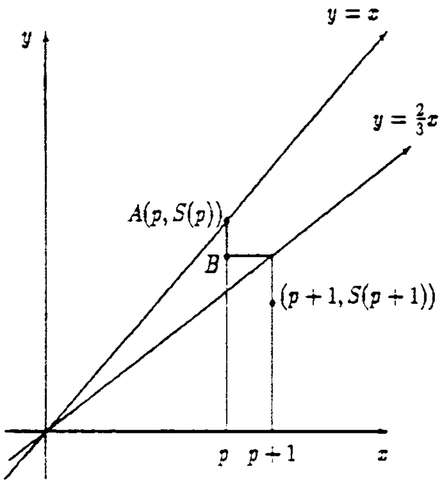
Collection of Problems On Smarandache Notions

Charles Ashbacher



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The graph on the first cover belongs to:
M. Popescu, P. Popescu, V. Seleacu, "About the behaviour of some new functions in the number theory" (to appear this year).

"Collection of Problems on Smarandache Notions",
by Charles Ashbacher

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Preface

The previous volume in this series, **An Introduction to the Smarandache Function**, also by Erhus University Press, dealt almost exclusively with some "basic" consequences of the Smarandache function. In this one, the universe of discourse has been expanded to include a great many other things.

A Smarandache notion is an element of an ill-defined set, sometimes being almost an accident of labeling. However, that takes nothing away from the interest and excitement that can be generated by exploring the consequences of such a problem. It is a well-known cliché among writers that the best novels are those where the author does not know what is going to happen until that point in the story is actually reached. That statement also holds for some of these problems. In mathematics, one often does not know what the consequences of a statement are. Unlike a novel however, there are no complete plot resolutions in mathematics as there are no villains to rub out. As the French emphatically say in another context, "Vive la différence!"

Hopefully, as you move through this book, some of the same spirit of exploration felt by the author will be part of your experience. For the reading of a book is a form of mind-joining, where the author tries to create the opportunity for a shared experience. And the creation of this book was very much an adventure for this author. If anything here gives you the urge to comment, feel free to do so at the address given below or to the people at Erhus University Press. Any comments regarding future directions or unsolved problems are especially solicited.

Again, I would like to thank R. Muller and all those who toil so diligently at Erhus University Press. Dr. Muller for his encouragement, support and frequent letters containing new material for study and everyone else for everything else. It may be a cliché to say that many work very hard behind the scenes to create the product you see, but in this case it is very true. As is always the case, it is the responsibility of the author to catch and remove any action of the ubiquitous error demons.

Rather than attempt to cite them in the text, I will now extend my gratitude to all those who combed the Smarandache archives for much of the material explored here:

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Thanks must again be extended to Brian Dalziel and Toufic Moubarek, my supervisors who agreed to let me use Decisionmark resources in the pursuit of my professional objectives. Sometimes, there is simply no substitute for significant computer power.

Others in my queue of those to thank include all who have helped me in the past. Special thanks go to Leo Lim, professor extraordinaire who truly believed in me and my abilities. It takes real effort to fail when people of his quality are teaching you.

Finally, I would like to dedicate this book to my mother Paula Ashbacher and my beautiful daughter Katrina. My mother, for her sense in teaching me things when I did not yet have the good sense to understand. If you are unable to hit a curveball or dunk a basketball, the next best skill is a love of reading, although I still fantasize about the first two. Katrina for just being Katrina. Much more than the apple of my eye, she is a complete orchard.

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Chapter 1

Smarandache Sequences

In the time since it was first published[1], the Smarandache function

$S(m) = n$, where n is the smallest number such that m divides n!.

and associated consequences has spawned many new branches of mathematics. A previous volume in this series[2] introduced the function and explored some of the problems derived from the definition.

In this book, we will explore several avenues of what are called Smarandache notions. The obvious question at this point is, "What is a Smarandache notion?" The answer is both simple and complex. A Smarandache notion is a problem in one of the following sets:

a) A problem posed by Florentin Smarandache.

b) A problem posed by someone else that is an extension of an element of set (a).

See **Some Notions and Questions in Number Theory**, edited by C. Dumitrescu and V. Seleacu, Erhus University Press, Glendale, 1994.

As should be clear from the statements above, a Smarandache notion may not directly or even indirectly involve the Smarandache function. In fact, most of those dealt with here do not. All of the problems presented in this book were either published in [3],[4] or [5] or appeared in a personal correspondence. The author has attempted to group them as much as possible, but there is no distinct order.

In this first chapter, we will concentrate on notions that involve sequences, moving on to other, more specific points in the second.

The first problem we will deal with is number (6) in [5].

Smarandache permutation sequence:

12, 1342, 135642, 13578642, 13579108642, 135791112108642, . . .

Or as a formula

$SPS(n) = 135 \ldots (2n-3)(2n-1)(2n)(2n-2)(2n-4) \ldots 642$

where SPS is an acronym for Smarandache Permutation Sequence.

Note on notation: As a general principle, acronyms will be created for all sequences. Consult appendix 1 for a complete list of all acronyms, the full names of the sequences and the page number where the sequence first appears. For purposes of notation the acronym will be used to denote the entire set and the acronym with a subscript will refer to a specific element of that set. For example,

SPS = { 12, 1342, 135642, 13578642, 13579108642, 135791112108642, . . . }

where

SPS(3) = 135642. □

Question: Is there any perfect power among these numbers? I. e. are there integers m, k and n such that

$$m^k = SPS(n)$$

With the hint:
The last digit must be 2 for exponents of the form 4k + 1 or 8 for exponents of the form 4k + 3.

And the final statement: "Smarandache conjectures: no!" □

The origin of the hint provides direction in the search for a proof, so it is repeated here.

Let m be an arbitrary integer. The terminal or rightmost digit d0 of this number must be in the set { 0,1,2,3,4,5,6,7,8,9 }. If m is taken to any integral power, the digit in the terminal position of the result is determined only by the remainder upon division by 10 of the terminal digit raised to that power. Taking each in turn and determining what values the powers take

0 to any power is zero
1 to any power is one
2^1 modulo 10 = 2
2^2 modulo 10 = 4
2^3 modulo 10 = 8
2^4 modulo 10 = 6
2^5 modulo 10 = 2
2^6 modulo 10 = 4

etc.

3^1 modulo 10 = 3

3^2 modulo 10 = 9

3^3 modulo 10 = 7

3^4 modulo 10 = 1

3^5 modulo 10 = 3

3^6 modulo 10 = 9

etc.

4^1 modulo 10 = 4

4^2 modulo 10 = 6

4^3 modulo 10 = 4

4^4 modulo 10 = 6

etc.

5^1 modulo 10 = 5

5^2 modulo 10 = 5

etc.

6^1 modulo 10 = 6

6^2 modulo 10 = 6

etc.

7^1 modulo 10 = 7

7^2 modulo 10 = 9

7^3 modulo 10 = 3

7^4 modulo 10 = 1

7^5 modulo 10 = 7

etc.

8^1 modulo 10 = 8

8^2 modulo 10 = 4

8^3 modulo 10 = 2

8^4 modulo 10 = 6

8^5 modulo 10 = 8

etc.

9^1 modulo 10 = 9

9^2 modulo 10 = 1

9^3 modulo 10 = 9

etc.

This cyclic behavior is typical and provides a powerful proof mechanism when the terminal numbers are known to be fixed. From this, the following lemma is obvious.

Lemma 1: The only candidates for a solution to the equation

$$m^k = SPS(n)$$

are when m terminates in 2 and k is of the form 4k + 1 or m terminates in 8 and k is of the form 4k + 3.

The search can be continued by examining the possibilities for all possible two digit combinations d1d0 for the last two digits of m to any integral power. However, the number of possibilities grows rapidly, so the search is best left to a computer. A simple program was written that allows for the examination of all possible values of

$$(10 * d + 2)^k \text{ modulo } 100$$

and

$$(10 * d + 8)^k \text{ modulo } 100$$

for d a decimal digit. All products cycle, for example

12^1 modulo 100 = 12
12^2 modulo 100 = 44
12^3 modulo 100 = 88
12^4 modulo 100 = 56
12^5 modulo 100 = 72
12^6 modulo 100 = 64
12^7 modulo 100 = 68
12^8 modulo 100 = 16
12^9 modulo 100 = 92
12^{10} modulo 100 = 4
12^{11} modulo 100 = 48
12^{12} modulo 100 = 76
12^{13} modulo 100 = 12

has a 12 member cycle.

In all cases examined, there was no combination where the value modulo 100 was 42. Since this search was exhaustive, we have proven our first theorem.

Theorem 1: There are no integers m,k and n such that

$$SPS(n) = m^k .$$

Unsolved problems (27) and (28) in [4] are similar to each other.

27. Smarandache Square Complements:

Definition 1: For each integer n, the Smarandache Square Complement SSC(n) is the smallest number k such that nk is a perfect square.

The first few numbers in the sequence are

1,2,3,1,5,6,7,2,10,11,3,14,15,1,17, . . . □

28. Smarandache Cube Complements:

Definition 2: For each integer n, the Smarandache Cube Complement SCC(n) is the smallest integer k such that nk is a perfect cube.

The first few numbers in this sequence are

1,4,9,2,25,36,49,1,3,100,121, . . . □

The computation of the elements of SSC and SCC are both straightforward. We start first with SSC.

Algorithm 1:

Input: A positive integer n.
Output: SSC(n), the smallest integer k such that kn is a perfect square.

Step 1: If n = 1 return k = 1.

Else:

Step 2: Factor n into the product of its' prime factors $n = p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_r^{\alpha r}$

Step 3: Set k = 1.

Step 4: For i=1 to r.

Step 4.1: If αi is odd then set $k = k * p_i$.

Step 4.2: End of for loop.

Step 5: Return k.

The correctness of this algorithm should be obvious from the definition. Clearly, if n is a perfect square, SSC(n) = 1.

Fixed points of SSC(n) are easily determined.

Theorem 2: The fixed points of SSC(n) are 1 and all numbers where every prime factor is to the first power.

Proof: Easily SSC(1) = 1. If n has a prime factorization where every prime is to the first power, then it is clear that SSC(n) must contain exactly one instance of every prime in n and SSC(n) = n.

Now, consider a number that has at least one prime factor that appears more than once. Call that prime p.

Case 1: P appears an even number of times in n.
Therefore, it is not necessary to add an instance of p to SSC(n) and since there is at least one prime factor they do not share, $n \neq SSC(n)$.

Case 2: P appears an odd number of times in n, where this number is at least three.
Therefore, the construction of SSC(n) will require one instance of p. However, since the number of instances of p is different in n and SSC(n), the values must differ. □

Another interesting property of this function concerns the range of incremental differences.

Theorem 3: Let $D = \{ d \mid d = | SSC(n+1) - SSC(n) | \}$. D is an infinite set or equivalently, there is no number M such that $M > d \ \forall d \in D$.

Proof: Contrary to the conclusion of the theorem, assume that such an M exists. Choose $p_k > M + 1$. Form the square number created by the product of all primes less than or equal to p_k.

$$r = p_1^2 \, p_2^2 \ldots p_k^2$$

where it is known that SSC(r) = 1.

Now consider the number r + 1. Since perfect squares are not sequential, r + 1 cannot also be a perfect square. Therefore, there must be at least one prime factor q of r + 1 that is to an odd power. Furthermore, $q > p_k$. Using the algorithm, we can build the inequality

$$SSC(r+1) \geq q > p_k > M + 1$$

and it follows that

$$SSC(r+1) - SSC(r) > M$$

contradicting the choice of M. □

Corollary 1: There are no positive integers M and k such that

$$|SSC(x) - SSC(y)| \leq M|x - y|^k.$$

In other words, SSC(x) does not satisfy the Lipschitz condition for any exponent k.

Proof: $|x - y| = 1$ in the previous theorem. □

Examining a few of the values of SSC(n) we find several instances where there are three fixed points in succession. For example,

SSC(5) = 5, SSC(6) = 6 = 2*3 , SSC(7) = 7

SSC(21) = 21 = 3*7, SSC(22) = 22 = 2*11, SSC(23) = 23

SSC(57) = 57 = 3*19 , SSC(58) = 58 = 2*29 , SSC(59) = 59

SSC(69) = 69 = 3*23 , SSC(70) = 2*5*7 , SSC(71) = 71

This is the maximum possible number of consecutive fixed points and that fact is the topic of the next theorem.

Theorem 4: There is no quadruple (m,m+1,m+2,m+3) such that all four are fixed points of SSC(n).

Proof: At least two of the numbers must be even. Without loss of generality, assume m and m+2 are both even. To be a fixed point of SSC(n), m must have every prime factor to the first power, including 2. Therefore, m = 2k, for k some odd integer. It then follows that m + 2 = 2k + 2 = 2(k + 1) where k + 1 is also even. M + 2 then must have more than one instance of 2 as a factor and cannot be a fixed point of SSC(n). □

The previous examples of triples of fixed points hint at a possible solution to the first of our list of unsolved problems.

Unsolved Problem 1: There are an infinite number of triples (m,m + 1,m + 2) such that each is a fixed point of SSC(n).

The algorithm to compute the values of SCC(n), the cubic complements of the integers. is similar to that of SSC(n).

Algorithm 2:

Input: A positive integer n.
Output: SCC(n), the smallest integer k such that kn is a perfect cube.

Step 1: If n = 1 return k = 1.

Else:

Step 2: Factor n into the product of its' prime factors $n = p_1^{\alpha 1} p_2^{\alpha 2} \dots p_r^{\alpha r}$

Step 3: Set k = 1.

Step 4: For i=1 to r.

Step 4.1: If αi is of the form 3j+1 then set $k = k * p_i * p_i$.
Else:
Step 4.2: If αi is of the form 3j+2 then set $k = k * p_i$.

Step 5: Return k.

Clearly, if n is a perfect cube, SCC(n) = 1.

The set of fixed points of SCC(n) is decidedly different from that of SSC(n).

Theorem 5: The only fixed point of SCC(n) is n = 1.

Proof: Clearly, SCC(1) = 1. If n $>$ 1, it can be factored into the product of its' prime factors. If we choose an arbitrary prime q in that factorization, there are three possible cases when building SCC(n).

Case 1: The exponent on q is evenly divisible by 3. Then no instance of q will be placed in SCC(n), so SCC(n) $\neq$ n.

Case 2: The exponent on q is of the form 3j+1. Then two instances of q will be placed in SCC(n). Since 2 is of the form 3j+2, SCC(n) $\neq$ n.

Case 3: The exponent on q is of the form 3j+2. Then one instance of q will be placed in SCC(n). Since 1 is of the form 3j+1, SCC(n) $\neq$ n. □

Theorem 6: Let $D = \{ d \mid d = | SCC(n+1) - SCC(n) | \}$. D is an infinite set or equivalently, there is no number M such that $M > d, \forall d \in D$.

Proof: Apply the reasoning of theorem 3, replacing all squares by cubes. □

Corollary 2: There are no positive integers M and k such that

$$| SCC(x) - SCC(y) | \leq M|x - y|^k.$$

In other words, SCC(x) does not satisfy the Lipschitz condition for any exponent k.

Proof: Same reasoning as that for corollary 1. □

The functions SSC(n) and SCC(n) share one property.

Theorem 7: There is no pair of integers (n,n+1) such that

$$SSC(n) = SSC(n+1) \qquad \text{or} \qquad SCC(n) = SCC(n+1).$$

Proof: $SSC(1) = 1$ and $SSC(2) = 2$, $SCC(1) = 1$ and $SCC(2) = 4$. Therefore, we can take $n > 1$ which is composed of prime factors. Clearly, n and n+1 cannot both be perfect squares or perfect cubes so it is not the case that $SSC(n) = 1 = SSC(n+1)$ or $SCC(n) = 1 = SCC(n+1)$. If either n or n+1 is a perfect square(cube) the other is not and therefore not equal to 1. Therefore, the only remaining possibility is when both n and n+1 are not perfect squares(cubes).

Clearly, n and n+1 have distinct prime factors. And so, when SSC(n) or SCC(n) is being computed, a prime factor q of n is encountered that is included in SSC(n) or SCC(n). Since q is not a factor of n+1, q will not appear in SSC(n+1) or SCC(n+1), forcing $SSC(n) \neq SSC(n+1)$ and $SCC(n) \neq SCC(n+1)$. □

The generic form of this problem appears as (24) in [5].

24. Smarandache m-Power Complements:

Definition 3: For each integer n, find the smallest integer k such that nk is a perfect m-power.

Additional problems derived from this definition will not be dealt with here.

Problem (71) in [5] deals with those numbers whose digits can be permuted to form a square.

71. Smarandache Pseudo-Squares of the Third Kind:

Definition 4: A number is a Smarandache Pseudo-Square of the third kind(SPS3) if some nontrivial permutation of the digits is a square.

Question: How many Smarandache Pseudo-Squares of the Third Kind are square numbers?

Conjecture: There are an infinite number. □

Theorem 8: There are an infinite number of perfect squares n such that a nontrivial permutation of the digits is a square.

Proof: The infinite family of numbers

$(101)^2 = 10201$
$(10001)^2 = 100020001$
$(1000001)^2 = 1000002000001$
etc.

are all palindromic and by definition a nontrivial permutation that is a reflection about the central digit 2 yields a square. Other such infinite palindromic families of squares exist. □

Problem (74) in [5] is similar.

74) Smarandache Pseudo-Cubes of the Third Kind:

Definition 5: A number is a Smarandache Pseudo-Cube of the Third Kind(SPC3) if some nontrivial permutation of the digits is a cube.

Question: How many Smarandache Pseudo-Cubes of the Third Kind are cubes?

Conjecture: There are an infinite number. □

Theorem 9: There are an infinite number of integers n such that a nontrivial permutation of the digits of n is a cube.

Proof: Each element of the infinite family of palindromic numbers

$(101)^3 = 1030301$
$(10001)^3 = 1000300030001$
$(1000001)^3 = 1000003000003000001$
etc.

can be nontrivially permuted to form a cube. Other such infinite families may exist. □

The general form of this problem is also given as (77)[5].

77. Smarandache Pseudo-m-Powers of the Third Kind:

Definition 6: A number is a Smarandache Pseudo-m-Power of the Third Kind if some nontrivial permutation of the digits is an m-power, $m \geq 2$.

Question: How many Smarandache pseudo-m-powers of the third kind are m-powers?

Conjecture: There are an infinite number. □

Using the same family as that for squares and cubes.

$(101)^4 = 104060401$
$(10001)^4 = 10004000600040001$
$(1000001)^4 = 1000004000006000004000001$
etc.

we have a solution for the case m = 4.

However, this family fails for powers greater than 4.

$(101)^5 = 10510100501$
$(101)^6 = 1061520150601$

In fact, no such family of palindromic powers is known for any power greater than 4.

Although, if we take the definitions literally, this problem has a trivial solution for all values of m.

10^m is an m power that can be nontrivially permuted into the number 1^m, provided we delete the leading zeros.

More generally, any number of the form 10^{km} can be nontrivially permuted into the number 10^m.

Modifications of this problem could include the restriction that there be no leading zeros in the result of the permutation.

An additional notion similar to that of the previous problems concerns odd numbers and all three appear in[5].

84) Smarandache Pseudo-Odd Numbers of the First Kind:

Definition 7: A number is said to be a Smarandache Pseudo-Odd Number of the First Kind(SPO1) if some permutation of the digits is an odd number. The identity permutation is allowed.

Clearly, a number need contain only one odd digit to be a member of this set. It is just as obvious that more numbers are pseudo-odd than not and the actual differences are the subject of the next theorem.

Theorem 10: If a positive integer n is chosen at random, the odds are overwhelming that n is a pseudo-odd number of the first kind. In fact, the limiting probability is 1.

Proof: Consider all numbers that are made of k digits, where the leading digit is nonzero. It is a simple matter to show that there are $9 \times 10^{k-1}$ such numbers. For our purposes here, we will determine how many of these numbers are not pseudo-odd. To do this, we need the following principle of counting.

Counting Principle 1: Given two independent tasks, the number of ways in which both can be done is the product of the number of ways each can be done separately. Specifically, if task a can be done m ways and b n ways, then the number of ways both a and b can be done is given by

$$m*n. \quad \square$$

We will now use counting principle 1 to determine how many k-digit numbers there are that contain no odd digit. Since zero cannot be the leading digit, there are four choices for the first one. Every even digit can be used in all other positions, so there are five choices for each additional position. Using the counting principle, we then have a total of

$$4 * 5^{k-1}$$

k-digit numbers containing no odd number. The probability of a random k-digit number containing all even numbers is then

$$\frac{4*5^{k-1}}{9*10^{k-1}} = \frac{4}{9*2^{k-1}}$$

which goes to zero in the limit as $k \to \infty$. $\square$

85) Smarandache Pseudo-Odd Numbers of the Second Kind.

Definition 8: An integer n is said to be a Smarandache Pseudo-Odd Number of the

Second Kind(SPO2) if it is even and some permutation of the digits is odd.

Theorem 11: If a positive even integer n is chosen at random, the odds are overwhelming that n is a pseudo-odd number of the second kind. In fact, the limiting probability is 1.

Proof: If we start with all k-digit numbers again, it is easy to show that

$$9 * 10^{k-2} * 5$$

of them are even. The number of k-digit numbers containing all even digits is still

$$4*5^{k-1}$$

so the fraction defining the probability is

$$\frac{4*5^{k-1}}{9*10^{k-2} * 5} = \frac{4}{9*2^{k-2}}$$

which also goes to zero in the limit as $k \to \infty$. □

The last of these three problems defines Smarandache Pseudo-Odd Numbers of the Third Kind.

86) **Definition 9:** A positive integer n is said to be a Smarandache pseudo-odd number of the third kind if some nontrivial permutation of the digits is an odd number.

It should be clear that the probability that a randomly chosen positive integer is pseudo-odd of the third kind is also one.

Problems (88), (89) and (90) of [5] all deal with similar definitions of pseudo-even numbers of the first, second and third kinds. Only the first definition will be given.

88) Smarandache Pseudo-Even numbers of the First Kind:

Definition 10: A number n is a Smarandache Pseudo-Even Number of the First Kind(SPE1) if some permutation of the digits of n is even.

The only problem that we will deal with involving this definition is the following.

Theorem 12: If a positive integer n is chosen at random, the probability that n is both a pseudo-odd number of the first kind and a pseudo-even number of the first kind is 1.

Proof: For n to be simultaneously a pseudo-odd and a pseudo-even number, it must contain both an odd and even digit. Using the basic principles of counting, the number of k-digit numbers containing only odd digits is given by

$$5^k.$$

The total number of k-digit numbers that are made up of all odd or all even digits is then

$$5^k + 4*5^{k-1}$$

so the probability that a random k-digit integer contains all even or all odd digits is

$$\frac{5^k + 4*5^{k-1}}{9*10^{k-1}}$$

which also goes to zero in the limit as $k \rightarrow \infty$. □

Similar conclusions can be reached for pseudo-odd and pseudo-even numbers of the second and third kind.

Problem (91) of [5] is similar and provides the lead in for an entire series of problems.

91) Smarandache Pseudo-Multiples of the First Kind (of 5):

> **Definition 11:** A number n is a Smarandache Pseudo-Multiple of the First Kind of 5(SPM15) if some permutation of the digits of n is a multiple of 5.

A number is a multiple of 5 if and only if it terminates with 0 or 5, so a number is a pseudo-multiple of 5 of the first kind if it contains either a 0 or a 5. It is sufficient to prove that nearly all numbers contain a 5 to verify that nearly all are divisible by 5.

Theorem 13: If a positive integer is chosen at random, the probability that it is a pseudo-multiple of 5 of the first kind is 1.

Proof: Let m be a k-digit positive integer and we will determine how many k-digit numbers do not contain a 5. There are 8 choices for the leading digit and 9 choices for the remaining k-1 digits. Using the principle of counting, the total number of ways one can construct a k-digit number without using the digit 5 is

$$8*9^{k-1}.$$

The number of ways that a k-digit number can be constructed is given by

$$9*10^{k-1}.$$

So the number of ways one can construct a number using at least one instance of the digit 5 is given by

$$9*10^{k-1} - 8*9^{k-1}.$$

And the probability that a k-digit integer chosen at random contains at least one 5 is given by the ratio

$$\frac{9*10^{k-1} - 8*9^{k-1}}{9*10^{k-1}}.$$

Which can be rewritten as

$$1 - \frac{8*9^{k-1}}{9*10^{k-1}}$$

and which goes to one in the limit as $k \to \infty$. □

Corollary 3: Let d be a decimal digit in the set { 0,1,2,3,4,5,6,7,8,9 }. If m is a positive integer chosen at random, then the probability that m contains d is 1.

Proof: If $d \in$ { 1,2,3,4,5,6,7,8,9 } then the counting process of theorem 13 is unaffected if 5 is replaced by d, so the same conclusion follows. If $d = 0$, the counting is slightly modified as zero cannot be the leading digit. The number of ways in which a k-digit number can be created without using the zero is given by

$$9*9^{k-1}$$

which has no affect on the final result. □

Coupling this with the results for the even numbers, we have the first instances of the following general problem, which is (94) of [5].

94) Smarandache Pseudo-Mulupies of the First Kind of p, where $p > 1$ is an integer.

Definition 12: A number m is a Smarandache Pseudo-Multiple of the First Kind of p(SPM1P) if some permutation of the digits of m is a multiple of p.

From previous work, we already know that nearly all numbers are pseudo-multiples of the first kind of 2 and 5. However, that does not hold in general, as can be seen from the following theorem.

Theorem 14: If m is a randomly chosen integer, the probability that m is a pseudo-multiple of the first kind of 3 is $\frac{1}{3}$.

Proof: The following number theoretic result is well-known.

A positive integer is evenly divisible by 3 if and only if the sum of the digits is evenly divisible by 3.

If an integer m is chosen at random, the probability that m is evenly divisible by 3 is known to be $\frac{1}{3}$. Since a permutation of the digits has no affect on the sum of the digits, the probability is unaffected by any permutation operation. □

Which is a lead in to the second unsolved problem.

Unsolved problem 2:
Given an arbitrary integer p, determine the probability that an integer chosen at random is a Smarandache Pseudo-Multiple of the First Kind of p.

Another sequence that appears in [5] deals with numbers and their divisors.

15) Smarandache Simple Numbers:

Definition 13: An integer n is said to be a Smarandache Simple Number if the product of its proper divisors is less than or equal to n.

Generally speaking, n has one of the forms $n = p$, $n = p^2$, $n = p^3$ or $n = pq$, where p and q are distinct primes. □

It is easy to prove the statement concerning the forms of the simple numbers.

Theorem 15: All Smarandache Simple Numbers are either a prime, square of a prime, cube of a prime or the product of two distinct primes.

Proof: Let $n > 1$ be an integer. The proof will be split into two subcases.

Case 1: $n = p^k$ where p is a prime. If $k = 1$, then 1 is the only proper divisor. When $k = 2$, the proper divisors are 1 and p, with product p. If $k = 3$, the proper divisors are 1, p and p^2, with product p^3. In general, if k is the exponent, the product of the proper divisors is p to the power

$$\sum_{i=1}^{k-1} i .$$

Where the sum is $((k-1) * k)/2$, known to be greater than k for $k > 3$.

Case 2: N is the product of at least two distinct primes. If $n = pq$, then the proper divisors are 1, p and q, with product pq. However, if n = pqr where all are primes and r is not necessarily distinct from p and q, the proper divisors are 1, p, q, r, pq, pr, and qr, with product $p^3q^3r^3$. This is clearly larger than n. □

Given that the terms of the sequence have these specific forms, it is easy to count how many simple numbers are less than a given number. The bound is defined by the number theoretic function

$$\pi(x)$$

which is the number of primes less than or equal to x.

Theorem 16: If x is any integer, then the number of Smarandache Simple Numbers less than or equal to x is given by the formula

$$\pi(x) + \binom{\pi(x)}{2} + \pi(\text{sqrt}(x)) + \pi(\text{cuberoot}(x))$$

Where $\binom{m}{n}$ is the number of independent ways n items can be chosen from a set of m items, sqrt(x) is the square root function and cuberoot(x) is the cube root.

Proof: This is an exercise in counting the number of integers that satisfy each of the four types in the previous theorem.

$\pi(x)$ is the number of primes less than or equal to x.

$\pi(\text{sqrt}(x))$ is the number of primes whose square is less than or equal to x.

$\pi(\text{cuberoot}(x))$ is the number of primes whose cube is less than or equal to x.

$\binom{\pi(x)}{2}$ is the number of ways 2 distinct primes can be chosen from all primes less than or equal to x.

By theorem 15, the sum of these four numbers is then the number of simple numbers less than or equal to x. □

Unsolved problem (44) of [4] deals with primes, but in this case how far away a given integer is from the nearest prime.

44) Smarandache Prime Additive Complements:

Definition 14: For each positive integer n, SPAC(n) is the smallest number k such that n+k is prime.

The first thirty numbers in the SPAC(n) sequence are:

1,0,0,1,0,1,0,3,2,1,0,1,0,3,2,1,0,1,0,3,2,1,0,5,4,3,2,1,0,1

Theorem 16: The range of SPAC(n) is all positive numbers.

Proof: It is well-known that there are arbitrarily large gaps between primes. Therefore, it is possible to find two primes p > q such that p - q > M for any positive integer M. From this SPAC(p+1) = p-q-1, SPAC(p+2) = p-q-2, . . . ,SPAC(q-1) = 1, creating a sequence from k = p-q-1 down to 1. Since k can be made arbitrarily large, all positive integers are included. □

Note 1: Notice that the above theorem does not state that every possible odd number can be used for k. Which would be equivalent to proving that every even number is the difference between two primes. And that problem is still unsolved[6].

Note 2: Given that the gap can be made arbitrarily large, the following result is unaffected if the definition is the nearest prime rather than the nearest prime greater than or equal to n.

Computing the values of the Smarandache function involves the values the factorial function

$$n! = n*(n-1)*(n-2)* \ldots *1.$$

Given the following well-known theorem.

Theorem 17: Let $m = p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_k^{\alpha k}$. Then

$$S(m) = \max \{ S(p_1^{\alpha 1}), S(p_2^{\alpha 2}), \ldots, S(p_k^{\alpha k}) \}.$$

It follows that if S(m) = n, then for any number k, where S(k) < n, S(km) = n. This leads to multiple solutions to equations involving the Smarandache function. One component of counting the number of solutions involves determining how many instances of a prime appear in a given factorial product.

The following sequence, (61) of [5], involves counting how many powers of two appear in the positive integers.

61) Smarandache Exponents of the Power 2.

Definition 15: SE2(n) = k if 2^k divides n but 2^{k-1} does not.
= 0 if 2 does not divide n.

The values of this sequence for the first 32 positive integers are:

0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4,0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,5, . . .

containing an obvious pattern.

Theorem 18: Let n be a positive integer. Then the number of instances of the prime number 2 in n! is given by

$$\sum_{i=1}^{n} SE2(i) .$$

Proof: By the definition, n! contains every number less than or equal to n as a factor. For any given number k, SE2(k) is the number of instances of 2 as a factor of k. The sum of all the values of SE2(i) is then the total number of instances of 2 in all numbers less than or equal to n. □

There is a very simple algorithm to compute the sum of theorem 18.

Algorithm 3:

Input: A positive integer n.

Output: Number_of_twos = $\sum_{i=1}^{n} SE2(i)$.

Variables: two_to_power is the current power of two in the division.

Step 1: Make the following integer assignments:
        number_of_twos = 0;
        two_to_power = 2;
Step 2: While(two_to_power $\leq$ n)
        Step 2.1: number_of_twos = number_of_twos + (n / two_to_power);
        Step 2.2: two_to_power = two_to_power*2;
        Step 2.3: End of loop.
Step 3: End of algorithm.

Proof of algorithm: Dividing n by 2 will count the number of times a number evenly divisible by 2 appears in the list

1, 2, 3, 4, . . . , n.

Dividing n by 4 will count the number of times a number evenly divisible by 4 appears in the list. Since one of those instances of 2 was already included in the previous step, we add only one per instance of divisibility by 4. Repeating by dividing by 8 will count the number of times a number appeared that was divisible by 8. Since two of those instances of 2 have already been counted, we need add only one. Continuing until two_to_power $>$ n is equivalent to continuing until there are no more powers of 2 to count. □

Note: The above algorithm can be used for any arbitrary prime p. Simply change the line

two_to_power = 2;   to   two_to_power = p;

and the line

Step 2.2: two_to_power = two_to_power*2;   to

Step 2.2: two_to_power = two_to_power*p; .

Which allows for the treatment of the general problem, (63) in [5].

63) Smarandache Exponents of Power p:

Definition 16: $SE_p(n) = k$ if p^k divides n and p^{k+1} does not.
$= 0$ if p does not divide n.

Definition 17: Use the notation $TP_p(n)$ where p is prime to denote the function

$$TP_p(n) = \sum_{i=1}^{n} SE_p(i) \quad \text{if } p \le n$$

$$= 0 \quad \text{if } p > n.$$

Which as has been mentioned before, would be the number of instances of the prime factor p that occur in n!. □

This function has many uses when dealing with some of the consequences of the Smarandache function S(n), most notably, the number of solutions to equations of the form

$$S(m) = n.$$

To demonstrate this, we need two other well-known theorems concerning the Smarandache function. For proofs of these theorems, see[2].

Theorem 19: If p is a prime, then $S(p) = p$.

Theorem 20: $S(m) \leq m$ for all positive integers m.

If p is a prime, we know by theorem 19 that S(p) = p. Let $TP_2(p) = m$. Then $2^m \mid p!$. Applying theorem 17, it follows that $S(2^j * p) = S(p) = p$, for $1 \leq j \leq m$. Therefore, there are m solutions to S(np) = p where n is a power of 2. Now, let $TP_3(p) = r$ so that there are r solutions to S(np) = p where n is a power of three. Furthermore, there would be mr solutions to the equation S(np) = p where n contains a power of two and a power of three as a factor.

This method can be continued for all primes less than p, but computing the number of solutions is not our point here. Suffice it to say that the number of integers m such that

$$S(m) = p$$

grows rapidly as p does.

Sometimes it is advantageous to use a computer to compute the terms of a sequence. Then, by examining the terms, it may be possible to discern a significant pattern. Peter Castini sent the author an unpublished manuscript[7] in which several sequences are defined. The stated challenge was to write computer programs to compute the terms of the given sequences.

Many of the series of sequences listed in the paper are called Smarandache-Recurrence Type Sequences and involve sums of powers.

Definition 18: The Smarandache-Recurrence Type Sequence for Sums of Two Squares(SS2) is recursively defined:

1) $1,2 \in SS2$.
2) If $b,c \in SS2$, then $a^2 + b^2 \in SS2$.
3) Only number constructed by rules 1 and 2 are in SS2.

A program to construct the terms of this sequence was written. The language was C++ and that program appears below. It is designed to compute all terms whose value is less than 100,000,000 and dump those terms into the file called smarseq.dat.The values are stored as long integers, so there is an inherent limit of slightly over 2,000,000,000 in the size of the terms. It is easy to modify this program to compute the terms of additional sequences and the locations of those changes are pointed out in the code.

Note on programs: All of the C++ programs appearing in this book were written using the Borland C++ compiler ver 4.5. The source files were always given the extension *.cpp so the compiler treated them as C++ files even when the code is essentially C.

The other programs were written in UBASIC ver. 7.25, an extended precision language very similar to original BASIC, line numbers and all. UBASIC is in the public domain so acquisition is easy. Anyone interested in obtaining an older version can contact the author, although later versions are no doubt available elsewhere. Such programs were used when the size of the numbers overflowed the storage capacity of C++ long integers. All primality checks and factoring were also done using UBASIC programs. □

```
#include<stdio.h>
// Given that such a large number of terms are to be computed, the items are all stored in a
// doubly linked list rather than an array. The following class contains a long integer which
// is the value of the term as well as the pointers to the previous and next terms in the list.

class sequence_member
{
  public:
  long value;
  sequence_member *pprev,*pnext;
};

void main()
{
// This is the pointer to the file.
 FILE *fp1;
// Terminate serves as a flag to exit the loop.
 unsigned char terminate;
// This integer serves to store the smallest value found in the search for the next term.
 long minterm;
// This integer is used to store the current candidate as an additional term in the sequence.
 long test;
// This integer is used to count the number of the term currently being computed.
 long lcount;
// Pointers to the head and tail of the list of terms in the sequence.
 sequence_member *head_of_list,*tail_of_list;
// Working pointers to move through the list of terms.
 sequence_member *pmember,*tmember;
 fp1=fopen("smarseq.dat","w");

// Create the first item in the sequence and assign it the value 1.
 pmember=new sequence_member;
 pmember->value=1;
 pmember->pprev=pmember->pnext=NULL;
 head_of_list=tail_of_list=pmember;
// Dumps the term to the file in the case where term number is not necessary.
```

```
 fprintf(fp1,"1\n");
 // Dumps the term to the file in the case where term number is desired.
// fprintf(fp1,"1 1\n");

// Create the second item in the sequence and assign it the value 2.
  pmember=new sequence_member;
 pmember->value=2;
 pmember->pprev=pmember->pnext=NULL;
 head_of_list=tail_of_list=pmember;
 // Dumps the term to the file in the case where term number is not necessary.
  fprintf(fp1,"2\n");
 // Dumps the term to the file in the case where term number is desired.
// fprintf(fp1,"2 2\n");
 lcount=3;
 terminate=1;
 while(terminate)
 {
// Initial setting of mingood that is infinity in this context. Guarantees that it will exceed
// the first value computed.
   mingood=2000000000L;
// Search through all items in the list and find the smallest number mingood that satisfies
// the following conditions:
//
// a) mingood is greater than all terms currently in the list
// b) mingood = a*a + b*b where a and b are terms already in the list.
// c) mingood is the smallest number satisfying conditions (a) and (b).

   pmember=head_of_list;
   while(pmember!=tail_of_list)
   {
    tmember=pmember->pnext;
    while(tmember!=NULL)
    {
     test=pmember->value*pmember->value+tmember->value*tmember->value;
     if((test>tail_of_list->value)&&(test<mingood))
     {
      mingood=test;
     }
     tmember=tmember->pnext;
    }
    pmember=pmember->pnext;
   }
```

```
// If the current value of mingood is within the desired bounds, add it to the list. Otherwise
// terminate the loop.
  if(mingood<100000000L)
   {
// Dumps only the value of the term to the file.
    fprintf(fp1,"%ld\n",mingood);
// Dumps the number and value of the term to the file.
//  fprintf(fp1,"%ld %ld\n",lcount,mingood);
    lcount++;
// Create the new term and add it to the tail of the list.
    pmember=new sequence_member;
    pmember->value=mingood;
    pmember->pprev=tail_of_list;
    pmember->pnext=NULL;
    tail_of_list->pnext=pmember;
    tail_of_list=pmember;
   }
  else
   {
    terminate=0;
   }
  }
 fclose(fp1);

// De-allocate the memory of the linked list.
 pmember=head_of_list;
 while(pmember!=NULL)
 {
   tmember=pmember;
   pmember=pmember->pnext;
  delete tmember;
  }
} // end of main function
```

If one wishes to change the initial two values of the sequence, it is only necessary to alter the assignments to the first two terms of the linked list.

Another sequence defined in the letter by Castini uses cubes rather than squares.

Definition 19: The Smarandache-Recurrence Type Sequence for the Sum of Two Cubes (CS2), is also recursively defined:

1) $1,2 \in CS2$.

2) If a,b $\in$ CS2, then $a^3 + b^3 \in$ CS2.
3) Only numbers formed using the rules (a) and (b) are in CS2.

Replacing the line

```
test = pmember->value*pmember->value + tmember->value*tmember->value;
```

by the line

```
test = pmember->value*pmember->value*pmember->value +
       tmember->value*tmember->value*tmember->value;
```

will alter the previous program so that it will compute the values of CS2.

However, one must be careful with these values. Computing large numbers of elements of this and any sequence like it will always eventually run one up to and beyond the limit of storage of the identifiers.

Another sequence defined in [7] is a modification of the definition of SS2.

Definition 20: The "converse" of the sequence SS2(n) is NSS2(n) and is defined using the following recurrence:

1) 1,2 $\in$ NSS2.
2) If b,c $\in$ NSS2, then $b^2 + c^2 \notin$ NSS2.
3) Only numbers obtained by rules 1) and 2) are in NSS2.

Or put another way, NSS2(n+1) is the smallest number, strictly greater than NSS2(n), which is not the sum of the squares of two previous distinct terms of the sequence.

A program to compute the terms of NSS2 can be created by adding a few features to the previous one, and the modified program follows.

```
// Program to compute the elements of the sequence NSS2(n).
#include<stdio.h>
// Given that such a large number of terms is to be computed, the items are all stored in a
// doubly linked list rather than an array. The following class contains a long integer which
// is the value of the term as well as the pointers to the previous and next terms in the list.

 class sequence_member
 {
  public:
  long value;
```

```
  sequence_member *pprev,*pnext;
 };

void main()
{
// This is the pointer to the file.
 FILE *fp1;
// Terminate serves as a flag to exit the loop.
 unsigned char terminate;
// This integer serves to store the smallest value found in the search for the next term.
 long minterm;
// This integer is used to store the current candidate as an additional term in the sequence.
 long test;
// This integer is used to count the number of term currently being computed.
 long lcount;
// This integer stores the value of mingood used to add items to the list.
 long lastmingood;
// This integer will be a counter to the last item added to the list and the smallest number
// that is the sum of two squares.
 long addem;
// Pointers to the head and tail of the list of terms in the sequence.
 sequence_member *head_of_list,*tail_of_list;
// Working pointers to move through the list of terms.
 sequence_member *pmember,*tmember;
 fp1=fopen("smarseq.dat","w");

// Create the first item in the sequence and assign it the value 1.
 pmember=new sequence_member;
 pmember->value=1;
 pmember->pprev=pmember->pnext=NULL;
 head_of_list=tail_of_list=pmember;
// Dumps the term to the file in the case where term number is not necessary.
 fprintf(fp1,"1\n");
// Dumps the term to the file in the case where term number is desired.
// fprintf(fp1,"1 1\n");
// Create the second item in the sequence and assign it the value 2.
 pmember=new sequence_member;
 pmember->value=2;
 pmember->pprev=pmember->pnext=NULL;
 head_of_list=tail_of_list=pmember;
// Dumps the term to the file in the case where term number is not necessary.
 fprintf(fp1,"2\n");
// Dumps the term to the file in the case where term number is desired.
```

```
// fprintf(fp1,"2 2\n");
 lcount=3;
 lastmingood=0;
 terminate=1;
 while(terminate)
 {
// Initial setting of mingood that is infinity in this context. Guarantees that it will exceed
// the first value computed.
  mingood=2000000000L;
// Search through all items in the list and find the smallest number mingood that satisfies
// the following conditions:
//
// a) mingood is greater than all terms currently in the list
// b) mingood = a*a + b*b where a and b are terms already in the list.
// c) mingood is the smallest number satisfying conditions (a) and (b).

  pmember=head_of_list;
  while(pmember!=tail_of_list)
  {
   tmember=pmember->pnext;
   while(tmember!=NULL)
   {
    test=pmember->value*pmember->value+tmember->value*tmember->value;
    if((test>tail_of_list->value)&&(test<mingood))
    {
     mingood=test;
    }
    tmember=tmember->pnext;
   }
   pmember=pmember->pnext;
  }

// If the current value of mingood is within the desired bounds, add all numbers between
// the value of the tail of the list and mingood to the list. Otherwise terminate the loop.
 if(mingood<100000000L)
 {
  for(addem=tail_of_list->value+1;addem<mingood;addem++)
  {
   fprintf(fp1,"%ld\n",addem);
//  fprintf(fp1,"%ld %ld\n",lcount,addem);
   lcount++;
// Create the new term and add it to the tail of the list.
   pmember=new sequence_member;
```

```
    pmember->value=addem;
    pmember->pprev=tail_of_list;
    pmember->pnext=NULL;
    tail_of_list->pnext=pmember;
    tail_of_list=pmember;
   }
  }
  else
  {
   terminate=0;
  }
 }
 fclose(fp1);

// De-allocate the memory of the linked list.
 pmember=head_of_list;
 while(pmember!=NULL)
 {
  tmember=pmember;
  pmember=pmember->pnext;
  delete tmember;
 }
} // end of main function
```

As was the case with the squares, there is a similar sequence defined for cubes[7].

Definition 21: The converse of CS2 is denoted by NCS2 and has the recursive definition:

1) 1,2 ϵ NCS2.
2) If c,d ϵ NCS2, then $c^3 + d^3 \notin$ NCS2.
3) Only numbers that can be constructed using steps 1 and 2 are in NCS2.

In other words, NCS2(n+1) is the smallest number strictly greater than NCS2(n) that is not the sum of two cubes of previous terms in the sequence.

Again, changing one line of the previous program is all that is necessary to compute this sequence.

Definition 22: The sequence SS122 defined by the author is a simple modification of SS2 and has the following recursive definition:

1) 1,2 ϵ SS122.
2) If a ϵ SS12, then a^2 ϵ SS122.

3) If a,b $\in$ SS12, then $a^2 + b^2$ $\in$ SS122.
4) Only numbers formed by steps (1), (2) or (3) are in SS122.

In other words, SS122(n+1) is the smallest number strictly greater than SS122(n) that is the sum of one or two squares of numbers already in SS122.

To compute the values of SS122, the only change necessary to the above program is to modify the assignments of the initial elements.

```
// Create the first item in the sequence and assign it the value 0.
 pmember=new sequence_member;
 pmember->value=0;
 pmember->pprev=pmember->pnext=NULL;
 head_of_list=tail_of_list=pmember;
 // Do NOT dump it to the file.
//  fprintf(fp1,"0\n");
 // Dumps the term to the file in the case where term number is desired.
// fprintf(fp1,"1 1\n");

// Create the second item in the sequence and assign it the value 1. This will be the first
// number dumped to the file. The numbering of the items of the sequence will be one less
// than the number of the item in the doubly-linked list used to create it.
  pmember=new sequence_member;
  pmember->value=1;
  pmember->pprev=pmember->pnext=NULL;
  head_of_list=tail_of_list=pmember;


 // Dumps the term to the file in the case where term number is not necessary.
  fprintf(fp1,"1\n");
 // Dumps the term to the file in the case where term number is desired.
// fprintf(fp1,"1 1\n");

// Create the third item in the sequence and assign it the value 2.
  pmember=new sequence_member;
  pmember->value=2;
  pmember->pprev=pmember->pnext=NULL;
  head_of_list=tail_of_list=pmember;
 // Dumps the term to the file in the case where term number is not necessary.
  fprintf(fp1,"2\n");
 // Dumps the term to the file in the case where term number is desired.
// fprintf(fp1,"2 2\n");
```

Where numbers that are the sum of one square are actually computed as the sum of two squares, one of which is zero.

Clearly, a similar modification to the program for cubes will allow for the computation of the elements of the sequence SS123.

Definition 23: The sequence SS123, a slight modification of CS2 is defined in the following way:

1) 1,2 ϵ SS123.
2) If a ϵ SS123, then a^3 ϵ SS123.
3) If a,b ϵ SS123, then $a^3 - b^3$ ϵ SS123.
4) Only numbers formed by rules (1), (2) and (3) are in SS123.

The given programs, along with the stated modifications will compute the elements of four of the sequences in the paper by Castini. That is enough for now, so it is time to move on to other things.

The following problem is defined in an unpublished manuscript sent to the author by A.Srinivas[8], a student at Arizona State University.

Definition 24: Smarandache Lucas-Partial Digital Sub-Sequence:

123 is a number where the sum of two initial groups of the digits is the same as the number formed by the remaining digits, i.e. 1 + 2 = 3. It is also a number in the Lucas sequence defined by

$$L(0) = 2,\ L(1) = 1,\ L(n+2) = L(n+1) + L(n) \quad \text{for } n \geq 2.$$

All numbers possessing these properties are members of the Smarandache Lucas-Partial Digital Sub-Sequence.

The manuscript author then posed the question:

Is 123 the only Lucas number that satisfies a Smarandache Type Partition?

The following C++ program was written to search for additional solutions to this problem. Unfortunately, the unsigned long integer of C++ only allows for numbers up through 4,000,000,000, which does not really allow for serious searches. It is easily modified to deal with other types of numbers, so it is included here.

```
#include<stdio.h>
```

```
void main()
{
// These three numbers are the three Lucas numbers used in the construction of additional
// numbers in the sequence.
 unsigned long l0,l1,l2;
// These arrays will store the digits of the number l2. They will first be placed in the array
//  first[..] in reverse order. They will then be copied in inverse order into the array
// digs[..] so that digs[0] is the most significant digit.
 int first[15],digs[15];
// Firsterm is the number constructed from the first group of digits, secondterm from the
// second group and thirdterm from the remaining group.
 unsigned long  firsterm,secondterm,thirdterm;
// Temp is used for temporary storage and power stores the current power of ten needed
// in the building of the integer.
 unsigned long temp,power;
// The following integers are all counters of one form or another.
 int m,i,j,i1,j1,k,count,lucascount;
// The first three digit Lucas number is L(10) = L(9) + L(8), formed from the sum
// 123 = 47 + 76. Lucascount is the subscript of the current Lucas number L(lucascount).
 lucascount=10;
 l0=47;
 l1=76;
 l2=l1+l0;
// The following while loop is used to terminate before overflow of the unsigned long.It
// must be used cautiously as the sum may overflow into the negative numbers before this
// test is done.
 while(l2 < 4000000000L)
 {
// Dump the current count to the screen, if desired.
//   printf("%ld\n",lucascount);
// Count is used to keep track of the number of digits in l2, the number of interest. Temp
// is used as the number to be split up and the digits of l2 are placed in the array first[..].
 count=0;
 temp=l2;
 while(temp>0)
 {
  first[count]=temp%10;
  temp=temp/10;
  count++;
 }

// M+1  is the number of digits in l2. Since arrays in C++ start at zero, the last filled
// position is first[m]. The next step is to place the digits into the array
```

```
// digs[..] in the proper order.
 m=count-1;
 for(i=0;i<=m;i++)
 {
  digs[i]=first[m-i];
 }

// The next step is to split the digits up into three groups. Firsterm will be the number
// formed from the first group and is constructed from the digits in the array positions
//  digs[0]...digs[i]. Secondterm is formed from the second group and is constructed
// from the array positions digs[i+1]...digs[j]. Thirdterm is the last group and is formed
// from array positions digs[j+1]...digs[m].

 for(i=0;i<m-1;i++)
 {
// Construct firsterm by going from position i to position 0 in the array.
   firsterm=0;
   power=1;
   for(i1=i;i1>=0;i1--)
   {
    firsterm=firsterm+digs[i1]*power;
    power=power*10;
   }
// Construct secondterm by going from position j to position i+1 in the array.
   for(j=i+1;j<=m-1;j++)
   {
    secondterm=0;
    power=1;
    for(j1=j;j1>i;j1--)
    {
     secondterm=secondterm+digs[j1]*power;
     power=power*10;
    }
// Construct thirdterm by going from position m to j+1.
     thirdterm=0;
     power=1;
     for(k=m;k>j;k--)
     {
      thirdterm=thirdterm+digs[k]*power;
      power=power*10;
     }
     sum=firsterm+secondterm;
```

```
// If the sums match dump the solution to the screen
      if(sum==thirdterm)
      {
       printf("lucascount %d\",lucascount);
       printf("%ld %ld %ld\n",firsterm,secondterm,thirdterm);
       printf("%ld\n",sum);
      }
    }
  }

// Increment lucascount and compute the value of the next Lucas number.
  lucascount++;
  l0=l1;
  l1=l2;
  l2=l1+l0;
 }
}
```

This program was run up through all values that the unsigned longs can store. The additional solution

L(35) = 20633239   206 + 33 = 239

was found.

In problems of this nature, limitations in the size of the numbers creates a very serious bottleneck. UBASIC is an extended precision language that is similar to original BASIC (it contains mandatory line numbers), that allows integers with large numbers of digits. It is also in the public domain, so it is readily available for use.

The following program is a UBASIC translation of the previous C++ program.

```
 10 dim first%(600),digs%(600)
 40 lucascount=10
 50 l0=47
 60 l1=76
100 l2=l1+l0
110 print lucascount
200 count=1
210 temp=l2
220 temp=temp\10
230 first%(count)=res
240 if temp<1 then goto 400
```

```
250 count=count+1
260 goto 220
400 for i=1 to count
410 digs%(count-i+1)=first%(i)
420 next i
500 for i=1 to count
510 firsterm=0
520 power=1
530 for i1=i to 1 step -1
540 firsterm=firsterm+digs%(i1)*power
550 power=power*10
560 next i1
580 for j=i+1 to count-1
590 secondterm=0
600 power=1
610 for j1=j to i+1 step -1
620 secondterm=secondterm+digs%(j1)*power
630 power=power*10
640 next j1
660 thirdterm=0
670 power=1
680 for k=count to j+1 step -1
690 thirdterm=thirdterm+digs%(k)*power
700 power=power*10
710 next k
```

This program was run up to lucascount = 442, where

L(442) = 235796302396688714339528028218191397124451051589940331822714896019626503374749639733405162203

and found only the two solutions already mentioned.

The Fibonacci sequence has a similar definition

$$F(0) = 0,\ F(1) = 1,\ F(n+2) = F(n+1) + F(n)$$

and the first three digit Fibonacci number is

$$F(12) = 144 = 89 + 55.$$

It is easy to modify the following two programs to search for solutions to the given problem that are Fibonacci numbers. Simply modify the initial lines to

```
lucascount=12
l0=55
l1=89
```

The UBASIC program was run up to lucasnumber=419, and the only solution discovered was

$F(30) = 832040 \qquad 8 + 32 = 040.$

The author is well aware that the performance of this algorithm can be dramatically improved. For example, if the number of digits in firsterm or secondterm exceeds the number in thirdterm, then it is already known that it is not possible for the sum to equal thirdterm. One immediate correction that can be done is to terminate the loop on i once more than half the digits in digs[..] are being used to create firsterm.

There are many other sequences of numbers that may satisfy the Smarandache type partition for some value. Some examples are:

Cullen numbers $C(n) = n * 2^n + 1$ for $n \geq 0$.

Catalan numbers $Ca(n) = 1$ if $n = 1$ $\frac{1}{n}\binom{2n-2}{n-1}$ if $n \geq 2$

Triangular numbers $T(n) = n(n+1)/2$

Tetrahedral numbers $Th(n) = \frac{n(n-1)(n-2)}{6}$

Factorials $Fact(n) = n!$

Smarandache numbers $S(n)$.

The programs are easy to modify to perform the search for any of these sequences of numbers. For example, the first three digit Cullen number is

$C(5) = 5*2^5 + 1 = 161$

Let powertwo be a new identifier and initialize it by

```
powertwo = 32
```

which will be the power of two segment of the number. The initialization for lucascount is

```
lucascount = 5
```

And the computation of the analog of l2, the number that is to be examined would then be

```
l2 = lucascount *powertwo + 1
```

The modification to prepare for the computation of the next Cullen number would then be

```
lucascount=lucascount+1
powertwo=powertwo*2
```

The UBASIC version of the program for the Cullen numbers was run up to lucascount = 282 and no solution was found.

After modifications to compute the values of the triangular numbers, the C++ version was run to search for solutions. Thirteen were found in the region $14 \le n \le 1099$ and a complete list of those solutions follows.

$T(25) = 325 \Rightarrow 3 + 2 = 5$
$T(77) = 3003 \Rightarrow 3 + 00 = 3 \Rightarrow 3 + 0 = 03$
$T(173) = 15051 = 1 + 50 = 51$
$T(214) = 23005 \Rightarrow 2 + 3 = 005$
$T(216) = 23436 \Rightarrow 2 + 34 = 36$
$T(286) = 41041 \Rightarrow 41 + 0 = 41$
$T(363) = 66066 \Rightarrow 6 + 60 = 66$
$T(479) = 114960 \Rightarrow 11 + 49 = 60$
$T(724) = 262450 \Rightarrow 26 + 24 = 50$
$T(819) = 335790 \Rightarrow 33 + 57 = 90$
$T(1011) = 511566 \Rightarrow 51 + 15 = 66$
$T(1095) = 600060 \Rightarrow 60 + 00 = 60 \Rightarrow 60 + 0 = 060$
$T(1099) = 604450 \Rightarrow 6 + 044 = 50$

The decision to terminate at 1099 was made solely because of the number of solutions discovered to that point. Notice that for two of the solutions, multiple sums are possible.

Modifications were then made to study the sequence of tetrahedral numbers, and the search was conducted for the range $8 \le n \le 2467$. Eleven solutions were found and a complete list follows.

$Th(22) = 2024 \Rightarrow 2 + 2 = 4$
$Th(76) = 76076 \Rightarrow 76 + 0 = 76$
$Th(77) = 79079 \Rightarrow 79 + 0 = 79$
$Th(274) = 3466100 \Rightarrow 34 + 66 = 100$
$Th(352) = 7331104 \Rightarrow 73 + 31 = 104$
$Th(368) = 8373840 \Rightarrow 837 + 3 = 840$

$Th(495) = 20337240 \Rightarrow 203 + 37 = 240$
$Th(560) = 29426320 \Rightarrow 294 + 26 = 320$
$Th(1188) = 280152180 \Rightarrow 28 + 152 = 180$
$Th(1804) = 980122220 \Rightarrow 98 + 122 = 220$
$Th(2467) = 2505440794 \Rightarrow 250 + 544 = 794$

Again, the decision to terminate at this point was made solely on the basis of the number of solutions discovered.

In all such sequences, the digits are not random, but repeat certain sequences according to the rules of arithmetic. With this in mind, the author makes the following conjectures.

Conjecture 1: There is no value of n such that the Cullen number

$$C(n) = n * 2^n + 1$$

satisfies the Smarandache type partition.

Conjecture 2: There are an infinite number of positive integers n such that the triangular number

$$T(n) = \frac{n(n+1)}{2}$$

satisfies the Smarandache type partition.

Conjecture 3: There are an infinite number of positive integers n such that the tetrahedral number

$$Th(n) = \frac{n(n+1)(n+2)}{6}$$

satisfies the Smarandache type partition.

It is readily conceded here that these conjectures are based on very little numeric evidence. They should be considered as items put forward for additional study by readers rather than as firm beliefs of the author.

The triangular numbers are the first particular instance of the general family of polygonal numbers. This is an infinite family of arithmetic sequences of the second order and the general formula for the family is

$$z_n = \frac{n}{2} [2 + (n - 1)d]$$

where d = 1,2,3, . . .

If d = 1, we have the triangular numbers used previously

$$z_n = \frac{n}{2}[\, 2 + (n - 1)\,] = n(n+1) / 2.$$

If d = 2, we have the square numbers

$$z_n = \frac{n}{2}[\, 2 + 2n - 2] = n^2.$$

If n = 3, we have the pentagonal numbers

$$z_n = \frac{n}{2}[\, 2 + 3n - 3] = n(3n-1)/ 2.$$

etc.

Of course, each of these additional sequences could form the basis for further exploration using the given computer programs.

Another sequence of numbers that can be investigated are the pyramidal numbers, given by the formula

$$P(n) = \frac{n(n-1)(2n-1)}{6}.$$

Reader Exercise 1: Search for values of n such that the corresponding pyramidal number satisfies the Smarandache type partition.

The name of the tetrahedral numbers is derived from the fact that if a triangular number of balls is placed as a base and all subsequently smaller triangular numbers piled on top, the construction is a tetrahedron. If one abstracts this into the fourth dimension, then one can start with a tetrahedral number as a base and "place" subsequently smaller tetrahedral numbers on top, creating a four dimensional "tetrahedron." The formula for this sequence is

$$FDT(n) = \frac{n(n-1)(n-2)(n-3)}{24}.$$

Reader exercise 2: Search for values of n such that the corresponding four dimensional tetrahedral number satisfies the Smarandache type partition.

There remains an enormous amount of unexplored territory here. At this time we will leave it and let the reader explore further if they choose. As was mentioned in the preface, the author is interested in hearing from any reader who makes progress in this area. Especially if anyone is able to verify or refute the conjectures.

Another sequence defined in [8] concerns splitting the numbers up into two pieces that satisfy a simple property.

Definition 25: The Smarandache Even-Digital Subsequence(SEDS) is the set of all numbers

$n = d_k \ldots d_2 d_1 d_0$

where n can be split into two pieces

$n1 = d_k \ldots d_i$ and $n2 = d_{i-1} \ldots d_1 d_0$

such that $2*n1 = n2$.

The first few numbers in the sequence are:

SEDS = { 12, 24, 36, 48, 510, 612, 714, . . . }

since 2*6 = 12 etc. Clearly, this set of numbers is infinite.

A question that is easy to answer concerns the number of elements in this set having a specified number of digits.

Theorem 21: For any number of digits k, the number of elements of SEDS depends on the parity of k.

a) If k = 2j, then the number of elements of SEDS having m digits is given by

$$\underbrace{400 \ldots 0}_{j-1\ 0's}.$$

b) If k = 2j+1, then the number of elements of SEDS having m digits is given by

$$\underbrace{500 \ldots 0}_{j-1\ 0's}.$$

Proof: Clearly, the number of digits in n1 must be less than or equal to the number of digits in n2. Furthermore, the difference in number of digits can be at most one.

a) With k = 2j, the principles listed above force the number of digits in n1 and n2 to both be j. Using simple arithmetic, all values in the range

$$n1 = 1\underbrace{00 \ldots 000}_{j-1\ 0's} \quad \text{to} \quad n1 = 4\underbrace{99 \ldots 999}_{j-1\ 9's}$$

yield a j-digit number when multiplied by 2.

b) With $k = 2j+1$, the principles listed above force the number of digits in n1 to be j and the number of digits in n2 to be j+1. Therefore, 2*n1 must yield a (j+1)-digit number. Again, using simple arithmetic, all numbers in the range

$$n1 = 5\underbrace{00 \ldots 0}_{j-1\ 0\text{'s}} \quad \text{to} \quad n1 = \underbrace{99 \ldots 9}_{j\ 9\text{'s}}$$

satisfy this condition. □

An additional sequence that is introduced in [8] deals with square numbers that can be partitioned into groups of digits that form square numbers.

Definition 26: The Smarandache Square-Partial-Digital Subsequence(SPDS) is the set of all numbers $n = d_k \ldots d_3 d_2 d_1 d_0$ such that $n = k^2$ for k some integer and there is some number i such that

$$n1 = d_k d_{k-1} \ldots d \qquad \text{and} \qquad n2 = d_{i-1} \ldots d_2 d_1 d_0$$

are both perfect squares. □

The first few numbers in this sequence are

49, 100, 144, 169, 361, 400, 441, ...

Note that the definition allows for the partition to have more than one part, as in

$441 \Rightarrow 4 = 2^2, 4 = 2^2$ and $1 = 1^2$.

Given that all numbers of the form

$$n = d\underbrace{00 \ldots 00}_{2k\ 0\text{'s}}$$

where d is a perfect square satisfy the conditions, it is clear that SPDS is an infinite set.

The definition is then followed by the question:

If numbers of the form

$$n = d\underbrace{000 \ldots 000}_{2k\ 0\text{'s}}$$

where d is a perfect square are removed, how many elements remain in SPDS?

Theorem 22: If all numbers of the form

$$n = d000 \ldots 000$$
$$\text{2k 0's}$$

where d is a perfect square are removed from SPDS, it remains an infinite set.

Proof: Consider the infinite family

$$102^2 = 10404$$
$$1002^2 = 1004004$$
$$10002^2 = 100040004$$
etc.

which remain in the set after removal of the first infinite family. Note that 100040004 can be split into squares several different ways

1^2 , $(0002)^2$ and $(0002)^2$
10^2, $(02)^2$ and $(0002)^2$
10^2, $(020)^2$ and $(02)^2$. □

It should be clear that as additional zeros are added to this family of solutions, the number of ways the number can be partitioned rises, going to infinity as the number of zeros goes to infinity.

The infinite family

$$212^2 = 44944$$
$$20102^2 = 404090404$$
$$2001002^2 = 4004009004004$$
$$200010002^2 = 40004000900040004$$

also satisfies the conditions of the problem.

Another problem also found in [8] has a similar definition.

Definition 28: A number n is a member of the Smarandache Square-Digital Subsequence (SSDS) if it satisfies the following conditions:

a) n is a perfect square.

b) All the digits of n are perfect squares, i.e. all digits of n are in the set { 0,1,4,9 }.

Notice that the numbers of the previous theorem can be used to verify that SSDS again remains infinite if all numbers of the form

$$n = d\underbrace{000\ \ldots\ 000}_{2k\ 0\text{'s}}$$

are removed.

The Smarandache Cube-Digital Subsequence(SCDS) of [8] has a definition similar to that of SSDS.

Definition 29: A number n is an element the Smarandache Cube-Digital Subsequence (SCDS) if it satisfies the following properties:

a) n is a perfect cube.

b) All the digits of n are a perfect cube, i.e. an element of the set { 0,1,8 }.

As can be seen from the initial elements of the set

{ 0,1,8,1000,8000,1000000,8000000, . . . }

SCDS is infinite.

Again the question can be asked:

If all numbers of the form

$$n = d\underbrace{000\ \ldots\ 000}_{3k\ 0\text{'s}}$$

where d $\in$ { 1,8 } are removed how many elements of SCDS remain?

A simple UBASIC program to search for elements of SCDS was written and run up to $n \leq 1 \times 10^{18}$ and no solution not an element of the infinite families

$$1\underbrace{000\ \ldots\ 00000}_{3k\ 0\text{'s}} \qquad 8\underbrace{000\ \ldots\ 000}_{3k\ 0\text{'s}}$$

was found.

Conjecture 4: There is no element of SCDS that is not of the form

$$d\underbrace{000\ldots00000}_{3k\ 0's}$$

where d = 1 or 8.

Rationale: From the computer evidence, it appears that there are no additional infinite families. As has been seen in previous work, as the number of digits in a number increases, the probability that the number contains only a certain small class of digits grows small very rapidly. Furthermore, a simple scan through all possible two digit endings

$02^3 = 08$ modulo 100
$12^3 = 28$ modulo 100
$22^3 = 48$ modulo 100
$32^3 = 68$ modulo 100
$42^3 = 88$ modulo 100
$52^3 = 08$ modulo 100
$62^3 = 28$ modulo 100
$72^3 = 48$ modulo 100
$82^3 = 68$ modulo 100
$92^3 = 88$ modulo 100

reveals that the two digit combination 18 cannot terminate a cube. It is also easy to verify that 10 also cannot terminate a cube.

Relaxing the restriction that all of the digits be cubes leads to a similar problem that is also defined in [8].

Definition 30: The Smarandache Cube-Partial-Digital(SCPD) sequence is the set of all numbers m satisfying the following properties:

a) m is a perfect cube.

b) It is possible to partition the digits of m into two or more groups that are themselves perfect cubes.

The first few numbers of SCPD are

1000, 8000, 10648, 27000, . . .

Clearly, all numbers of the form

m000 . . . 000
3k 0's
where m is a perfect cube are elements of SCPD.

With the inclusion of 10648, it is clear that not all elements of SCPD are of the form of those above. This brings up an obvious next question which is resolved below.

Theorem 23: There are an infinite number of elements of the set SCPD that are not of the form

m000 . . . 000
3k 0's

where m is a perfect cube.

Proof: If we examine the family of cubes

303^3 = 27818127
3003^3 = 27081081027
30003^3 = 27008100810027
300003^3 = 27000810008100027
3000003^3 = 27000081000081000027
etc.

it is clear that there is an infinite family of solutions. □

Note once again that the number of ways the cube can be partitioned goes to infinity as the number of zeros goes to infinity. Since no extensive search for such families was performed, it is very possible that additional such families exist.

If we replace cubes by primes, a much harder problem is created, which also appeared in [8].

Definition 31: A number m is an element of the Smarandache Prime-Digital Subsequence(SPDS) if it satisfies the following set of properties:

a) m is a prime.

b) All of the digits of m are prime, i.e. they are all elements of the set { 2,3,5,7 }.

The first few elements of SPDS are:

2,3,5,7, 23, 37, 53, 73, . . .

As a follow up, there was the conjecture:

Conjecture 5: SPDS is an infinite set.

Since so little is known about specific sequences of primes, that it should come as no surprise that this problem is as yet unsolved. This question is similar to another that also remains unsolved.

Unsolved problem 3: How many primes are there of the form

$$\underbrace{111 \ldots 111}_{k\ 1\text{'s}}$$

where of course k is odd.

It is very likely that any machinery used to resolve the unsolved problem will also have relevance in the search for a solution to the previous conjecture.

A short UBASIC program was written that counts the number of prime numbers as well as those that are elements of SPDS. The program was run for all numbers up to 1,000,000 and the counts were

78498 primes < 1,000,000

578 members of SPDS < 1,000,000.

Which is what one would expect. Assuming some form of even distribution of the primes, not allowing an initial digit of 1, 4, 6, 8 or 9 immediately eliminates $\frac{5}{9}$ of the primes. Adding additional digits places additional restrictive parameters on the numbers. All of which leads to the following question:

Unsolved problem 4: Let SPDSN(n) represent the number of elements of SPDS that are less than or equal to n. Then

$$\lim_{n \to \infty} \frac{\text{SPDSN}(n)}{\pi(n)} = 0$$

Note that a proof of the above is not a proof that SPDS is a finite set.

An additional problem similar to the above also appears in [8].

Definition 32: A number n is said to be a member of the set of the Smarandache Prime-Partial Digital Sequence(SPPDS) if it satisfies the following properties:

a) n is prime.

b) It is possible to partition n into groups of digits so that each group is prime. □

The first few elements of this sequence are:

23, 37, 53, 73, 113, 137, 173, . . .

With the corresponding conjecture that SPPDS is an infinite set.

Clearly, SPDS $\subset$ SPPDS, so any proof that SPDS is infinite implies that SPPDS is also infinite.

It seems very likely that SPPDS is indeed an infinite set. Since such a high percentage, $\frac{21}{90}$ of two digit numbers are prime and $\frac{143}{900}$ of three digit numbers are also prime, if a nonzero digit d is chosen at random, the probability is quite good that d is a component of either a two or three digit prime.

Example:
Assume that the digit 3 is embedded in a list of digits where all other digits are randomly distributed.

$$. . . . d_1d_23d_3d_4 . . . .$$

Since 03, 13, 23, 43, 53, 73 and 83 are all prime, the probability is $\frac{7}{10}$ that d_23 is prime, where three of the leading digits are not prime. The probability that $3d_3$ is prime is $\frac{2}{10}$.

There are 100 possible combinations for the leading digits d_1d_2 and 42 of those combinations yield a prime number that terminates in 3, so the probability that d_1d_23 is prime is $\frac{42}{100}$. Of the 100 possible combinations for d_3d_4, 16 yield a prime number of the form $3d_3d_4$.

Example:
Assume that the digit 8 is embedded in a list of digits that are randomly distributed

$$. . . . d_1d_28d_3d_4 . . . .$$

Clearly, there is no combination of d_28 or d_1d_28 that is prime. There are three choices for d_3 where $8d_3$ is prime and 15 choices for d_3d_4 that yield a prime number $8d_3d_4$.

Sieving the natural numbers is a well-known process. One way the prime numbers can be approached is to consider them to be the result of the sieving process known as the Sieve of Erasthosthenes.

Algorithm 4: Sieve of Erasthosthenes

1) Start at p=2.
2) Remove all multiples of p.
3) Move to the next number remaining in the list and call it p.
4) Go to step 2.

Other sieving methods modify steps (2) and (3) so that different classes of numbers are removed from the list. The most widely known example are the so-called Lucky numbers defined by the following by the following process.

Using the natural numbers

$$N = \{ 1, 2, 3, 4, \ldots \}$$

1) Set p = 2, start= 1.
2) Starting at start, strike out every p-th number in the remaining list.
3) Move p to the smallest number remaining in the list larger than the current p.
4) Goto step 2.

Those that remain after this process is carried out are said to be Lucky.

Note that the first step eliminates all numbers of the form 2k, and the second step all those of the form 6k-1. The key point is to realize that unlike the Sieve of Erasthosthenes, the elimination of numbers is not based on being evenly divisible. Therefore, the final result contains both prime and composite numbers.

The first few Lucky numbers are

1, 3, 7, 9, 13, 15, 19, 21, 25, 27, 31, 33, 37, 43, 49, 51, 63, . . .

It is pointed out in [8] that $L_3 = 7$ and 37 is a Lucky number and $L_4 = 9$ and 49 is also a Lucky number.

The question is then posed:

How many other Lucky numbers satisfy these conditions?

The Lucky numbers are generated by a process that eliminates numbers based on positions more than the properties of those numbers. Each Lucky number L(k) = m also generates a number km that is also based more on position than properties. Therefore, one would expect the probability that any number km where L(k) = m is a Lucky number to be based largely on the distribution of the Lucky numbers in the region of km. To investigate this further, a computer program to search for additional solutions was written in the language C++. The program was run for all Lucky numbers L(k) $\leq$ 100,000 and a complete list of solutions appears below.

L(3) = 7 and 37 is a Lucky number
L(4) = 9 and 49 is a Lucky number
L(6) = 15 and 615 is a Lucky number
L(9) = 31 and 931 is a Lucky number
L(15) = 63 and 1563 is a Lucky number
L(20) = 79 and 2079 is a Lucky number
L(21) = 87 and 2187 is a Lucky number
L(26) = 115 and 26115 is a Lucky number
L(28) = 129 and 28129 is a Lucky number
L(35) = 169 and 35169 is a Lucky number
L(40) = 201 and 40201 is a Lucky number
L(42) = 211 and 42211 is a Lucky number
L(54) = 285 and 54285 is a Lucky number
L(57) = 303 and 57303 is a Lucky number
L(63) = 339 and 63339 is a Lucky number
L(68) = 385 and 68385 is a Lucky number
L(80) = 475 and 80475 is a Lucky number
L(85) = 495 and 85495 is a Lucky number
L(88) = 519 and 88519 is a Lucky number
L(90) = 535 and 90535 is a Lucky number
L(95) = 577 and 95577 is a Lucky number
L(96) = 579 and 96579 is a Lucky number

Given that at least 22 out of the first 100 Lucky numbers satisfy the conditions, the following conjecture seems a safe one:

Conjecture 6: There are an infinite number of integers k such that Lucky number k is m (L(k) = m) and km is also a Lucky number. Note that in this case, km is k concatenated with m and not k multiplied with m.

Another form of sieve that can be used on the natural numbers is called the binary sieve in [5].

Using the natural numbers

$$N = \{ 1, 2, 3, 4, \ldots \}$$

1) Set p = 2, start= 1.

2) Starting at start, strike out every p-th number in the remaining list.
3) Multiply p by 2.
4) Goto step 2.

And the first few numbers in this sequence are:

1, 3, 5, 9, 11, 13, 17, 21, 25, 27, 29, 33, 35, 37, 43, . . .

There were two conjectures associated with this sieve.

a) There are an infinite number of primes in this sequence.

b) There are an infinite number of composite numbers in this sequence.

Part (b) is fairly easy to resolve, and the method of solution will also apply to similar problems.

Consider the list of natural numbers

1, 2, 3, 4, 5, 6, 7, 8, . . . , n

where n is very large.

The first step in the sieve deletes every second number, so the number left after this step is $\frac{n}{2}$. The next step removes every fourth number, so $\frac{3}{4}$ of those left after the first step will remain after the second. Continuing, $\frac{1}{8}$ are removed by the third step, $\frac{1}{16}$ by the fourth etc. The number remaining after performing all of the operations is then given by the product

$$n * [\frac{1}{2} * \frac{3}{4} * \frac{7}{8} * \frac{15}{16} * \ldots * \frac{2^k-1}{2^k}]$$

The term surrounded by the square brackets gives us the percentage of the natural numbers that remain after the sieve is performed. This product is non-zero and is slightly greater than 0.25. This will allow us to answer part (b).

The function $\pi(x)$ is the number of primes less than or equal to x and by the well known prime number

$$\pi(x) \approx \frac{x}{\log(x)}$$

for x large.

The percentage of numbers from 1 to x that are prime is then given by the ratio

$$\frac{\frac{x}{\log(x)}}{x}$$

which is clearly less than 0.25 for large x. Therefore, since there are a higher percentage of numbers left after the binary sieve than there are primes, the sieve leaves composite numbers. Taking the percentages to infinity it follows that the number of composite numbers must be infinite. □

Another sieve found in [5] is based on 3 rather than 2 and is called the trinary sieve.

Start with the natural numbers

$$N = \{ 1, 2, 3, 4, \ldots \}$$

1) Set p = 3, start= 1.
2) Starting at start, strike out every p-th number in the remaining list.
3) Multiply p by 3.
4) Goto step 2.

Where the first few numbers that remain are:

1, 2, 4, 5, 7, 8, 10, 11, 14, 16, 17, 19, 20, 22, 23, 25, 28, 29, 31, 32, 34, 35, 37, 38, 41, 43

Notice that for the trinary sieve, there are 26 numbers less than 44 and for the binary sieve there are 15. That holds in general and is easily explained. The first deletion in this case leaves $\frac{2}{3}$ of the numbers, the second $\frac{8}{9}$ of that , the third $\frac{26}{27}$ of the remainder and so on. The final product in this case is then

$$n * [\tfrac{2}{3} * \tfrac{8}{9} * \tfrac{26}{27} * \tfrac{80}{81} * \quad \ldots \quad * \tfrac{3^k - 1}{3^k}]$$

where the product of the fractions is clearly larger than the corresponding term for the binary sieve.

From this, it is clear that the trinary sieve also leaves an infinite number of composite numbers.

The more general problem is also defined in [5] and is called the n-ary sieve.

Start with the natural numbers

$$N = \{ 1, 2, 3, 4, \ldots \}$$

For n a positive integer greater than 1.

1) Set p = n, start= 1.
2) Starting at start, strike out every p-th number in the remaining list.
3) Multiply p by n.
4) Goto step 2.

It should also be obvious that if any n-ary sieve is performed on the natural numbers where $n > 3$, the set of numbers that are left after the sieve contains an infinite number of composite numbers.

Again, the numbers removed by the actions of the n-ary sieve are done on the basis of position rather than any divisibility properties that they possess. So, it seems reasonable that the likelihood of an r-digit number of the remaining numbers being prime is based more on the number of primes with r digits than anything else. Therefore, the conjecture that there are an infinite number of primes remaining after the action of any n-ary sieve appears to be a safe one.

Conjecture 7: If we start with the natural numbers

$$N = \{ 1, 2, 3, 4, 5, \dots \}$$

and perform the action of any n-ary sieve, the list of remaining numbers contains an infinite number of primes.

The first problem in [5] deals with the sequence of numbers formed by successively appending the natural numbers to the least significant end of the previous number in the sequence.

Definition 32: The Smarandache Consecutive Sequence(SCS) consists of all numbers satisfying the following properties:

1) $1 \in$ SCS.
2) If $n \in$ SCS where n consists of the first k natural numbers concatentated in the order

12345.....(k-1)k

then $n(k+1) \in$ SCS, where the operation is concatenation. □

The first few members of the sequence are

1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, 123456789, . . .

The question is then asked:

How many primes are contained in SCS?

Clearly, every other number is even and cannot be prime. Using another modulo argument, we can refine this much further.

Theorem 24: If p = 123456 ... k ϵ SCS, where p is prime, then $k \equiv 1$ modulo 3.

Proof: The following theorem is well-known

An integer n is divisible by 3 if and only if the sum of the digits of n are divisible by 3.

If we start at the number

$$n = 1234 \ldots k$$

where $k \equiv 0$ modulo 3 and n is evenly divisible by 3, the sum of the digits of

$$n1 = 1234 \ldots k(k+1)$$

will be congruent to 1 modulo 3. Since k+2 will then be congruent to 2 modulo 3, appending it to n1 will construct a number the sum of whose digits are evenly divisible by 3 and therefore divisible by 3. Appending a number with digit sum evenly divisible by 3 will then also create a number evenly divisible by 3 and we repeat the cycle.

The initial or basis number for this repeated cycling is

$$n = 1234.$$

Therefore, all numbers of this form where the terminal number is either 0 or 2 modulo 3 are evenly divisible by 3 and cannot be prime. □

Furthermore, every other number of the form 3k + 1 is even. So, in a search for primes, we are reduced to starting with an initial number

$$n = 1234 \ldots k$$

where $k \equiv 1$ modulo 3 and k is odd, and examining the sequence

$$n = 1234 \ldots k \ldots (k+6j).$$

Following this, the first number that can possibly be prime is

$$n = 1234567.$$

However the first nine numbers of this sequence are not prime, as

1234567 = 127 * 9721
12345678910111213 = 113 * 125693 * 869211457
12345678910111213141516171819 = 13 * 43 * 79 * 281 * 1193 * 8339294576458675 63.
12345678910111213141516171819202122232425 is evenly divisible by 5.
1234 . . . 28293031 = 29 * k
1234 . . . 353637 = 71*12378 * k
1234 . . . 414243 = 7 * 17 * 449 * k
1234 . . . 474849 = 23 * 109 * k
1234 . . . 535455 = 5 * k

Where the last number also shows that the every fifth number in the sequence to be searched is evenly divisible by 5.

Which leads to the following question, perhaps much easier than the similar one posed earlier:

Unsolved problem 5: Find the first prime member of SCS or prove that none are prime.

Another, similar sequence also found in [5] is called the Smarandache Symmetric Sequence(SSS).

1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 123454321, 1234554321, 12345654321, 123456654321, 1234567654321, 12345677654321, 123456787654321, 1234567887654321, . . .

With the corresponding question:

How many primes are there in this sequence?

Clearly, there is at least one, namely 11. As was the case in the previous sequence, examining the values of the numbers modulo 3 provides a way to eliminate many of the numbers.

Theorem 25: For any number of the form

$$n = 1234 \ldots (k-1)kk(k-1) \ldots 321$$

if $k \equiv 2$ modulo 3, then n is evenly divisible by 3.

Proof: If we examine any number of the form m = j(j+1)(j+2)

where j +2 is evenly divisible by 3, it is clear that m is evenly divisible by 3. Since the digit sum is also divisible by 3, any permutation into the form

m1 = (j+2)(j+1)j

does not alter the divisibility by 3.

Therefore, the digit sums of

1234 . . . (k-2) and (k-2)(k-3) . . . 321

are both evenly divisible by 3.

Examining the remaining quadruplet

(k-1)kk(k-1)

k - 1 $\equiv$ 1 modulo 3 and k $\equiv$ 2 modulo 3. Adding this up, we get
1 + 1 + 2 + 2 $\equiv$ 0 modulo 3. Therefore, the digit sum of n is evenly divisible by 3 and therefore so is n. □

Corollary 4: For any number of the form

n = 1234 . . . (k-1)k(k-1) . . . 4321

or

n = 1234 . . . (k-1)kk(k-1) . . . 4321

if k $\equiv$ 0 modulo 3, then n is evenly divisible by 3.

Proof: By the previous theorem, all numbers of the form

n = 1234 . . . (k-1)(k-1) . . . 4321

have a digit sum evenly divisible by 3. Since k has a digit sum evenly divisible by 3, the insertion of either k or kk yields a number also having digit sum evenly divisible by 3. □

Corollary 5: If n is of the form

n = 1234 . . . k . . . 4321

or

n = 1234 . . . kk . . . 4321

where k $\equiv$ 1 modulo 3 , then n is not evenly divisible by 3.

Proof: By the previous theorem

m = 1234 . . . (k-1)(k-1) . . . 4321

has a digit sum evenly divisible by 3. Insertion of one instance of k yields a number with digit sum congruent to one modulo 3 and two instances of k a number congruent to 2 modulo k. Therefore, neither of those numbers can be evenly divisible by 3. □

Corollary 6: If n is of the form

n = 1234 . . . k . . . 4321

where k $\equiv$ 2 modulo 3, then n is not evenly divisible by 3.

Proof: By the previous corollary,

m = 1234 . . . (k-1)(k-1) . . . 4321

is congruent to 2 modulo 3. Insertion of one instance of k yields a number congruent to 1 modulo k, and therefore cannot be evenly divisible by 3. □

All numbers in SSS up through

1234 . . . 111211 . . . 4321

were examined and the two additional primes

12345678910987654321 and 1234567891010987654321

were discovered.

The Smarandache Mirror Sequence(SMS) is a sequence of palindromic numbers defined in problem 5 of [5]

1, 212, 32123, 4321234, 543212345, 65432123456, 7654321234567, 876543212345678, 98765432123456789, 109876543212345678910, . . .

With the corresponding question

How many of the numbers in this sequence are prime?

Clearly, half of the numbers are even and cannot be prime. If we examine the digit sums of the numbers in this sequence, we are unable to eliminate any additional numbers, as

1 ≡ 1 modulo 3
212 ≡ 2 modulo 3
32123 ≡ 2 modulo 3
4321234 ≡ 1 modulo 3
543212345 ≡ 2 modulo 3
65432123456 ≡ 2 modulo 3
7654321234567 ≡ 1 modulo 3

and the pattern repeats indefinitely.

The odd numbers in the list up through

191817 . . . 32123 . . . 171819

were examined and the only prime discovered was

131211 . . . 32123 . . . 111213.

And so, there appears no reason to doubt that there are more primes in this sequence. As to whether there are an infinite number, nothing as yet would indicate that there is only a finite number of primes in this sequence.

Reader exercise 3: Find the next prime in the Smarandache Mirror Sequence or prove that none exist.

Problem (19) of [5] is called the Smarandache Pierced Chain(SPC). In this case, the sequence is

101, 1010101, 10101010101, 101010101010101, . . .

Clearly, each number in this sequence is evenly divisible by 101, so the posed question here is

How many elements of SPC(n) / 101 are prime?

The first few elements of this sequence are:

SPC(1)/101 = 101/101 = 1
SPC(2)/101 = 1010101/101 = 10001 = 73 * 137
SPC(3)/101 = 10101010101/101 = 3 * 7 * 13 * 37 * 9901
SPC(4)/101 = 101010101010101/101 = 17 * 73 * 137 * 5882353
SPC(5)/101 = 1010101010101010101/101 = 41 * 271 * 3541 * 9091 * 27961
SPC(6)/101 = 10101010101010101010101/101 = 3 * 7 * 13 * 37 * 73 * 137 * 9901 * 99990001

which points out some patterns.

a) If k is evenly divisible by 3, then so is SPC(k) as the number of 1's is also evenly divisible by 3.

b) SPC(2k) is evenly divisible by 73 for k = 1, 2, 3, 4, . . .

Proof: SPC(2) = 73 * 101 *137. Since SPC(4) is formed by appending the character string 01010101 to SPC(2) and

01010101 / 73 = 13837

it follows that SPC(4) is also evenly divisible by 73. Repeating this will guarantee that SPC(2k) is always evenly divisible by 73. □

c) SPC(3 + 4k) is evenly divisible by 37 for k = 1, 2, 3, 4, . . .

Proof: SPC(3) is evenly divisible by 37. SPC(7) is formed by appending the character string 010101010101 to SPC(3).

10101010101 = 37 * 273000273

so if SPC(3) is evenly divisible by 37, SPC(7) must be as well. Repeating this process gives the general result. □

Many similar results can also be verified, all of which indicate that it is unlikely that there are any primes in this sequence. Therefore, the question put forward is:

Unsolved problem 6: Find the first prime in the SPC sequence or prove that none exist.

Chapter 2

While the first chapter was devoted to Smarandache notions concerning sequences, the purpose of chapter two is to examine problems that do not involve sequences. Problems relating the Smarandache function to magic squares have also been posed by Mike Mudge in his regular 'Numbers Count' column in **Personal Computer World**.

Definition 33: A magic square is a collections of numbers $a_1, a_2, \ldots, a_k$ such that $k = n*n$ is a perfect square and the numbers can be placed in an n x n array where the sum on any column, row or diagonal is the same. For example, the numbers { 1, 2, 3, . . . , 16 } can be placed in the array

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

such that the common sum is 34.

It is well-known that magic squares can be constructed using only primes, so it is also known that there exist positive integers $a_1, a_2, \ldots, a_k$ such that the corresponding Smarandache Numbers $S(a_1), S(a_2), \ldots, S(a_k)$ can be used to construct a magic square. For example, consider the 3 x 3 magic square

101	29	83
53	71	89
59	113	41

composed only of primes. Since $S(p) = p$ for p a prime, the Smarandache values would form the same 3 x 3 magic square.

A logical extension of this problem places a restriction on the numbers.

Theorem 26: It is possible to find a set of numbers $a_1, a_2, \ldots, a_k$ where $k = n^2$ and not all a_i are prime such that the corresponding set of Smarandache numbers $S(a_1), S(a_2), \ldots S(a_k)$ can be used to construct an n x n magic square. In fact, there are an infinite number of such sets.

Proof: In the previous volume of this series[2], it was proven that the range of the S(n) contains all non-negative numbers except 1. It is also well-known that all 3 x 3 magic squares follow the pattern:

a+5b+2c	a	a+4b+c
a+2b	a+3b+c	a+4b+2c
a+2b+c	a+6b+2c	a+b

with magic sum $3a + 9b + 3c$.

Since a can be chosen as composite, we are guaranteed to have at least one composite entry in the array. Of course, this will guarantee several composite entries in the array. □

We continue our treatment of magic squares with the following problem:

Problem 1: Is it possible to find a set of numbers $A = \{ a_1, a_2, \ldots, a_k \}$ where $k = n^2$ and not all a_i are prime such that an n x n magic square can be constructed using the elements of A and it is also possible to construct an n x n magic square using the elements of the set $SA = \{ S(a_1), S(a_2), \ldots, S(a_k) \}$? Note that this does NOT mean that the positions of a_i and $S(a_i)$ are the same.

At first glance, this problem may appear to be complex, but in fact it has a simple solution.

Again, consider the 3 x 3 magic square

101	29	83
53	71	89
59	113	41

constructed of nine primes.

The smallest prime in this list is 29, so if we take any prime $p < 29$, we can multiply every entry in this square by that prime and the square will remain magic. If p and q are distinct primes, S(pq) = largest of p and q, so the Smarandache values will remain unaltered. Therefore, the resulting magic square satisfies the conditions of the problem.

For example, if we take p = 3, the square

303	87	249
159	213	267
177	339	123

is also magic with sum 639. The square of corresponding values of the Smarandache function would be the same as the previous.

In fact, any prime in the set { 2, 3, 5, 7, 11, 13, 17, 19, 23 } will yield a solution.

In 1988, Harry L. Nelson used a CRAY supercomputer to find the following 3 x 3 magic square composed of 9 consecutive primes[9].

1480028201	1480028129	1480028183
1480028153	1480028171	1480028189
1480028159	1480028213	1480028141

Clearly, multiplying the entries by any prime p $<$ 1480028129 will yield a solution to the given problem. Even that is not the end of the matter. If m is any number where S(m) $<$ 1480028129, then the entries of this square can be multiplied by m and the result is also a solution. Which means that the number of solutions to the problem rises rapidly as the size of the primes increases.

Another notion culled from the Smarandache archives is the concept of the Smarandache Bad Numbers.

Definition 34: A number n is said to be a Smarandache Bad Number if there are no integers r and s such that

$$n = | r^3 - s^2 |.$$

In the text of the manuscript sent to the author, it is stated that

5, 6, 7, 10, 13, 14

are probably bad numbers

and

1, 2, 3, 4, 8, 9, 11, 12, 15

are not as

$1 = | 2^3 - 3^2 |$ $2 = | 3^3 - 5^2 |$ $3 = | 1^3 - 2^2 |$ $4 = | 5^3 - 11^2 |$
$8 = | 1^3 - 3^2 |$ $9 = | 6^3 - 15^2 |$ $11 = | 3^3 - 4^2 |$ $12 = | 13^3 - 47^2 |$
$15 = | 4^3 - 7^2 |$.

And the problem concludes with a challenge to write a computer program to search for numbers that are not Smarandache bad numbers.

The following simple UBASIC program was written to search for numbers that are not bad Smarandache numbers.

```
 10 testnum=10
 20 cube=1
 30 print testnum
 40 t1=cube*cube*cube
 50 square=1
 60 t2=square*square
 70 t3=t1-t2
 80 t3=abs(t3)
 90 if t3<>testnum then goto 140
100 print testnum,t1,t2
110 print cube,square
120 input z%
130 goto 230
140 square=square+1
150 t2=square*square
160 t3=t1+testnum
170 if t2>t3 then goto 190
180 goto 70
190 cube=cube+1
200 if cube>1000 then goto 230
210 t1=cube*cube*cube
220 goto 50
230 testnum=testnum+1
240 goto 20
```

The program searches for solutions to the equation

testnum = cube * cube * cube - square * square.

For each value of testnum the search range of values for cube is

$$1 \leq \text{cube} \leq 1000.$$

And once the value of cube is fixed, the range for square is

$$1 \leq \text{square} < \text{cube*cube*cube} + \text{testnum}$$

The program uses a planned infinite loop, and if a solution is found, it is printed out and the program waits for a user response. Once the user responds by typing in an integer, the program proceeds to the next number. Clearly, if no solution is found, this does not mean that the number is Smarandache bad, just that there is no solution where the value of r is less than or equal to 1000.

The program was run for all values of testnum in the range [5,100]. The numbers 7 and 13, given as probably bad in the paper, are not as can be seen from

$$7 = 2^3 - 1^2 \qquad 13 = 17^3 - 70^2$$

Additional numbers within the examined range that are also probably Smarandache bad are

16, 21, 27, 29, 31, 32, 33, 34, 42, 43, 46, 50, 51, 52, 58, 59, 62, 66, 69, 70, 75, 77, 78, 82, 84, 85, 86, 88, 90, 91, 93, 96.

Problem (126) of [5] can easily be proven using well-known results of number theory.

126) Smarandache Divisibility Theorem:
If a and m are integers, and $m > 0$, then

$$(a^m - a)(m - 1)!$$

is divisible by m.

Proof: The following theorem in number theory is well-known,

If p is prime and a is a positive integer, then $a^p - a$ is evenly divisible by p.

which deals with the case of m being prime. So suppose, m is not prime. It follows that the prime factors of m are all less than m and by definition, each is found in $(m-1)!$. □

Problems (107), (108) and (109) of [5] all deal with the same concept.

107) Smarandache Paradoxist Numbers:

There exist a few "Smarandache" number sequences.

A number n is called a "Smarandache paradoxist number" if and only if n does not belong to any of the Smarandache defined numbers.

Solution?

1) If a number k is a Smarandache paradoxist number, then k does not belong to any of the Smarandache defined numbers, therefore k does not belong to the Smarandache paradoxist numbers either.

2) If a number k does not belong to any of the Smarandache defined numbers, then k is a Smarandache paradoxist number. Therefore, k belongs to a Smarandache defined number sequence, because the sequence of Smarandache paradoxist numbers is also in the same category. Which is a contradiction.

Dilemma: Is the Smarandache paradoxist number sequence empty? □

108) Non-Smarandache Numbers:
A number n is called a "non-Smarandache number" if and only if n is neither a Smarandache paradoxist number nor any of the Smarandache defined numbers.

Question: Find the non-Smarandache number sequence.

Dilemma 1: Is the non-Smarandache number sequence empty too?
Dilemma 2: Is a non-Smarandache number equivalent to a Smarandache paradoxist number? □

109) The paradox of Smarandache numbers:
Any number is a Smarandache number, the non-Smarandache numbers too.
This is deduced from the following paradox (see the reference[..]):

"All is possible, the impossible too!" □

Like many other statements, implied assumptions or ambiguous language are used to give a paradoxical appearance. At this time, we will attempt to resolve the apparent conflicts.

For purposes of notation, let SPN denote the set of Smarandache Paradoxist Numbers.

Also, let S_k denote an element of the collection of sequences of Smarandache defined numbers as sets. Furthermore, let SU represent the union of all these sequences

$$SU = \cup S_k$$

and SK the collection of sequences S_k

$$SK = \{ S_k \mid S_k \text{ is a Smarandache defined sequence } \}.$$

We have no explicitly defined set of discourse, so let U represent the universal set.

For each of the sets S_k, x is an element of that set by virtue of having satisfied the properties defining S_k. Therefore, any object x is an element of SU because it satisfies the properties defining at least one of the sets S_k.

Since there is no explicit definition of SK, we split the treatment into two cases, depending on whether or not SPN is an element of SK.

Case 1: SPN $\notin$ SK.

In this case, the results are simple. An object x is in SPN if it is not in any of S_k and by definition of SPN

$$SPN \cup SU = U.$$

If SU = U then SPN $= \emptyset$.

Note that there is a clear causality here, in that all of the objects S_k must be defined before SPN can be determined.

Case 2: SPN ϵ SK

Let S_i denote the element of SK that is SPN. Choose an arbitrary element x ϵ U and attempt to determine if x ϵ SPN. To do this, execute the following algorithm:

Step 1: Set j = 1.
Step2: If x ϵ S_j then x $\notin$ SPN and exit.
Step 3: Increment j by 1 if possible, if not exit with x $\in$ SPN.
Step 4: Go to step 2.

and at some point, we must reach the key set S_i. And it is here that there is a problem. When step 2 is performed, the question is reduced to

If x $\in$ SPN then x $\notin$ SPN

which is a contradiction of the laws of set theory if x is indeed an element of SPN. A statement equivalent to this would be

If x is even, then x is odd.

Which is a contradiction only if x is even and not a paradox.

Since all elements of U will be tested in this fashion, the only way to avoid a contradiction is to have SPN $= \emptyset$. This result does not violate any principles of number theory or any of the notation used here.

For example, the statement:

The set of integers that are both even and odd.

is not nonsense, it simply describes a set with no elements. Each item in the list S_k is a set, so we are notationally correct as well. □

This result then eliminates the doubts raised in the "solution?" that follows problem (107), where the implied assumption is that SPN $\in$ SK. In the case where SPN is empty, the if labeled with (1) is satisfied in the vacuous sense, i.e.

If a number is a Smarandache paradoxist number ....

is true because there are no such numbers.

The if labeled (2) is also satisfied in the vacuous sense as this is just a restatement of the previous if.

For problem (108) it is clear that the set of non-Smarandache numbers must be empty, as

$$SPN \cup SK = U.$$

by the definitions of the two sets. Note that this result is independent of whether or not SPN $\in$ SK.

This resolves the first dilemma following this problem.

Since it is a direct consequence of the definitions that there no such numbers, the answer to the second dilemma is actually yes, but in the vacuous sense.

Finally, problem (109) is not a paradox at all. Since the set of non-Smarandache numbers is empty, all non-Smarandache numbers are vacuously Smarandache numbers. This is a result of the way satisfaction is interpreted when there are no elements to test.

The supposed "paradox" following problem (109) is not a paradox, but another instance where vacuous satisfaction is taken to be a paradox.

Let E denote the set of all possible events. Define the subsets

$$P = \{ e \mid e \in E \text{ and } E \text{ is possible} \}$$

$$I = \{ e \mid e \in E \text{ and } E \text{ is impossible} \}$$

or equivalently

$I = \{ e \mid e \in E \text{ and } e \notin P \}$.

Note that by the choice of sets, $P \cup I = E$.

The truth of the statement

"All is possible, the impossible too!"

then follows from the laws of logic and set theory.

Proof: If the initial phrase

"All is possible ..."

is true then $I = \emptyset$ and the entire statement is true since the set I is empty, and the second component is vacuously true.

If the initial phrase is false, then by the laws of logic the entire statement is true as one can deduce any result from a false premise. □

And finally, problem (10) in [45] is sufficiently different from all others and is a good way to terminate this work.

10) Smarandache Logic:
Is it true that for any question there is at least an answer? Reciprocally: Is any assertion the result of at least a question?

This problem may or may not have a solution, depending on how the terms are defined. In a mathematical sense, the word assertion is used to refer to a statement that can be assigned a value of either true or false. Using that interpretation, it is always possible to create a question having that assertion as a result.

However, the well-known result of Godel's Incompleteness Theorem states that in any system powerful enough to perform arithmetic, there will always be statements that are true, but a proof is not possible. Sometimes this will mean that the proof is simply impossible and in other cases it will mean that a proof would require an infinite amount of time to carry out.

For example, the question:

The digits in the decimal expansion of π are randomly distributed.

is either true or false, but the answer may always remain indeterminate. At this time the lack of suitable theory requires that it would take an infinite amount of time to resolve.

From this, it then falls into the definition of the term answer. If indeterminate answers are not allowed, then it is a direct consequence of Godel's Theorem that there are questions for which it is mathematically impossible to form an answer. Therefore, the answer to the first part of the question is no.

At this point, it is time to stop once again. There are still unexplored regions in that area of mathematics loosely defined as the Smarandache notions, but they will be left for a future time.

It is again the authors fervent hope that you, the reader, have gained by examining this book. If anyone should feel an overpowering urge to comment on any conclusion expressed here, feel free to contact the author at the address given in the preface.

Appendix A

The following is a summary of all Smarandache notions that are represented as acronyms and the page where they are introduced.

SPS(n) - Smarandache Permutation Sequence, page 5.
SSC(n) - Smarandache Square Complements, page 9.
SCC(n) - Smarandache Cube Complements, page 9.
SPS3 - Smarandache Pseudo-Square of the third kind, page 14.
SPC3 - Smarandache Pseudo-Cube of the third kind, page 14.
SPO1 - Smarandache Pseudo-Odd number of the first kind, page 16.
SPO2 - Smarandache Pseudo-Odd number of the second kind, page 16.
SPE1 - Smarandache Pseudo-Even number of the first kind, page 17.
SPM15 - Smarandache Pseudo-Multiple of the first kind of 5, page 18.
SPM1P - Smarandache Pseudo-Multiple of the first kind of p, page 19.
SPAC(n) - Smarandache Prime Additive Complements, page 21.
SE2(n) - Smarandache Exponents of the Power 2, page 22.
$SE_p(n)$ - Smarandache Exponents of power p, page 24.
SS2 - Smarandache-Recurrence Type Sequence for sums of two squares, page 25.
CS2 - Smarandache-Recurrence Type Sequence for sums of two cubes, page 28.
NSS2(n) - converse of SS2, page 29.
NCS2(n) - converse of CS2, page 32.
SS122 - modification of SS2 where the sum is of one or two squares, page 33.
SS123 - modification of CS2 where the sum is of one or two cubes, page 34.
SLPSS - Smarandache Lucas-Partial Digital Sub-Sequence, page 34.
SEDS - Smarandache Even-Digital Subsequence, page 43.
SPDS - Smarandache Square-Partial-Digital Subsequence, page 44.
SSDS - Smarandache Square-Digital Subsequence, page 45.
SCDS - Smarandache Cube-Digital Subsequence, page 46.
SCPD - Smarandache Cube-Partial-Digital sequence, page 47.
SPDS - Smarandache Prime-Digital Subsequence, page 48.
SPPDS - Smarandache Prime-Partial Digital Sequence, page 49.
SCS - Smarandache Consequtive Sequence, page 55.
SMS - Smarandache Mirror Sequence, page 59.
SPC - Smarandache Pierced Chain, page 60.
SPN - Smarandache Paradoxist Number, page 67.

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In this book, we will explore several avenues of what are called Smarandache notions. The obvious question at this point is, "What is a Smarandache notion?" The answer is both simple and complex. A Smarandache notion is a problem in one of the following sets:

a) A problem posed by Florentin Smarandache.

b) A problem posed by someone else that is an extension of an element of set (a).

See **Some Notions and Questions in Number Theory**, edited by C. Dumitrescu and V. Seleacu, Erhus University Press, Glendale, 1994.

A Smarandache notion is an element of an ill-defined set, sometimes being almost an accident of labeling. However, that takes nothing away from the interest and excitement that can be generated by exploring the consequences of such a problem.

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