min) and solar (1440) day. It is similar with the high probability in different geographic points at the same local (longitude) time. More recently investigations were carried out with collimators, cutting out separate flows of total α-particles flying out at radioactive decay of ^{239}Pu. These experiments revealed sharp dependence the histogram structure on the direction of α-particles flow.

In the presented work measurements were made with collimators rotating in the plane of sky equator. It was shown that during rotation the shape of histograms changes with periods determined by number of revolution. These results correspond to the assumption that the histogram shapes are determined by a picture of the celestial sphere, and also by interposition of the Earth, the Sun and the Moon.

1 Introduction

It has been earlier shown, that the fine structure of statistical distributions of measurement results of processes of various nature depends on cosmo-physical factors. The shape of corresponding histograms changes with the period equal to sidereal and solar day, i. e. 1436 and 1440 minutes [1, 2, 3, 4].

These periods disappeared at measurements of alpha-activity of ^{239}Pu samples near the North Pole [5]. These results corresponded to the assumption of association of the histogram shapes with a picture of the celestial sphere, and also with interposition of the Earth, the Sun and the Moon.

However, at measurements at latitude 54° N (in Pushchino), absence of the daily period [9] also was revealed when using collimators restricting a flow of the alpha particles of radioactive decay at the direction to the north celestial pole. This result meant, that the question is not about dependence on a picture of the celestial sphere above a place of measurements, but about a direction of alpha particles flow.

In experiments with two collimators, directed one to the East and another to the West, it was revealed, that histograms of the similar shape at measurements with west collimator appear at 718 minutes (half of sidereal day) later then ones registered with East collimator [9]. Therefore, as acquired, the space surrounding the Earth is highly anisotropic, and this anisotropy is connected basically to a picture of the celestial sphere (sphere of distant stars).

This suggestion has been confirmed in experiments with collimators, rotated counter-clockwise, west to east (i. e. in a direction of rotation of the Earth), as well as clockwise (east to west). The description of these experiments is given further.

2 Methods

As well as earlier, the basic object of these of research was a set of histograms constructed by results of measurements of alpha-activity of samples ^{239}Pu.

Experimental methods, the devices for alpha-radioactivity measurements of ^{239}Pu samples with collimators, and also construction of histograms and analysis of its shapes, are described in details in the earlier publications [2, 3, 8]. Measurements of number of events of radioactive decay were completed by device designed by one of the authors (I. A. R.). In this device the semi-conductor detector (photo diode) is placed after collimator, restricting a flow of the alpha particles in a certain direction. Results of measurements, consecutive numbers of events of the decay registered by the detector in 1-second intervals, are stored in computer archive.

Depending on specific targets, a time sequence of 1-second measurements was summarized to consecutive values of activity for 6, 15 or 60 seconds. Obtained time series were separated into consecutive pieces of 60 numbers in each. A histogram was built for each piece of 60 numbers. Histograms were smoothed using the method of moving averages for the greater convenience of a visual estimation of similarity of their shapes (more details see in [8, 9]). Comparison of histograms was performed using auxiliary computer program by Edwin Pozharski [8].

A mechanical device designed by one of the authors (V. A. Sh.) was used in experiments with rotation of collimators. In this device the measuring piece of equipment with collimator was attached to the platform rotated in a plane of Celestial Equator.

3 Results

Three revolutions of collimator counter-clockwise in a day.

The diurnal period of increase in frequency of histograms with similar shape means dependence of an observable picture on rotation of the Earth.

The period of approximately 24 hours or with higher resolution 1436 minutes is also observed at measurements using collimators restricting a flow of alpha particles in a certain direction [9, 10]. Therefore, the fine structure of distribution of results of measurements depends on what site of celestial sphere the flow of alpha particles is directed to. Studies of shapes of histograms constructed by results of measurements using rotated collimators testify to the benefit of this assumption.

The number of the "diurnal" cycles at clockwise rotation should be one less then numbers of collimator revolutions because of compensation of the Earth rotation.

At May 28 through June 10, 2004, we have performed measurements of alpha-activity of a sample ^{239}Pu at 3 collimator revolutions a day, and also, for the control, simultaneous measurements with motionless collimator, directed to the West. Results of these measurements are presented on Fig. 1–4. At these figures a dependence of frequency histograms of the same shape on size of time interval between similar histograms is shown.

Fig.1 shows results of comparison of 60-minute histograms, constructed at measurements with motionless collimator. A typical dependence repeatedly obtained in earlier studies is visible at the Fig. 1: histograms of the same shape most likely appear at the nearest intervals of time ("effect of a near zone") and in one day (24 hours).

Fig. 2 presents the result of comparison of 60-minute histograms constructed at measurements with collimator rotated 3 times a day counter-clockwise in a plane of celestial equator.

As you can see at the Fig. 2, at three revolutions of collimator counter-clockwise, the frequency of similar histograms fluctuates with the period of 6 hours: peaks correspond to the intervals of 6, 12, 18 and 24 hours.

24-hour period at a higher resolution consists of two components. It is visible by comparison of one-minute histograms shown at Fig. 3 for measurements with motionless collimator and at Fig. 4 for measurements at 3 collimator revolutions counter-clockwise. At measurements with motionless collimator (Fig. 3) there are two peaks — one corresponds to sidereal day (1436 minutes), the second, which is less expressed, corresponds to solar day (1440 minutes).

You can see at Fig. 4 that 6-hour period at measurements with three revolutions of collimator also has two components. The first 6-hour maximum has two joint peaks of 359 and 360 minutes. The second 12-hour maximum has two peaks of 718 and 720 minutes. The third maximum (18 hours) has two peaks of 1077 and 1080 minutes. And the fourth one (24 hours) has two peaks of 1436 and 1440 minutes.

Results of these experiments confirm a conclusion according to which a change in histogram shape is caused by change in direction of alpha particles flow in relation to distant stars and the Sun (and other space objects). This conclusion is supported also by results of experiments with rotation of collimator clockwise.

In these experiments collimator made one revolution a day clockwise, east to west, i. e. against daily rotation of the

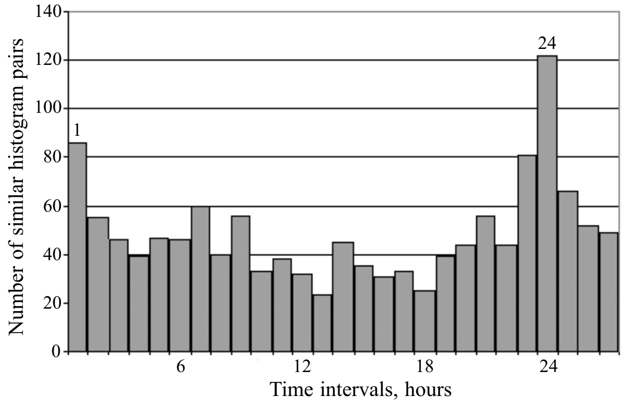

Fig. 1: Frequency of similar 60-minute histograms against the time interval between histograms. Measurements of alpha-activity of a ^{239}Pu sample by detector with motionless collimator directed to the West, June 8–30, 2004.

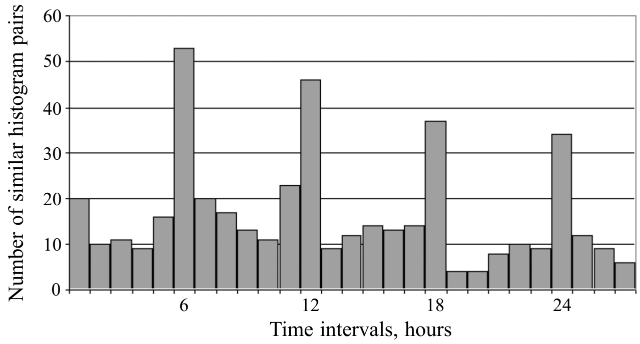

Fig. 2: Frequency of similar 60-minute histograms against the time interval between histograms. Measurements of alphaactivity of a ^{239}Pu sample by detector with collimator, making three revolutions counter-clockwise (west to east) in a day.

Earth. As a result, the flow of alpha particles all the time was directed to the same point of celestial sphere. We expected in this case disappearance the diurnal period of frequency of similar histograms. This expectation was proved to be true.

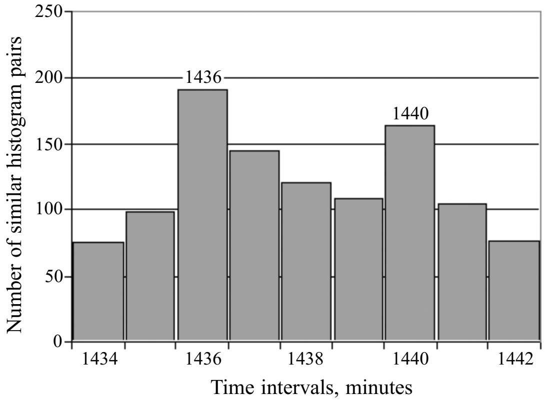

Fig. 3: 24-hour period of frequency of similar histograms with the one-minute resolution. Measurements of May 29 — June 1, 2004 by detector with motionless collimator directed to the West.

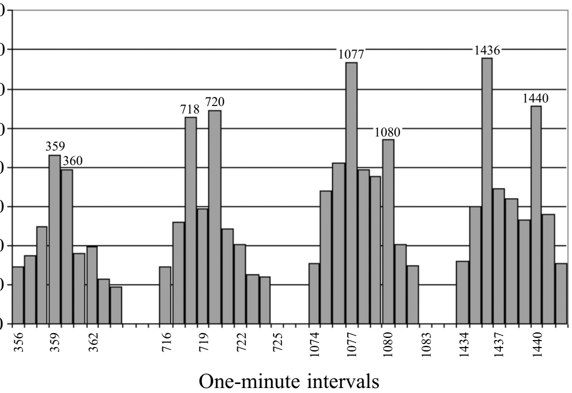

Fig. 4: Experiments with rotated collimators. Frequency of similar 1-minute histograms by time interval between them. Three revolutions a day counter-clockwise. Two components of the 6-hour period: sidereal and solar.

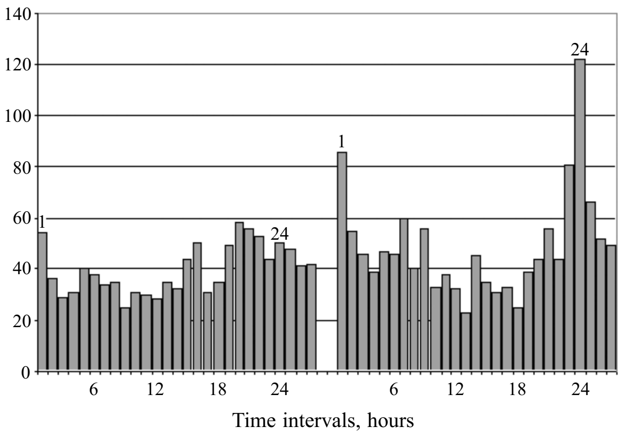

Fig. 5: 60-minutes histograms. Left: 1 revolution clockwise. Right: control, motionless collimator.

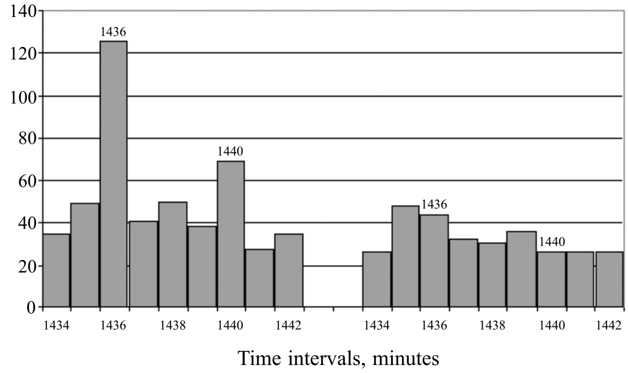

Fig. 6: One-minute histograms. Left: control, motionless collimator. Right: rotation 1 revolution clockwise (east to west).

On Fig. 5 and 6 one can see that in such experiments frequency of appearance of similar 60 minute and 1-minute histograms does not depend on time. At the same time at synchronous measurements with motionless collimator the usual dependence with the diurnal period and near zone effect is observed.

4 Discussion

Results of measurements with rotated collimator confirm a conclusion about dependence of fine structure of statistical distributions on a direction in space. This fine structure is defined by a spectrum of amplitudes of fluctuations of measured values. Presence of "peaks" and "hollows" at corresponding histograms suggests presence of the primary, allocated, "forbidden" and "permissible" values of amplitudes of fluctuations in each given moment [4]. Thus, a fine structure of statistical distributions presents a spectrum of the permissible amplitudes of fluctuations, and dependence of it on a direction in space shows sharp anisotropy of space.

It is necessary to emphasize, that the question is not about influence on the subject of measurement (in this case on radioactive decay). With accuracy of traditional statistical criteria, overall characteristics of distribution of radioactive decay measurements compliant with Poisson distribution [3]. Only the shape of histogram constructed for small sample size varies regularly. This regularity emerges in precise sidereal and solar periods of increase of frequency of similar histograms.

As shown above, the shape of histograms constructed by results of measurements of alpha-activity of samples ^{239}Pu, varies with the period determined by number of revolutions in relation to celestial sphere and the Sun. In experiments with collimator, which made three revolutions counter-clockwise, the "diurnal" period was equal to 6 hours (three revolutions of collimator and one revolution of the Earth was observed — in total 4 revolutions in relation to celestial sphere and the Sun give the period equal $24/4 = 6$ hours).

The result obtained in experiments with one revolution of collimator clockwise is not less important. The Earth rotation is compensated and a flow of alpha particles is directed all the time to the same point of celestial sphere. In these experiments the diurnal period was not observed at all.

The obtained results, though very clear ones, cause natural bewilderment.

Really, it is completely not obvious, by virtue of what reasons the spectrum of amplitudes of fluctuations of number of alpha particles, may depend on a direction of their flow in relation to celestial sphere and the Sun. The explanation of these phenomena probably demands essential change in general physical conceptions.

In such situation a dominant problem is to validate a reliability of the discussed phenomena. In aggregate of performed studies, we believe this task was completed .

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There Is No Speed Barrier for a Wave Phase Nor for Entangled Particles

Florentin Smarandache

Dept. of Mathematics, University of New Mexico, 200 College Road, Gallup, NM 87301, USA

E-mail: fsmarandache@yahoo.com; smarand@unm.edu

In this short paper, as an extension and consequence of Einstein-Podolski-Rosen paradox and Bell's inequality, one promotes the hypothesis (it has been called the Smarandache Hypothesis [1, 2, 3]) that: There is no speed barrier in the Universe and one can construct arbitrary speeds, and also one asks if it is possible to have an infinite speed (instantaneous transmission)? Future research: to study the composition of faster-than-light velocities and what happens with the laws of physics at faster-than-light velocities?

This is the new version of an early article. That early version, based on a 1972 paper [4], was presented at the Universidad de Blumenau, Brazil, May–June 1993, in the Conference on "Paradoxism in Literature and Science"; and at the University of Kishinev, in December 1994. See that early version in [5].

1 Introduction

What is new in science (physics)?

According to researchers from the common group of the University of Innsbruck in Austria and US National Institute of Standards and Technology (starting from December 1997, Rainer Blatt, David Wineland et al.):

- Photon is a bit of light, the quantum of electromagnetic radiation (quantum is the smallest amount of energy that a system can gain or lose);
- Polarization refers to the direction and characteristics of the light wave vibration;
- If one uses the entanglement phenomenon, in order to transfer the polarization between two photons, then: whatever happens to one is the opposite of what happens to the other; hence, their polarizations are opposite of each other;
- In quantum mechanics, objects such as subatomic particles do not have specific, fixed characteristic at any given instant in time until they are measured;
- Suppose a certain physical process produces a pair of entangled particles A and B (having opposite or complementary characteristics), which fly off into space in the opposite direction and, when they are billions of miles apart, one measures particle A; because B is the opposite, the act of measuring A instantaneously tells B what to be; therefore those instructions would somehow have to travel between A and B faster than the speed of light; hence, one can extend the Einstein-Podolsky-Rosen paradox and Bell's inequality and assert that the light speed is not a speed barrier in the Universe.

Such results were also obtained by: Nicolas Gisin at the University of Geneva, Switzerland, who successfully teleported quantum bits, or qubits, between two labs over 2 km of coiled cable. But the actual distance between the two labs was about 55 m; researchers from the University of Vienna and the Austrian Academy of Science (Rupert Ursin et al. have carried out successful teleportation with particles of light over a distance of 600 m across the River Danube in Austria); researchers from Australia National University and many others [6, 7, 8].

2 Scientific hypothesis

We even promote the hypothesis that:

> There is no speed barrier in the Universe, which would theoretically be proved by increasing, in the previous example, the distance between particles A and B as much as the Universe allows it, and then measuring particle A.

It has been called the *Smarandache Hypotesis* [1, 2, 3].

3 An open question now

If the space is infinite, is the maximum speed infinite?

"This Smarandache hypothesis is controversially interpreted by scientists. Some say that it violates the theory of relativity and the principle of causality, others support the ideas that this hypothesis works for particles with no mass or imaginary mass, in non-locality, through tunneling effect, or in other (extra-) dimension(s)." Kamla John, [9].

Scott Owens' answer [10] to Hans Gunter in an e-mail from January 22, 2001 (the last one forwarded it to the author): "It appears that the only things the Smarandache hypothesis can be applied to are entities that do not have real mass or energy or information. The best example I can come up with is the difference between the wavefront velocity of a photon and the phase velocity. It is common for the phase velocity to exceed the wavefront velocity c, but that does not mean that any real energy is traveling faster than c. So, while it is possible to construct arbitrary speeds from zero in infinite, the superluminal speeds can only apply to purely imaginary entities or components."

Would be possible to accelerate a photon (or another particle traveling at, say, $0.99c$ and thus to get speed greater than c (where c is the speed of light)?

4 Future possible research

It would be interesting to study the composition of two velocities v and u in the cases when:

$$\mathrm{v} < c \;\text{ and }\; u = c;$$
$$\mathrm{v} = c \;\text{ and }\; u = c;$$
$$\mathrm{v} > c \;\text{ and }\; u = c;$$
$$\mathrm{v} > c \;\text{ and }\; u > c;$$
$$\mathrm{v} < c \;\text{ and }\; u = \infty;$$
$$\mathrm{v} = c \;\text{ and }\; u = \infty;$$
$$\mathrm{v} > c \;\text{ and }\; u = \infty;$$
$$\mathrm{v} = \infty \;\text{ and }\; u = \infty.$$

What happens with the laws of physics in each of these cases?

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